Algebraic Coarse Spaces for Overlapping Schwarz Preconditioners

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Clark R. Dohrmann
Sandia National Laboratories

joint work with Axel Klawonn and Olof Widlund

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Additive Schwarz

$$M^{-1}r = \underbrace{R_0^T \tilde{A}_0^{-1} R_0 r}_{\text{global "coarse" problem}} + \sum_{i=1}^{N} \underbrace{R_i^T \tilde{A}_i^{-1} R_i r}_{\text{local problems}}$$

global "coarse" problem **local problems**

- **Goal: Simplicity**

Outline

- **Existing Coarse Spaces**
  - overlapping methods
  - iterative substructuring
- **"New" Coarse Spaces**
  - generalization of DSW (1994)
  - comparisons with BDD & BDDC
- **Application to Overlapping Schwarz Preconditioners**
  - some theory
  - numerical examples
- **Summary & Conclusions**

Coarse Spaces (Overlapping Methods)

- **Geometric:**
  - conceptually simple
  - applicable to 2nd and 4th order PDEs
  - requires coarse mesh
- **Smoothed Aggregation:**
  - applicable to 2nd and 4th order PDEs
  - generous overlap: Brezina & Vanek (1999)
  - small overlap: Jenkins, et al. (2001)

theory for "nice" coefficients

- **Partition of Unity:**
  - 2nd order PDEs: Sarkis, et al. (2002-2003)
  - harmonic overlap variants: coefficient jumps
  - 4th order PDEs: works well, not pretty, no theory D (2003)

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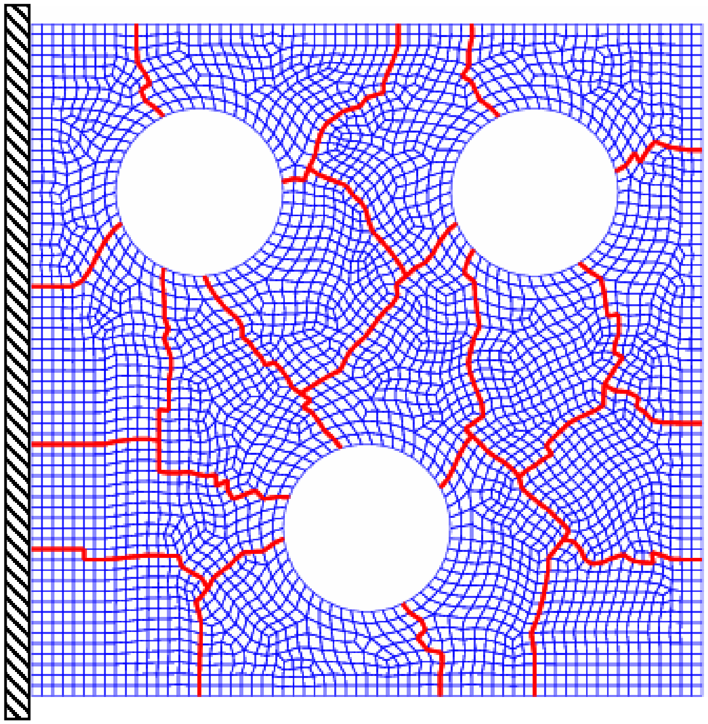
Coarse Spaces (Iterative Substructuring)

- **FETI/BDD:**
  - uses rigid body modes of subdomains
  - works well for 2nd order PDEs
  - "conforming" coarse basis functions

- **FETI-DP/BDDC:**
  - flexibility in choosing coarse dofs (corner, edge, face)
  - works well for 2nd and 4th order PDEs
  - "nonconforming" coarse basis functions

- **"Face-Based" Approach (Section 5.4.3 of T&W):**
  - introduced by Dryja, Smith, Widlund (1994)
  - one coarse dof for each vertex, edge, and face
  - "conforming" coarse basis functions

