Homogenization in Electrostatic and Piezoelectric Transducers

DFG Junior Research Group

Inverse Problems in Piezoelectricity

Tom Lahmer *

Joachim Schöberl **

*) Department of Sensor Technology, University of Erlangen

**) CCES, RWTH Aachen

Overview

- Motivation
- Electrostatic Interdigital Sensors and
- Piezoelectric Stack Actuators - Forward Problem (FEM)

- Homogenization – 2 Scale Approach
    - Micro Model - Unit Cell Problem
    - Macro Model – Homogenized Structure

- Numerical results
- Summary and Outlook

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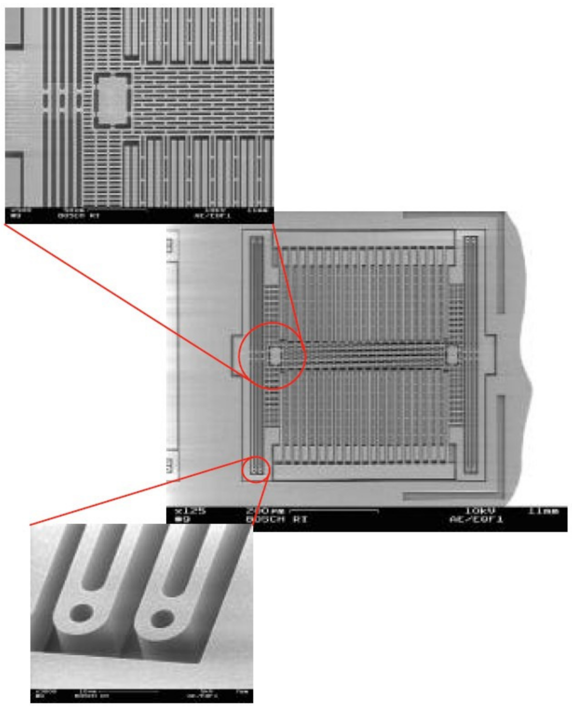
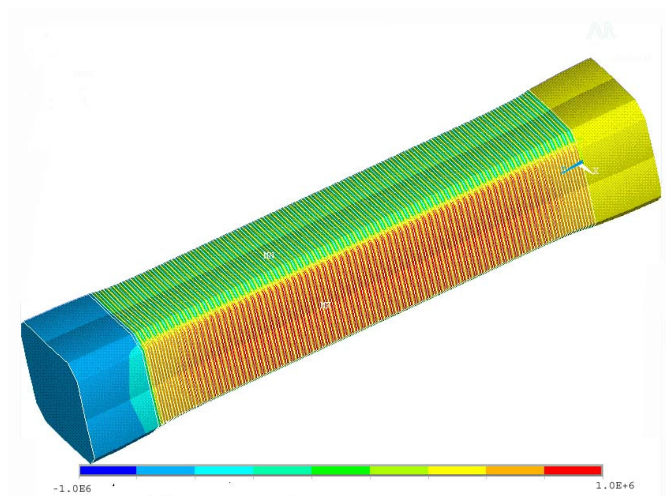
TASK

Find efficient solution method for electro and piezoelectric transducers with periodic structures





Homogenization in Composites / Piezoelectricity

- Calculation of effective piezoelectric material parameters: (*Berger, Gabbert, Köppe, Rodriguez - Ramos, Bravo - Castillero, Guinovart-Diaz, Otero, Maugin, …)*

- Classical homogenization: (double scale asymptotic expansion)
  (Sanchez – Palencia, Levy, …)

- Bloch approximation:

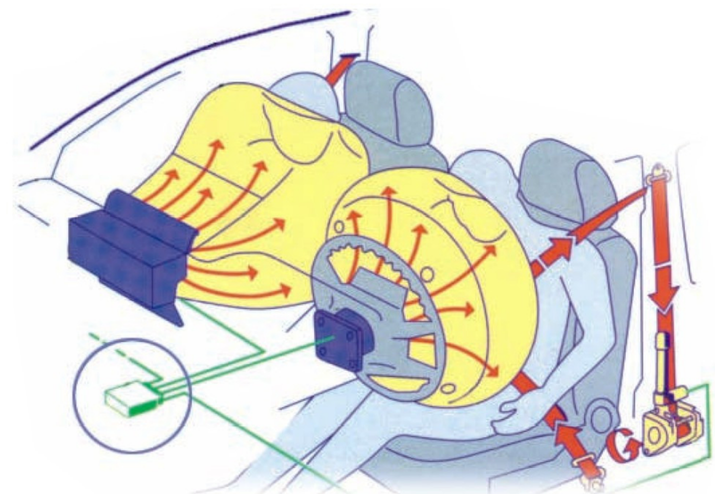
  Elliptic operators: (*Conca, Natesan, Vanninathan, …)*
  PDEs with periodic coefficients: *(Bensoussan, Lions, Papanicoloaou)*
  Piezoelectricity: *(Turbé, Maugin)*
  SAW – Filters: *(Zaglmayr, Schöberl, Langer)*

- Generalized FEM for homogenization problems:

  (A. M. Matache, C. Schwab, …)

First Application: Electrostatic Sensor

Sensor reacts with a change in capacitance while the electric field is changed by some outer impact



Application areas:

Force and acceleration sensors
Airbag deployment
Rotary capacitor

Source: http://www.semiconductors.bosch.de/

Two Scale Homogenization (Electrostatics)

$$
\begin{aligned}
-div(\varepsilon\nabla\phi) &= q \quad \text{in} \\
\phi &= 0 \quad \text{on } \Gamma_0 \\
\phi &= \phi_e \quad \text{on } \Gamma_{elec}
\end{aligned}
$$

$$\phi(x) = \phi(x+\epsilon)$$

$$\epsilon := diam(\Omega_m^\epsilon)$$

$$\phi \text{ ... elec. potential}$$

<u>2 scale series expansion:</u>

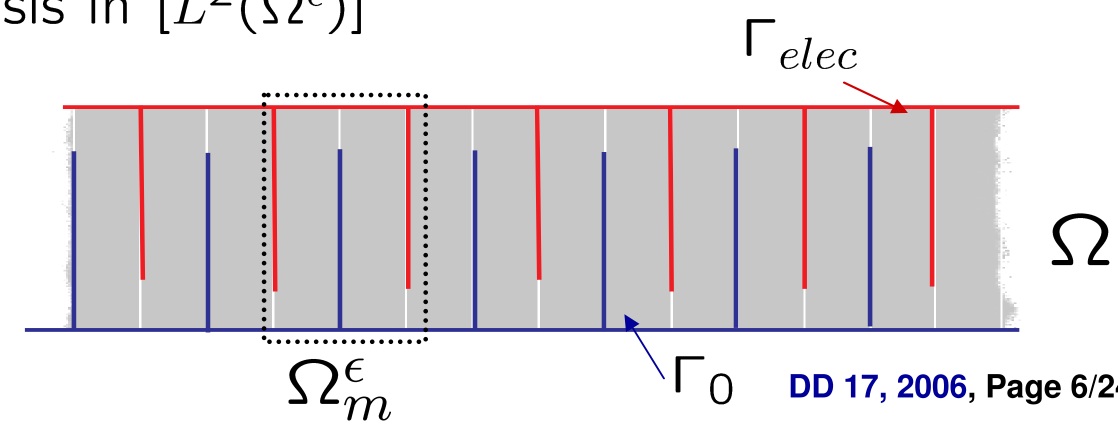
$$\phi = \sum_{i=0}^{N} \phi_i \psi_i \quad \text{with } \psi_0 \equiv 1 \quad \text{in } \Omega$$

ψ_i–eigensolutions on unit cell Ω_m^ϵ – micro scale

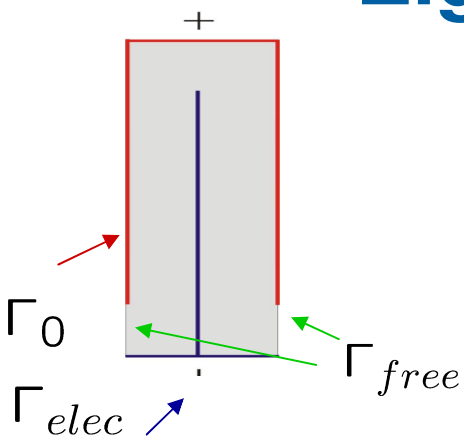
ψ_i - orthonormal basis in $[L^2(\Omega^\epsilon)]$

$\phi_i \in H_0^1(\Omega)$,

$\Omega := \bigcup_{m=0}^{M} \Omega_m^\epsilon$



The Quasi Periodic Electrostatic Eigenvalue Problem



$$
\begin{aligned}
-div(\varepsilon \nabla \psi) &= \lambda \psi \quad \text{in } \Omega_m^\epsilon \\
\psi &= 0 \quad \text{on } \Gamma_0 \\
\psi &= c_1 \quad \text{on } \Gamma_{elec} \\
\psi &= c_2 \quad \text{on } \Gamma_{free}
\end{aligned}
$$

Unit cell problem:

$$
\begin{aligned}
K_{\psi\psi} \psi^h &= \lambda \psi^h, \\
(k_{\psi\psi})_{pq} &= \int_{\Omega_e} \varepsilon (\nabla N^p)^T (\nabla N^q) \, d\Omega_e
\end{aligned}
$$

N^p, N^q ... nodal shape fcts

solution with LAPACK - EV Solver

Series expansion:

$$\phi^H = \sum_{i=0}^N \phi_i^H \psi_i^h$$

$h << H$, $h-$mesh size micro cell, $H-$mesh size macro cell

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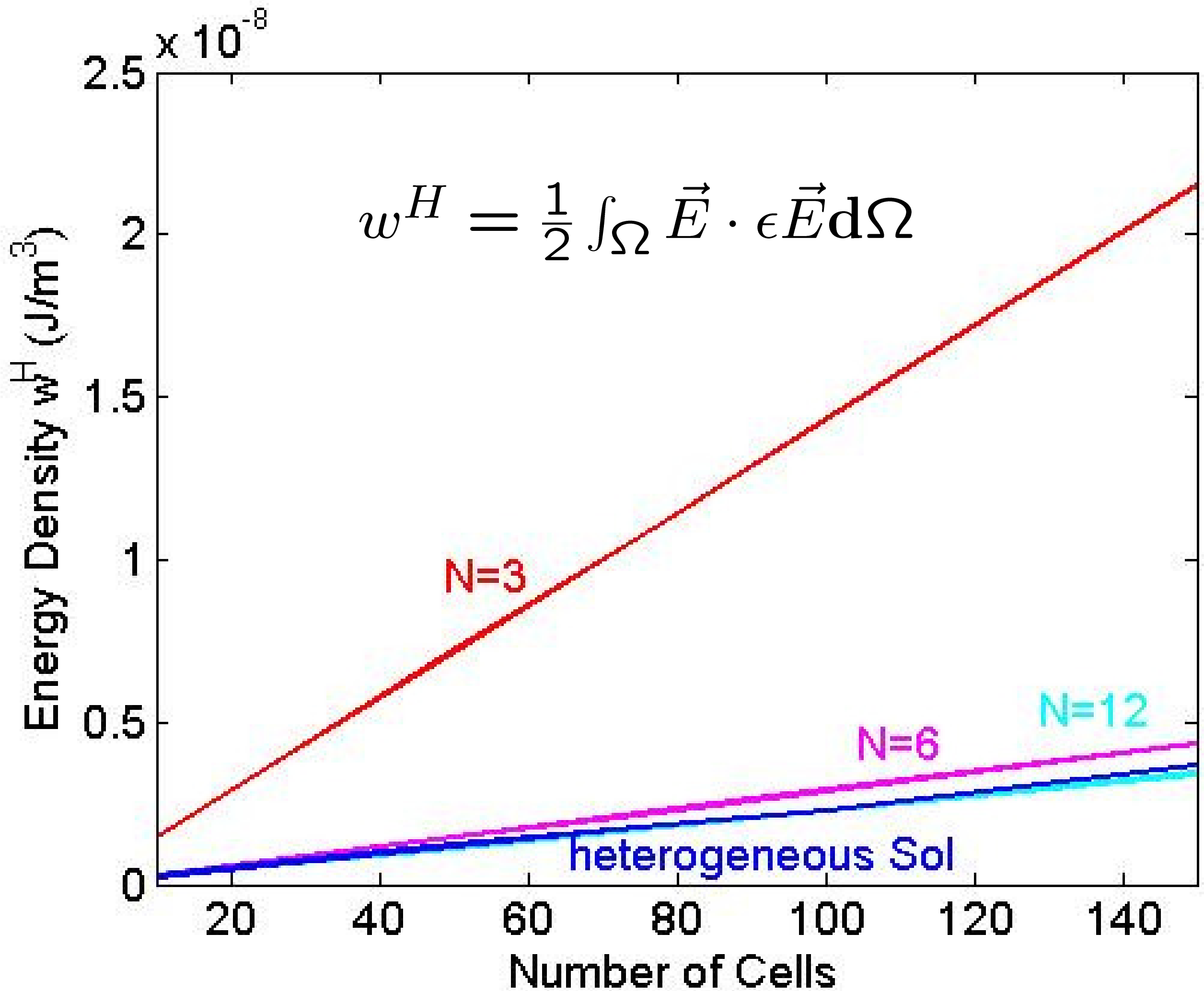
Numerical Results Electrostatics (quadratic elements)



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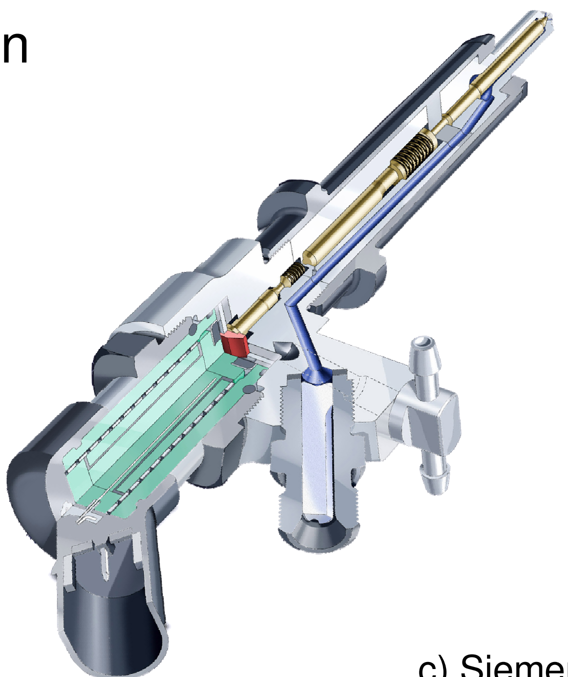
Second application: Piezoelectric Stack - Actuator

Actuator reacts with a deformation in longitudinal direction by application of an electric field



c) Siemens

Application areas:

Injection valves (common – rail)

Optics, Laser Tuning

<u>General:</u>

High mechanical precision steering

High frequency driving

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Piezoelectric Effect

$$\vec{\sigma} = \mathbf{c}^E \vec{S} - \mathbf{e}^T \vec{E}$$
$$\vec{D} = \mathbf{e}\vec{S} + \varepsilon^S \vec{E}$$

$\vec{\sigma}$ ... mechanical stress

$\vec{S} = \mathcal{B}\vec{u}$ ... mechanical strain

$\vec{E} = -\nabla\phi$ ... electric field

$\vec{D}$ ... dielectric displacement

$\vec{u}$ ... mechanical displacement

ϕ ... electric potential

$\mathbf{c}^E, \mathbf{e}, \varepsilon^S$ ... material tensors

Newton's law: $\mathcal{B}^T \vec{\sigma} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$

