Optimized Schwarz Methods for Problems with Discontinuous Coefficients

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Motivation

Flow in heterogeneous media has many applications, for example

- oil recovery,
- earthquake prediction,
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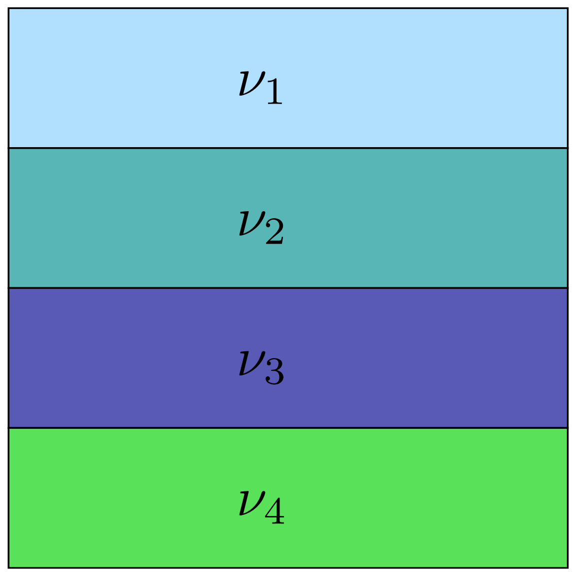

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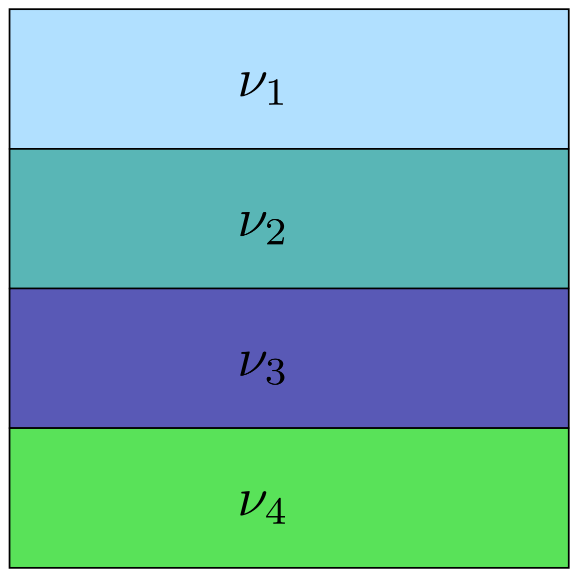


$\Rightarrow$ This suggests a natural nonoverlapping domain decomposition.

Outline

1. Motivation
2. Introduction to the model problem
3. Schwarz iteration and optimal operators
4. Optimized transmission conditions
   (a) one-sided Robin conditions (two versions)
   (b) two-sided Robin conditions
   (c) second order conditions
5. Asymptotic performance for strong heterogeneity
6. Numerical experiments
7. Conclusions and work in progress