“New” Coarse Spaces

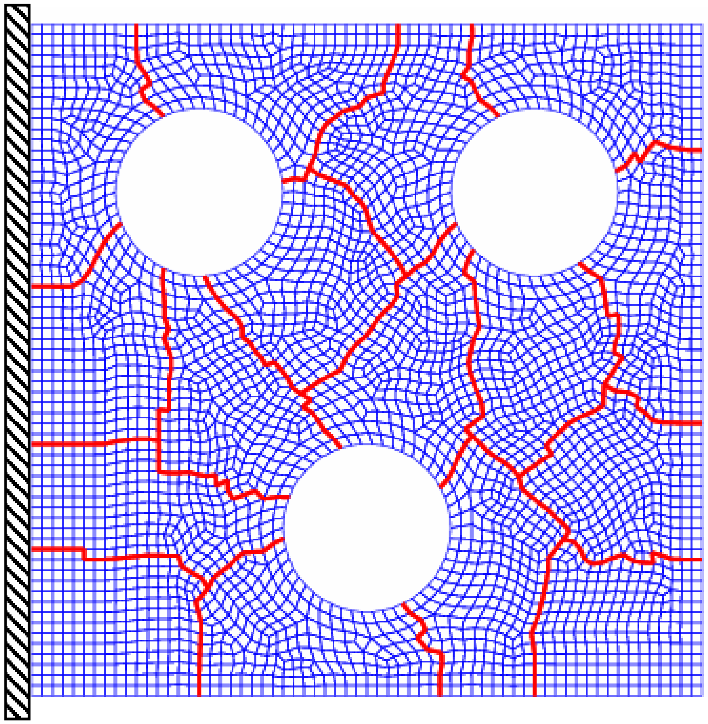


interface Γ shown in red

partition nodes of Γ into corners, edges, faces

Input: Coarse matrix N_Γ

“New” Coarse Spaces



$$\mathcal{R}(N_{\Gamma_j}) = \mathcal{R}(R_{\Gamma_j} N_{\Gamma})$$

$$\det(N_{\Gamma_j}^T N_{\Gamma_j}) \neq 0$$

interface Γ shown in red

partition nodes of Γ into corners, edges, faces

Input: Coarse matrix N_{Γ}

$$u_{\Gamma_j} = R_{\Gamma_j} u_{\Gamma}$$

$$u_{\Gamma_{jc}} = N_{\Gamma_j} q_j$$

$$u_{\Gamma_c} = \sum_j R_{\Gamma_j}^T u_{\Gamma_{jc}}$$

$$u_c = \mathcal{H}(u_{\Gamma_c})$$

$N_{\Gamma} = e \Rightarrow$ identical to DSW (1994)

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Comparisons with BDD and BDDC

	BDD	BDDC	GDSW
2nd order problems	yes	yes	yes
4th order problems	no	yes	yes
conforming coarse space	yes	no	yes
"nice" coarse problem sparsity	no	yes	yes
subdomain matrices required	yes	yes	no
null space information required	yes	no	yes
"easy" multilevel extensions	no	yes	yes
theory for coefficient jumps	yes	yes	yes
3D elasticity coarse dimension	6N	9N	36N
near incompressible elasticity	yes	yes	yes

Some Theory (Overlapping Schwarz)

- **Poisson Equation & Compressible Elasticity:**
  - **Coarse matrix N_Γ spans rigid body modes**

$$\kappa(M^{-1}A) \le C(1 + H/\delta)(1 + \log(H/h))$$

  - **N_Γ enriched w/ linear functions, no property jumps**

$$\kappa(M^{-1}A) \le C(1 + H/\delta)$$

- **Nearly Incompressible Elasticity (discontinuous pressure):**
  - **Coarse matrix N_Γ spans rigid body modes. Preliminary theory (2D) suggests**

$$\kappa(M^{-1}A) \le C(1 + H/\delta)(1 + \log(H/h))^2$$

  - **result not too surprising considering coarse space is richer than stable elements like Q_2–P_0**

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Numerical Examples (AOS)

- **Poisson Equation & Compressible Elasticity:**
  - **no surprises, consistent with theory**

- **Nearly Incompressible Elasticity (2D plane strain):**

  Q_2-P_{-1} elements, H/h = 8, δ = H/4, rtol = 10^{-8}

N	ν = 0.3		ν = 0.4999		ν = 0.4999999	
	iter	cond	iter	cond	iter	cond
4	19	5.4	23	6.8	25	7.1
16	24	6.8	29	9.1	34	9.2
36	25	7.6	31	9.8	36	10.1
64	26	8.1	32	9.9	37	10.1