Gauss' law: $\mathrm{div}\vec{D} = 0$

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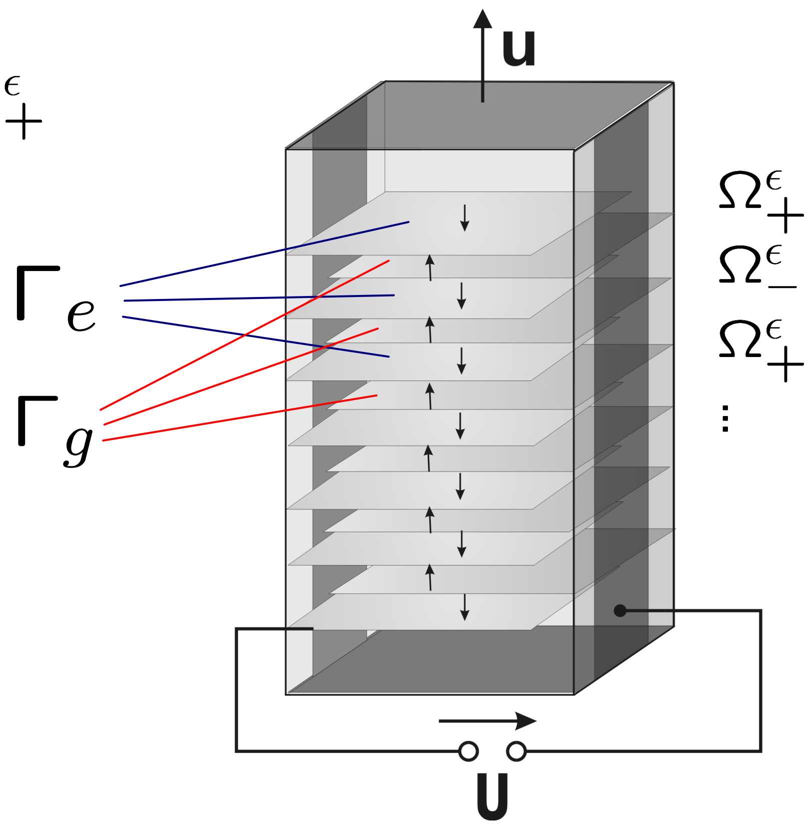
Piezoelectric PDEs (Fourier Transformed)

$$\begin{aligned} -\rho\omega^2\vec{\hat{u}} - \mathcal{B}^T\left(\mathbf{c}^E\mathcal{B}\vec{\hat{u}} + \mathbf{e}^T\nabla\hat{\phi}\right) &= 0 \in \Omega \\ -\mathrm{div}\left(\mathbf{e}\mathcal{B}\vec{\hat{u}} - \varepsilon^S\nabla\hat{\phi}\right) &= 0 \in \Omega \end{aligned}$$

$$\Omega := \cup_{m=0}^{M}\Omega_m^\epsilon, \quad \Omega_m^\epsilon := \Omega_-^\epsilon \cup \Omega_+^\epsilon$$

$$\mathbf{e} := -\mathbf{e} \text{ in } \Omega_- \quad \text{(polarization)}$$



Boundary conditions:

$$\begin{aligned} N^T\sigma &= 0 && \text{on } \partial\Omega \\ \hat{\phi} &= 0 && \text{on } \Gamma_g \\ \hat{\phi} &= \hat{\phi}^e && \text{on } \Gamma_e \\ \vec{D}\cdot\vec{N} &= 0 && \text{on } \partial\Omega \end{aligned}$$

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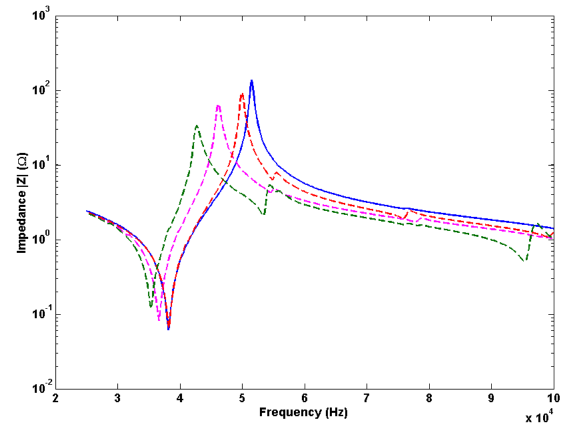
Heterogeneous (scale resolving) 3D Model (computing times)