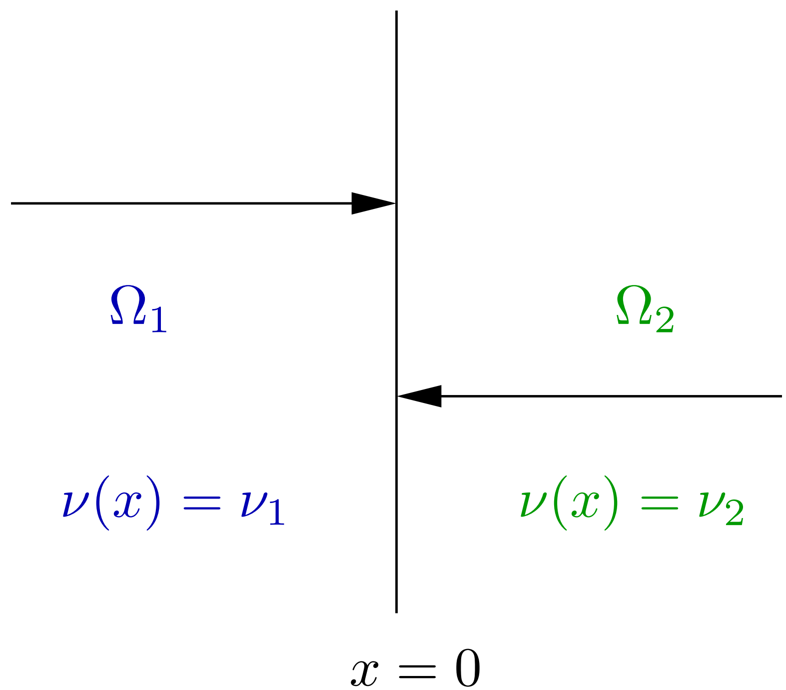
Some recent work

- Y. Maday and F. Magoulès. Multilevel optimized Schwarz methods without overlap for highly heterogeneous media. Research report at Laboratoire Jacques-Louis Lions, 2005.
- Y. Maday and F. Magoulès. Improved ad hoc interface conditions for Schwarz solution procedure tuned to highly heterogeneous media. *Applied Mathematical Modelling*, 30(8):731-743, 2006.

The model problem

We consider a simple diffusion problem with a discontinuous coefficient

$$\begin{cases} -\nabla \cdot (\nu(x)\nabla u) = f \text{ on } \mathbb{R}^2, \\ u \text{ is bounded at infinity.} \end{cases} \tag{P}$$



A general Schwarz iteration

The solution to problem (P) satisfies the matching conditions

$$u(0^-, y) = u(0^+, y), \quad \nu_1 \frac{\partial u}{\partial n}(0^-, y) = \nu_2 \frac{\partial u}{\partial n}(0^+, y).$$

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Consider a general Schwarz iteration of the form

$$\begin{cases} -\nabla \cdot (\nu_1 \nabla u_1^{n+1}) = f & \text{on } \Omega_1 = (-\infty, 0) \times \mathbb{R}, \\ \left(\nu_1 \frac{\partial}{\partial x} + \mathcal{S}_1\right) u_1^{n+1} = \left(\nu_2 \frac{\partial}{\partial x} + \mathcal{S}_1\right) u_2^n & \text{at } x = 0, \end{cases}$$

$$\begin{cases} -\nabla \cdot (\nu_2 \nabla u_2^{n+1}) = f & \text{on } \Omega_2 = (0, \infty) \times \mathbb{R}, \\ \left(-\nu_2 \frac{\partial}{\partial x} + \mathcal{S}_2\right) u_2^{n+1} = \left(-\nu_1 \frac{\partial}{\partial x} + \mathcal{S}_2\right) u_1^n & \text{at } x = 0. \end{cases}$$

- u_i^n = approximate solution in subdomain Ω_i, at iteration n.
- $\mathcal{S}_i$ are linear boundary operators acting in y only

Fourier analysis

Fourier transform in y:

$$\mathcal{F}_y[u(x,y)] = \hat{u}(x,k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-iyk} dy$$

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Fourier symbols for the transmission operators $\mathcal{S}_i$:

$$\mathcal{F}_y[\mathcal{S}_i u(x,y)] = \sigma_i(k)\hat{u}(x,k), \quad \text{for } i = 1,2.$$

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Convergence factor of the Schwarz iteration in Fourier space:

$$\boxed{\rho(k,\sigma_1,\sigma_2) := \left|\frac{\hat{u}_i^{n+1}(0,k)}{\hat{u}_i^{n-1}(0,k)}\right| = \left|\frac{(\sigma_1 - \nu_2|k|)(\sigma_2 - \nu_1|k|)}{(\sigma_1 + \nu_1|k|)(\sigma_2 + \nu_2|k|)}\right|.}$$

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Optimal choice of operators:

$$\sigma_1^{opt}(k) = \nu_2|k|, \quad \sigma_2^{opt}(k) = \nu_1|k|.$$

Optimized Schwarz methods

Find the “best” transmission conditions from a class of local operators $\mathcal{C}$,

$$\min_{\sigma_1,\sigma_2\in\mathcal{C}}\left(\max_{k_1\leq k\leq k_2}\rho(k,\sigma_1,\sigma_2)\right).$$