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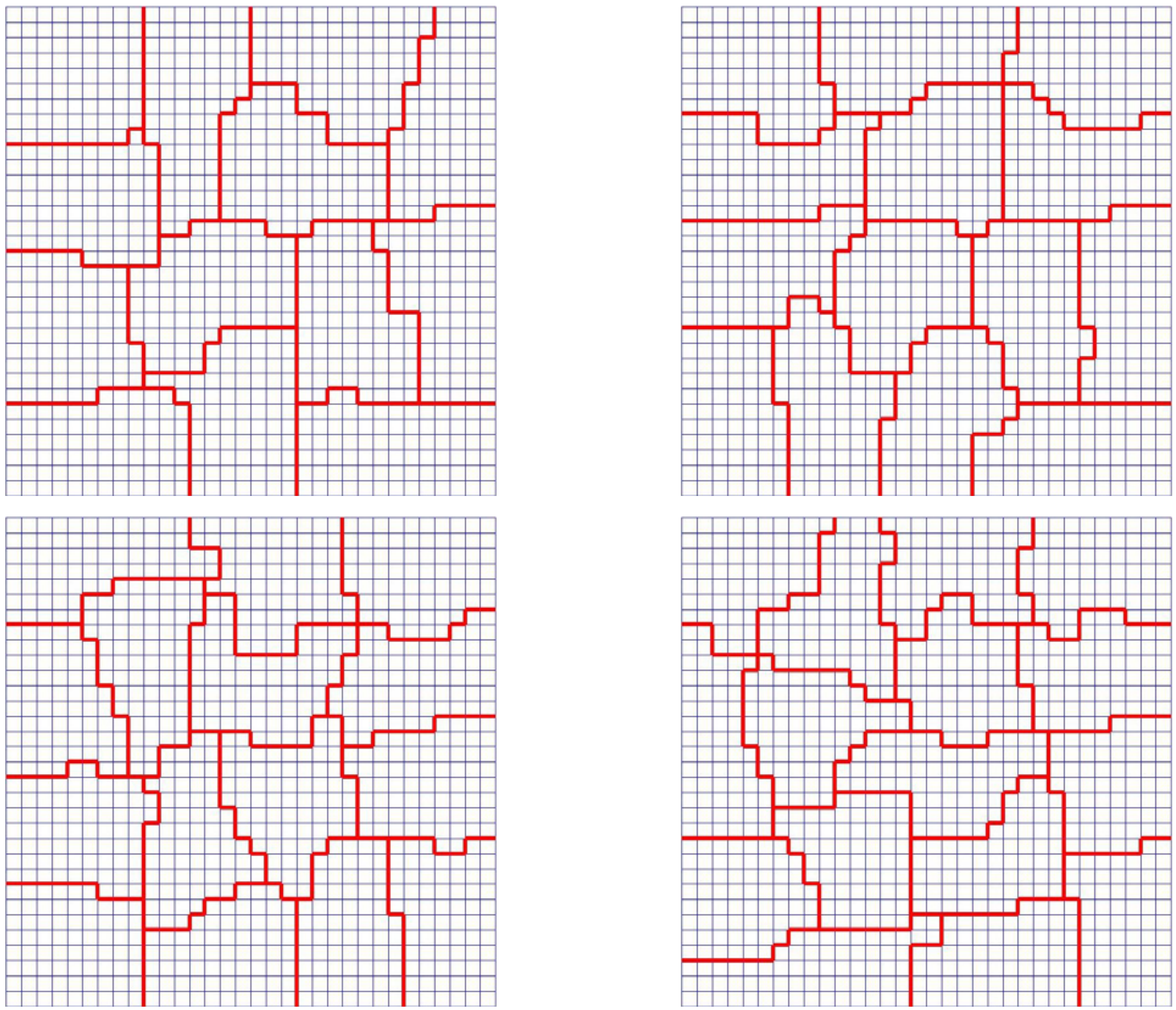
Numerical Examples (AOS)

- **2D plane strain (continued):**

  Q_2-P_{-1} elements, N = 16, δ = H/4, rtol = 10^{-8}

H/h	ν = 0.3		ν = 0.4999		ν = 0.4999999	
	iter	cond	iter	cond	iter	cond
4	23	6.5	29	8.1	33	8.5
8	24	6.8	29	9.1	34	9.2
12	23	6.9	30	9.6	34	9.8
16	24	7.0	30	10.1	34	10.3
20	23	7.0	30	10.4	34	10.6

Unstructured Meshes



N = 13

N = 14

N = 15

N = 16

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Numerical Examples (AOS)