- Stack 200 Layers - one frequency step ~ 5 min

- Calculation of impedance curve – 100 frequency steps ~ 7.13 hrs



- Simulation based parameter identification for composite

$F(p) = y^{meas}$, $F-$solution operator of piezo PDEs

$p-$piezoelectric parameters, $y^{meas}-$electrical measurements

(evaluation at 15 frequencies x 10 parameters x 10 Newton steps) ~ 1 week

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Two Scale Homogenization

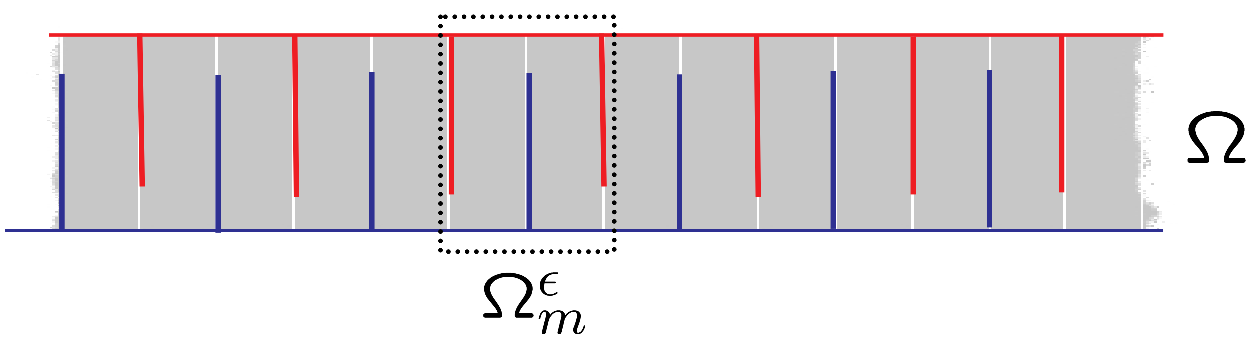
2 scale series expansion:

$$\vec{\tilde{u}} = \sum_{i=0}^{N} \vec{\tilde{u}}_i \psi_i^{mech} \quad \text{with } \psi_0^{mech} \equiv 1 \quad \text{in } \Omega$$

$$\hat{\phi} = \sum_{i=0}^{N} \hat{\phi}_i \psi_i^{elec} \quad \text{with } \psi_0^{elec} \equiv 1 \quad \text{in } \Omega$$

$(\psi_i^{mech}, \psi_i^{elec})-$eigensolutions of unit cell - micro scale
orthonormal basis in $[L^2(\Omega_m^\epsilon)]^3$

$$(\vec{\tilde{u}}_i, \hat{\phi}_i) \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega), \;\; \Omega := \cup_{m=0}^{M} \Omega_m^\epsilon$$



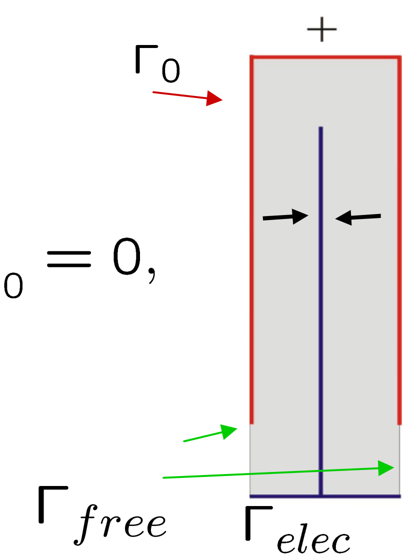
Piezoelectric Eigenvalue Problem

Find $(\vec{u}, \phi, \lambda) \in [H^1_{per,0}(\Omega)]^2 \times [H^1_{per,0}(\Omega)] \times \mathbb{R}$
such that

$$\begin{aligned}
\int_\Omega (\mathcal{B}\vec{v})^T \mathbf{c}^E \mathcal{B}\vec{u}\; d\Omega + \int_\Omega (\mathcal{B}\vec{v})^T e^T \nabla\phi\; d\Omega &= -\lambda\rho \int_\Omega \vec{v}\vec{u}\; d\Omega \\
\int_\Omega (\nabla w)^T e \mathcal{B}\vec{u}\; d\Omega - \int_\Omega (\nabla w)^T \varepsilon^S \nabla\phi\; d\Omega &= 0
\end{aligned}$$

for all $\vec{v} \in [H^1_0(\Omega)]^2$ and $w \in H^1_0(\Omega)$

$$\begin{aligned}
H^1_{per,0}(\Omega) &:= \{\phi \mid ||\phi||_{L_2} + ||\nabla\phi||_{L_2} < \infty, \phi_{|\Gamma_0} = 0, \\
&\qquad \phi_{|\Gamma_e} = c_1,\; \phi_{|\Gamma_{free}} = c_2\} \\
[H^1_{per,0}(\Omega)]^2 &:= \{u \mid ||u||_{L_2} + ||\mathcal{B}u||_{L_2} < \infty\}
\end{aligned}$$



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Piezoelectric Eigenvalue Problem (discretized)

$$\begin{pmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{\phi u} & -\mathbf{K}_{\phi\phi} \end{pmatrix} \begin{pmatrix} u^h \\ \phi^h \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{M}_{uu} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^h \\ \phi^h \end{pmatrix}$$

$$\begin{aligned}
(m_{uu})_{ij} &= \int_\Omega \rho \underline{N}_i^{uT} \underline{N}_j^u \, d\Omega && \text{(mass matrix)} \\
(k_{uu})_{ij} &= \int_\Omega (\mathcal{B}\underline{N}_i^u)^T \mathbf{c}^E (\mathcal{B}\underline{N}_j^u) \, d\Omega && \text{(stiffness matrix)} \\
(k_{u\phi})_{ij} &= \int_\Omega (\mathcal{B}\underline{N}_i^u)^T \mathbf{e} (\nabla N_j^\phi) \, d\Omega && \text{(piezo. coupling matrix)} \\
(k_{\phi\phi})_{ij} &= \int_\Omega (\nabla N_i^\phi)^T \varepsilon^S (\nabla N_j^\phi) \, d\Omega && \text{(permittivity matrix)}
\end{aligned}$$

N_i^u, N_i^ϕ - nodal shape functions

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Piezoelectric Eigenvalue Problem

Solution of the eigenvalue problem by ARPACK using the implicitly restarted Arnoldi iteration