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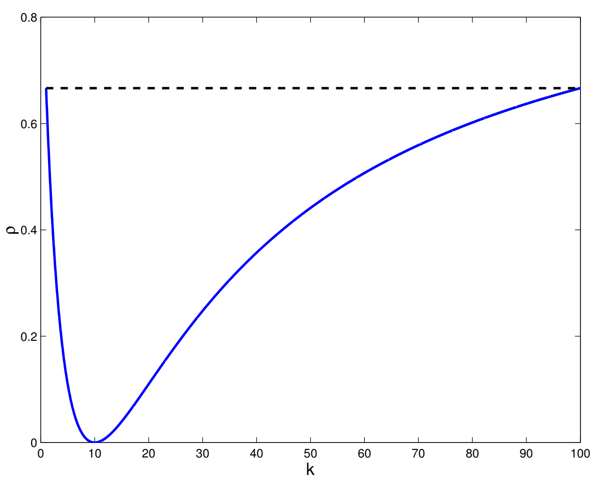
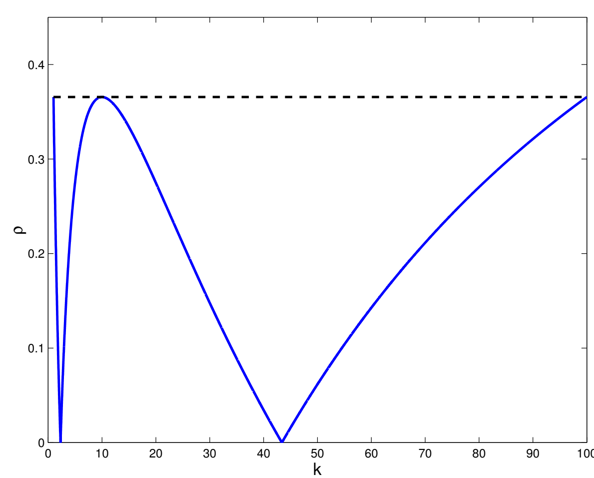
Equioscillation principle: often, the solution of this min-max problem is characterized by equioscillation of the convergence factor ρ at the local maxima, e.g.



1 parameter



2 parameters

Optimized Robin conditions (v. 1)

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Convergence factor:

$$\rho(k, p) = \left| \frac{(p - \nu_1|k|)(p - \nu_2|k|)}{(p + \nu_1|k|)(p + \nu_2|k|)} \right| .$$

Uniform minimization of the convergence factor:

$$\min_{p \in \mathbb{R}} \left(\max_{k_1 \leq k \leq k_2} \rho(k, p) \right) . \qquad (M_1)$$

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We will state our result in terms of

$$\mu := \frac{\max(\nu_1, \nu_2)}{\min(\nu_1, \nu_2)}, \quad k_r = \frac{k_2}{k_1}.$$

Optimized Robin conditions (v. 1)

Theorem 1. ***Solution of the min-max problem*** *(M_1). Let*

$$f(\mu) := \frac{(\mu+1)^2 + (\mu-1)\sqrt{\mu^2+6\mu+1}}{4\mu}.$$

- *If $k_r \geq f(\mu)$, then one minimizer of (M_1) is $p^* = \sqrt{\nu_1\nu_2 k_1 k_2}$. This minimizer p^* is unique when*