- **2D plane strain for unstructured meshes:**

  Q_2-P_{-1} elements, $H/h \approx 8$, $\delta \approx H/4$, rtol = 10^{-8}

N	$\nu = 0.3$		$\nu = 0.4999$		$\nu = 0.4999999$	
	iter	cond	iter	cond	iter	cond
13	26	7.2	32	11.5	36	11.9
14	26	7.0	33	13.3	38	13.8
15	27	7.2	34	11.8	38	12.3
16	25	6.7	33	11.0	38	11.4

Numerical Examples (AOS)

- **2D plate bending (4th order problem):**

DKT elements, H/h = 8, δ = H/4, rtol = 10^{-8}

N	iter	cond
4	29	10.2
16	41	17.7
64	48	19.8
256	52	21.1

Numerical Examples (AOS)

- **2D plate bending (4th order problem):**

DKT elements, δ = H/4, rtol = 10^{-8}

H/h	N = 16		N = 64	
	iter	cond	iter	cond
8	41	17.7	48	19.8
16	46	23.4	57	27.6
24	47	26.2	61	31.5
32	50	28.0	need more patience	
40	51	29.4		

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Numerical Examples (AOS)

- **Problems in H(curl;Ω):**

$$(u,v)_\Omega = \int_\Omega (u \cdot v)\, dx$$

$$a(u,v) = \sum_{i=1}^{N} a(u,v)_{\Omega_i} \qquad u,v \in H(curl;\Omega)$$

$$a(u,v)_{\Omega_i} = a_i(\nabla \times u, \nabla \times v)_{\Omega_i} + b_i(u,v)_{\Omega_i}$$

Examples: $a_i = \alpha$ and $b_i = \beta$ for $i = 1,\ldots,N$

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Numerical Examples (AOS)

- **2D problems in H(curl;Ω):** N_Γ **has one column**

edge elements, H/h = 8, δ = H/8, β = 1, rtol = 10^{-8}

N	$\alpha = 0$	$\alpha = 10^{-2}$	$\alpha = 1$	$\alpha = 10^{2}$	$\alpha = 10^{4}$
4	5 (3.0)	16 (4.4)	22 (7.0)	23 (7.2)	25 (7.2)
16	6 (3.0)	20 (5.3)	25 (7.4)	28 (7.5)	30 (7.5)
36	6 (3.0)	22 (6.0)	26 (7.5)	28 (7.5)	31 (7.5)
64	6 (3.0)	23 (6.4)	26 (7.5)	29 (7.6)	31 (7.6)
100	6 (3.0)	24 (6.8)	27 (7.6)	30 (7.6)	32 (7.6)
144	6 (3.0)	24 (7.0)	27 (7.6)	30 (7.6)	32 (7.6)

Numerical Examples (AOS)

- **2D problems in H(curl;Ω):**

edge elements, N = 16, δ = H/8, β = 1, rtol = 10^{-8}

H/h	$\alpha = 0$	$\alpha = 10^{-2}$	$\alpha = 1$	$\alpha = 10^{2}$	$\alpha = 10^{4}$
8	6 (3.0)	20 (5.3)	25 (7.4)	28 (7.5)	30 (7.5)
16	4 (3.0)	20 (5.0)	22 (5.8)	23 (5.9)	25 (5.9)
24	3 (3.0)	20 (4.9)	22 (5.6)	23 (5.6)	24 (5.6)
32	3 (3.0)	20 (4.8)	22 (5.4)	23 (5.5)	24 (5.5)
40	3 (3.0)	20 (4.8)	21 (5.3)	23 (5.4)	25 (5.4)
48	3 (3.0)	20 (4.8)	21 (5.3)	23 (5.4)	23 (5.4)

where are you logs?

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Summary/Conclusions

- “New” coarse spaces give bounds independent of material property jumps for classic overlapping Schwarz preconditioners
- Coarse spaces can be constructed from assembled problem matrix
- Dimensions of coarse spaces generally larger than those for BDD or BDDC
- Accommodating nearly incompressible materials very straightforward
- Theory and specification of coarse matrix N_{Γ} remain open for some problem types

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Humor if needed

Why do people in ship mutinies always ask for "better treatment?" I'd ask for a pinball machine, because with all that rocking back and forth you'd probably be able to get a lot of free games. ---
Jack Handy