With $\mu := \frac{1}{\lambda - \lambda_s}$ (shift of spectrum)

$$K_{uu}u^h - \lambda_s M_{uu}u^h + K_{u\phi}\phi^h = \frac{1}{\mu}M_{uu}u^h$$
$$K_{\phi u}u^h + K_{\phi\phi}\phi^h = 0$$

$$\mu \begin{pmatrix} u^h \\ \phi^h \end{pmatrix} = \begin{pmatrix} K_{uu} - \lambda_s M_{uu} & K_{u\phi} \\ K_{\phi u} & -K_{\phi\phi} \end{pmatrix}^{-1} \begin{pmatrix} M_{uu} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^h \\ \phi^h \end{pmatrix}$$

we have a form amenable to the Lanczos algorithm

(eigenvectors are invariant under spectral transformation, eigenvalues might be recovered as $\lambda = \frac{1}{\mu} + \lambda_s$)

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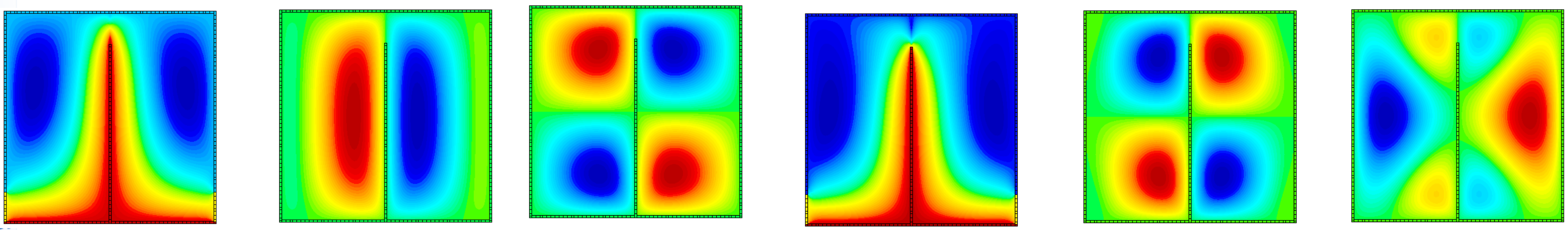
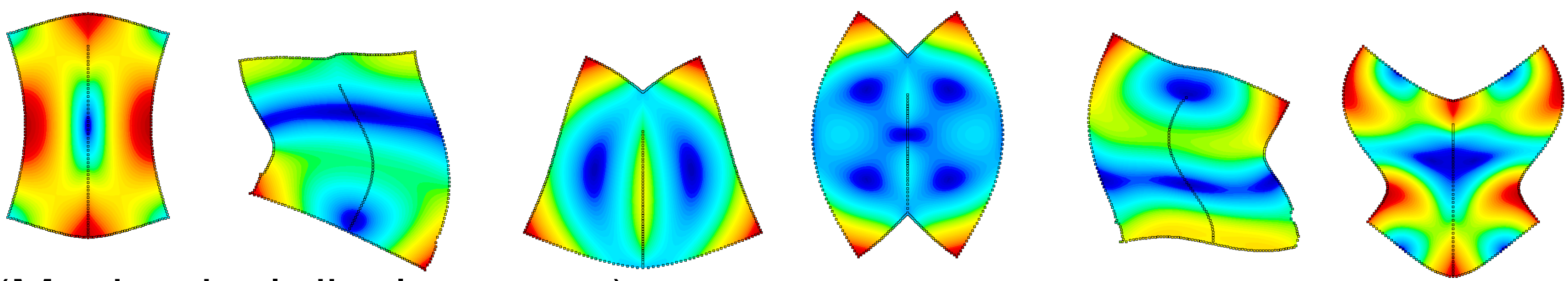
Eigensolutions



(Electric potential)

1 2 3 4 5 6



(Mechanical displacement)

Scaled material parameters: $\mathbf{c}_{ij}^{E} \approx 1.0e-11\mathbf{c}_{ij}^{E}, \;\; \varepsilon_{kl}^{S} \approx 1.0e+08\varepsilon_{kl}^{S}$

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Weak form homogenized Piezo PDE

Find $(\vec{u}_i, \phi_i) \in [H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]$ such that

$$
\begin{aligned}
& \int_\Omega (\mathcal{B} \sum_{i=0}^{N} \vec{\bar{v}}_i \psi_i^{mech})^T \mathbf{c}^E \mathcal{B} \sum_{i=0}^{N} \vec{\bar{u}}_i \psi_i^{mech} \, d\Omega \\
+ & \int_\Omega (\mathcal{B} \sum_{i=0}^{N} \vec{\bar{v}}_i \psi_i^{mech})^T e^T \nabla \sum_{i=0}^{N} \phi_i \psi_i^{elec} \, d\Omega \\
- & \rho\omega^2 \int_\Omega \sum_{i=0}^{N} \vec{\bar{v}}_i \psi_i^{mech} \sum_{i=0}^{N} \vec{\bar{u}}_i \psi_i^{mech} \, d\Omega = 0
\end{aligned}
$$

———— ———— ———— ———— ————

$$
\begin{aligned}
& \int_\Omega (\nabla \sum_{i=0}^{N} w_i \psi_i^{elec})^T e \mathcal{B} \sum_{i=0}^{N} \vec{\bar{u}}_i \psi_i^{mech} \, d\Omega \\
- & \int_\Omega (\nabla \sum_{i=0}^{N} w_i \psi_i^{elec})^T \varepsilon^S \nabla \sum_{i=0}^{N} \phi_i \psi_i^{elec} \, d\Omega = 0
\end{aligned}
$$