  $$\rho(k_1, p^*) \geq \rho\left(\frac{p^*}{\sqrt{\nu_1\nu_2}}, p^*\right).$$

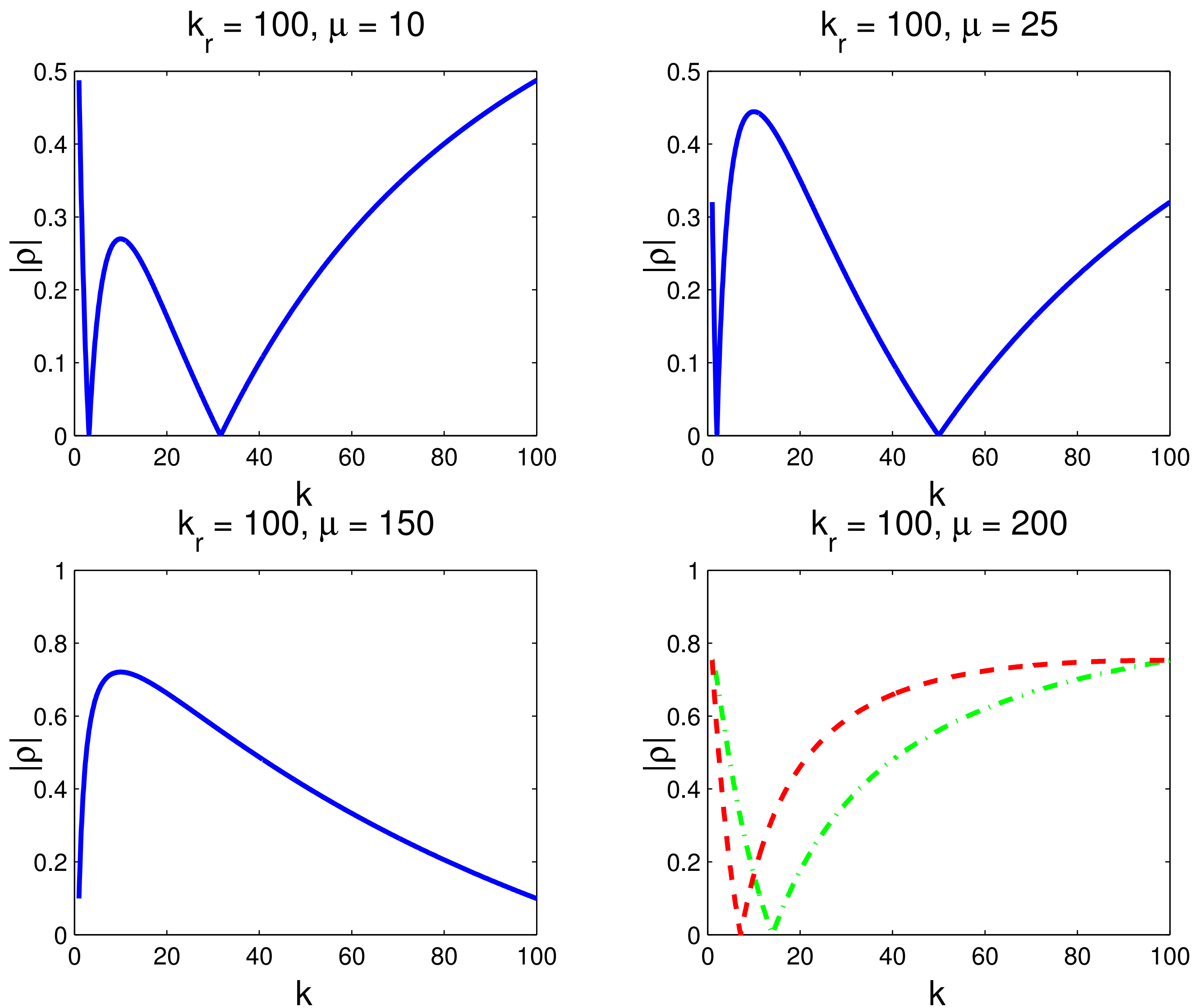
  Otherwise, the minimum is also attained for any p chosen in some closed interval containing p^*.

- *If $k_r < f(\mu)$, then there are two minimizers given by the two positive roots of*

  $$p^4 + \left[\nu_1\nu_2(k_1^2+k_2^2) - k_1k_2(\nu_1+\nu_2)^2\right]p^2 + (\nu_1\nu_2k_1k_2)^2.$$

  Both of these two values yield equioscillation, i.e. $\rho(k_1, p^*) = \rho(k_2, p^*)$.

Optimized Robin conditions (v. 1)



Optimized Robin conditions (v. 2)

Recall the optimal symbols are

$$\sigma_1^{opt}(k) = \nu_2|k|, \quad \sigma_2^{opt}(k) = \nu_1|k|.$$

This suggests a different scaling in the Robin conditions,

$$\sigma_1(k) = \nu_2 p, \quad \sigma_2(k) = \nu_1 p, \quad \text{for } p \in \mathbb{R}.$$

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Convergence factor:

$$\rho(k, p) = \frac{(p-k)^2}{(p+\mu k)(p+k/\mu)}.$$

Uniform minimization of the convergence factor:

$$\min_{p\in\mathbb{R}} \left(\max_{k_1 \leq k \leq k_2} \rho(k, p) \right). \qquad (M_2)$$

Optimized Robin conditions (v. 2)

Theorem 2. ***Solution of the min-max problem*** (M_2)**.**

The optimization problem (M_2) *has a unique minimizer, given by*

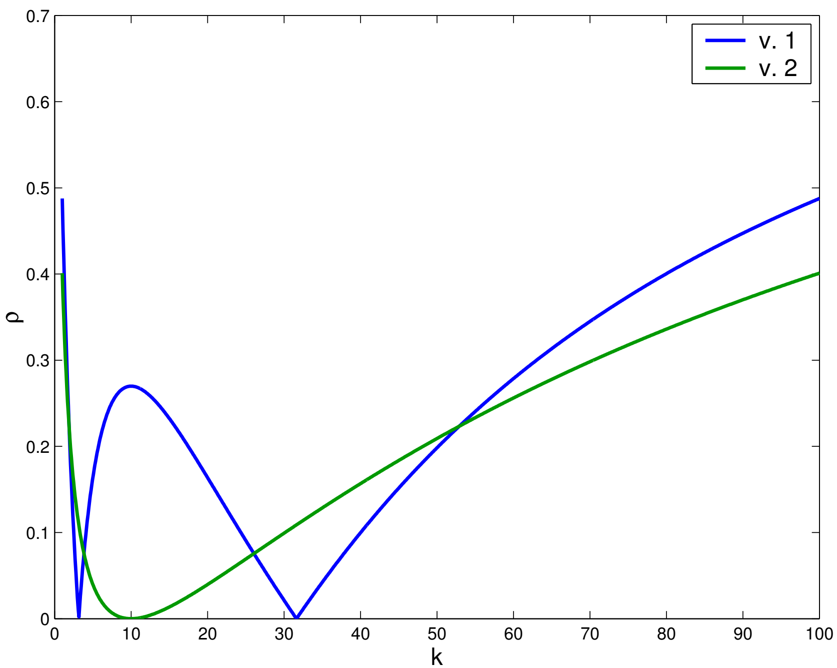
$$p^* = \sqrt{k_1 k_2}.$$

This value always gives the equioscillation $\rho(k_1, p^*) = \rho(k_2, p^*)$.