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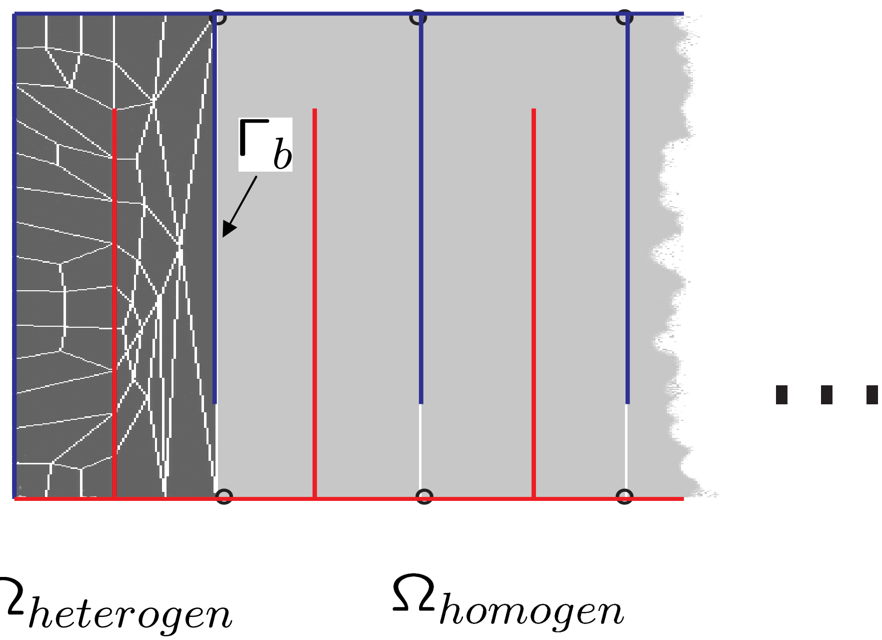
Treatment of boundary

Scale resolution close to boundary



$u = \sum_{i=0}^{N} u_i \psi_i,\ \psi_0 \equiv 1,\ \psi_i \equiv 0$ for $i > 0$ in $\Omega^{heterogen}$

$+$ appropriate essential boundary conditions at Γ_b

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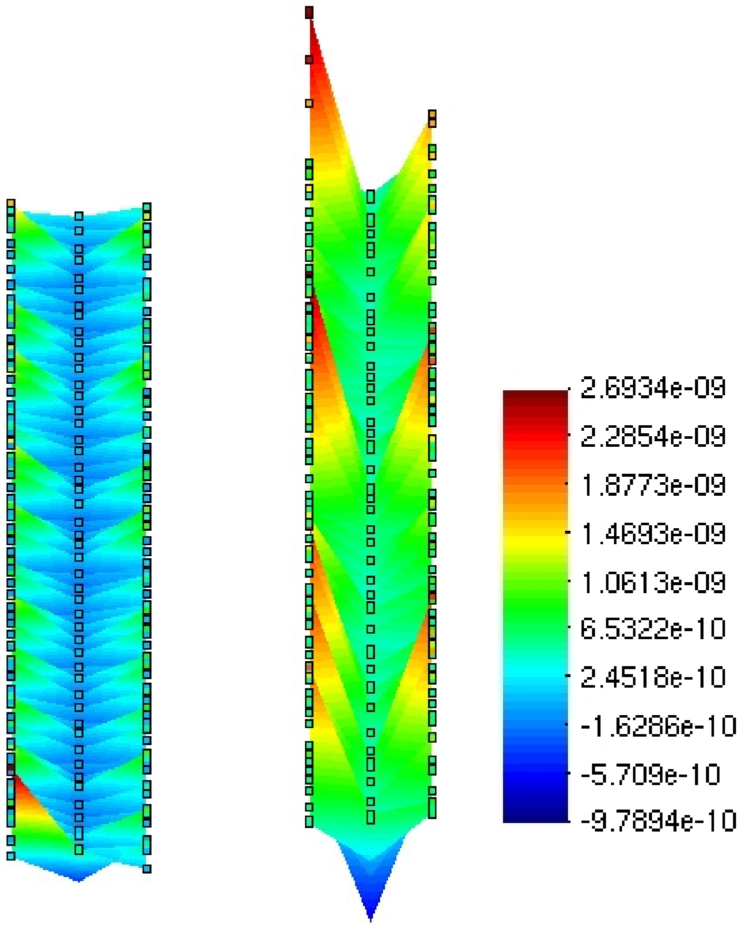
Visualization of Homogenized Solution of Piezoelectric Stack Actuator

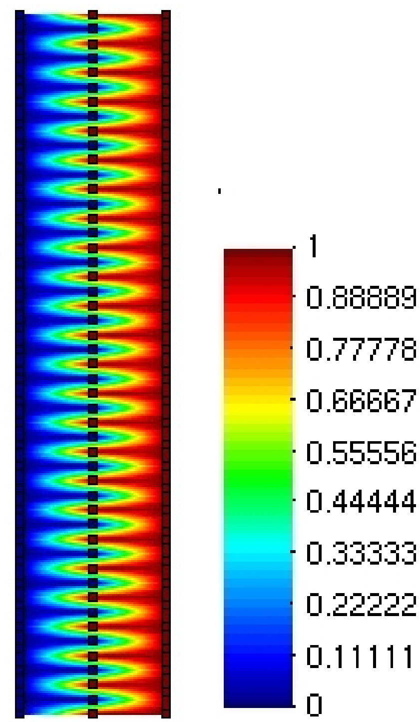
Mechanical Displacement (m):