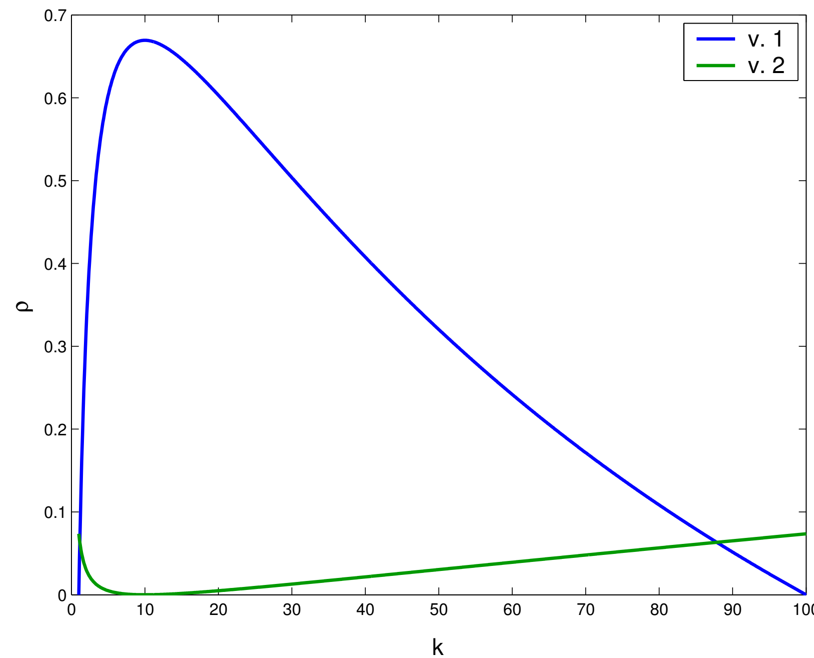
Optimized Robin conditions

Comparison of optimized convergence factors for version 1 and 2:

$\mu = 10$



$\mu = 100$



Asymptotic performance

Theorem 3. *When $k_2 = \frac{\pi}{h}$, and as $h \to 0$ (keeping ν_1 and ν_2 constant), we find the asymptotic expansions:*

- *Optimized Robin conditions, v. 1*

$$\max_{k_1 \le k \le \pi/h} |\rho(k, p^*)| = 1 - 2\left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)\left(\frac{k_1 h}{\pi}\right)^{\frac{1}{2}} + O(h)$$

- *Optimized Robin conditions, v. 2*

$$\max_{k_1 \le k \le \pi/h} |\rho(k, p^*)| = 1 - \frac{(\mu+1)^2}{\mu}\left(\frac{k_1 h}{\pi}\right)^{\frac{1}{2}} + O(h)$$

Optimized two-sided Robin conditions

We now consider two-sided Robin conditions, with two free parameters

$$\sigma_1(k) = \nu_2 p, \quad \sigma_2(k) = \nu_1 q, \quad \text{for } p, q \in \mathbb{R}.$$

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Convergence factor:

$$\rho(k, p, q) = \left| \frac{(p-k)(q-k)}{(p + \frac{\nu_1}{\nu_2} k)(q + \frac{\nu_2}{\nu_1} k)} \right|.$$

Uniform minimization of the convergence factor:

$$\min_{p,q\in\mathbb{R}} \left(\max_{k_1 \le k \le k_2} \rho(k, p, q) \right). \qquad (M_3)$$

Optimized two-sided Robin conditions

Let $\lambda := \frac{\nu_1}{\nu_2}$.

Theorem 4. ***Solution of the min-max problem*** (M_3)***.***

- *When* $\lambda \geq 1$*, the unique minimizing pair* (p^*, q^*) *of problem* (M_3) *is the unique solution to the system of equations*

$$p^* q^* = k_1 k_2, \qquad (E_1)$$

$$|\rho(k_1, p^*, q^*)| = |\rho(\sqrt{p^* q^*}, p^*, q^*)|, \qquad (E_2)$$

  satisfying $k_1 < p^* < q^*$*.*

- *When* $\lambda \leq 1$*, the unique minimizing pair* (p^*, q^*) *of* (M_3) *is the solution of the above equations* (E_1)*-*(E_2) *satisfying* $k_1 < q^* < p^*$ *instead.*

The optimized convergence factor always has the equioscillation property at the frequencies k_1*,* k_2 *and* $k_c = \sqrt{pq}$*.*

Optimized two-sided Robin conditions

Computing p^* reduces to finding the unique real root of the quartic

$$(p+\lambda k_1)(p+\lambda k_2)(\sqrt{k_1 k_2}-p)^2-(p-k_1)(k_2-p)(p+\lambda\sqrt{k_1 k_2})^2$$

in the interval $(k_1, \sqrt{k_1 k_2})$ if $\lambda \geq 1$, or in the interval $(\sqrt{k_1, k_2}, k_2)$ if $\lambda \leq 1$.

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in the interval $(k_1, \sqrt{k_1 k_2})$ if $\lambda \geq 1$, or in the interval $(\sqrt{k_1, k_2}, k_2)$ if $\lambda \leq 1$.

Theorem 5. *When $k_2 = \frac{\pi}{h}$, and as h approaches 0, the asymptotic performance of the optimized two-sided Robin conditions is*

$$\max_{k_1 \leq k \leq \pi/h} \rho(k, p^*, q^*) = \frac{1}{\mu} - \frac{4(\mu+1)}{\mu(\mu-1)}\sqrt{\frac{k_1}{\pi}} h^{\frac{1}{2}} + O(h).$$

Optimized 2nd order conditions

We can also consider transmission conditions that are second order in the tangential direction

$$\left[\nu_1 \frac{\partial}{\partial x} + \nu_2 \left(p - q\frac{\partial^2}{\partial y^2}\right)\right] u_1^{n+1} = \left[\nu_2 \frac{\partial}{\partial x} + \nu_2 \left(p - q\frac{\partial^2}{\partial y^2}\right)\right] u_2^{n},$$

$$\left[-\nu_2 \frac{\partial}{\partial x} + \nu_1 \left(p - q\frac{\partial^2}{\partial y^2}\right)\right] u_2^{n+1} = \left[-\nu_1 \frac{\partial}{\partial x} + \nu_1 \left(p - q\frac{\partial^2}{\partial y^2}\right)\right] u_1^{n}.$$

The corresponding symbols in Fourier space are

$$\sigma_1(k) = \nu_2(p + qk^2), \quad \sigma_2(k) = \nu_1(p + qk^2).$$

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The corresponding symbols in Fourier space are

$$\sigma_1(k) = \nu_2(p + qk^2), \quad \sigma_2(k) = \nu_1(p + qk^2).$$

This leads to the min-max problem

$$\min_{p,q\in\mathbb{R}} \left(\max_{k_1 \le k \le k_2} \left| \frac{(p + qk^2 - k)^2}{(p + qk^2 + \mu k)(p + qk^2 + k/\mu)} \right| \right). \qquad (M_4)$$

Optimized 2nd order conditions

Conjecture 1. Solution of the min-max problem (M_4).
The min-max problem (M_4) has a unique solution, given by the equioscillation of the convergence factor at the frequencies k_1, k_2 and $k_c = \sqrt{\frac{p}{q}}$. This gives the formulas

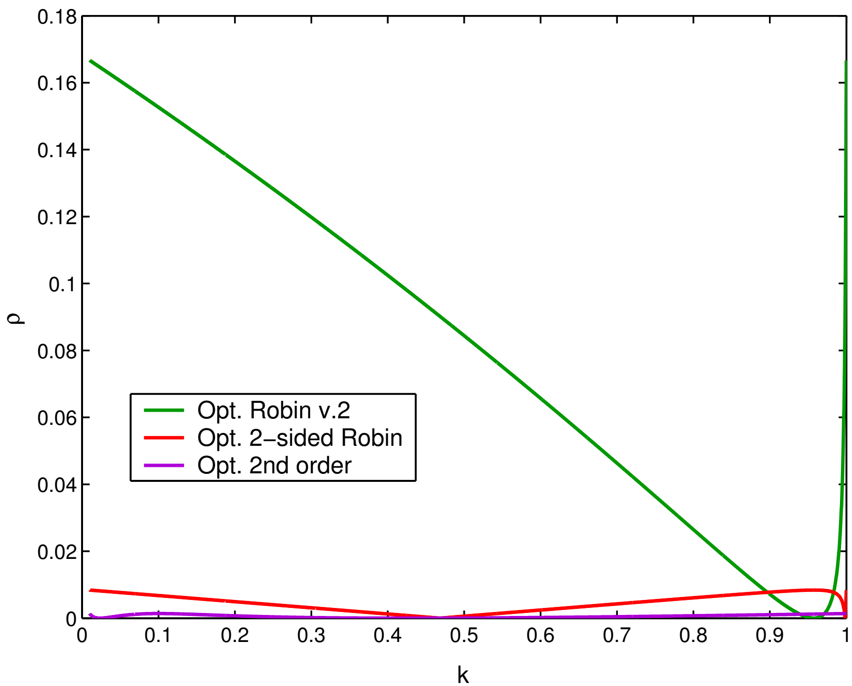
$$p^* = \frac{(k_1 k_2)^{\frac{3}{4}}}{\sqrt{2(k_1 + k_2)}}, \quad q^* = \frac{1}{\sqrt{2(k_1 + k_2)}(k_1 k_2)^{\frac{1}{4}}}.$$