$f = 6.0e+05, f = 3.7e+5$

Electric Potential (V):

(thickness of each cell 0.2 mm)



$f = 6.0e+05$

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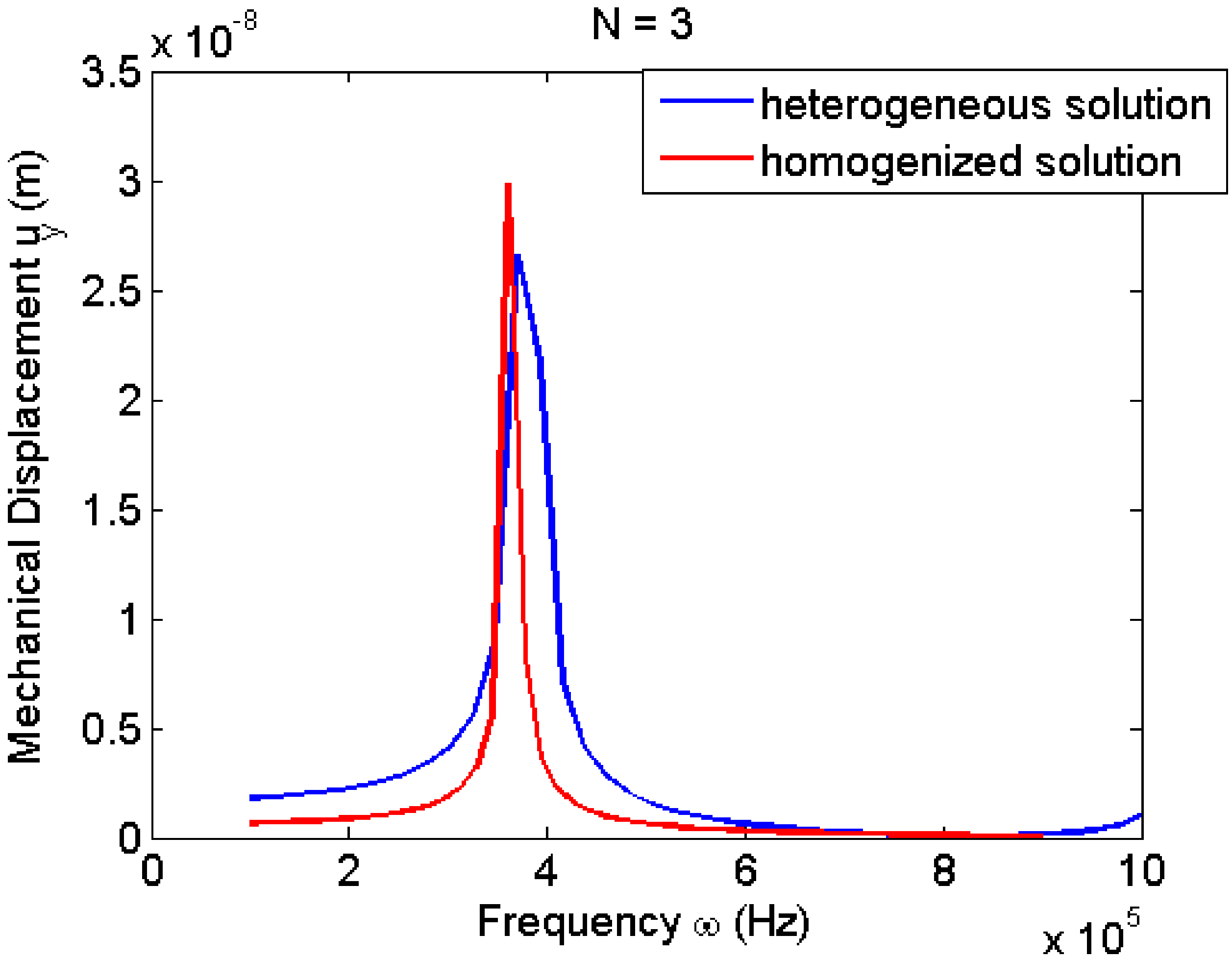
Mechanical Displacement (50 cells)



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CPU Times
(50 cells, calculation of one frequency step)

Model	Number of Nodes	Number of Equations	CPU Times
Heterogeneous	50030	144840	58.4
Unit Cell (EV Pb.) (calculation of 12 EVs)	7701	22621	20.23
Homogeneous	408	1818 (N=2)	2.85
	408	3636 (N=4)	4.64
	408	5454 (N=6)	9.65
	408	7272 (N=8)	17.7

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Summary and Outlook

- ✓ Implemented a scheme which effectively resolves oscillatory behavior of a periodic structure
- ✓ Analyzed corresponding eigenvalue problems
- ✓ Homogenization scheme works with electrostatics and piezoelectricity

- ➢ Improve convergence with hp-FEM
- ➢ Extend model to 3D case
- ➢ Consider boundary conditions, e.g. pre-stressed stack
- ➢ Embed homogenized calculation in parameter identification method

Selected References:

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Have we seen a movie yet?