When $k_2 = \frac{\pi}{h}$ and h tends to 0, we find the asymptotic performance

$$\max_{k_1 \leq k \leq \pi/h} \rho(k, p^*, q^*) = 1 - \sqrt{2}\left(2 + \mu + \frac{1}{\mu}\right)\left(\frac{k_1}{\pi}\right)^{\frac{1}{4}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}).$$

Comparison of convergence factors

$\mu = 100$



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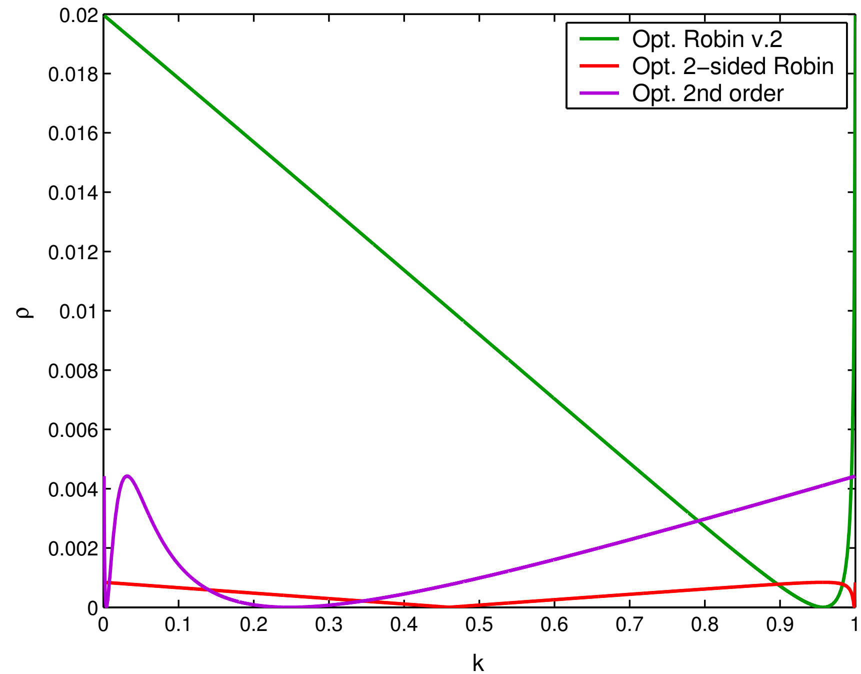


Asymptotics for strong heterogeneity

Let us now consider the case when h is small but held fixed, and μ is large

$$\mu = \frac{\max(\nu_1, \nu_2)}{\min(\nu_1, \nu_2)} >> 1.$$

The asymptotics become

- Optimized Robin conditions, v. 1

$$\max_{k_1 \leq k \leq \pi/h} |\rho(k, p^*)| = 1 - 2\sqrt{\frac{k_1 h}{\pi}} + \ldots - O\left(\frac{h^{-\frac{1}{2}}}{\mu}\right).$$

- Optimized Robin conditions, v. 2

$$\max_{k_1 \leq k \leq \pi/h} |\rho(k, p^*)| = \sqrt{\frac{\pi}{k_1 h}}\left(\frac{1}{\mu}\right) - O\left(\frac{1}{\mu}\right).$$

Asymptotics for strong heterogeneity

- Optimized two-sided Robin conditions

$$\max_{k_1 \le k \le \pi/h} |\rho(k, p^*, q^*)| = \frac{1}{\mu} - O\left(\frac{h^{\frac{1}{2}}}{\mu}\right).$$

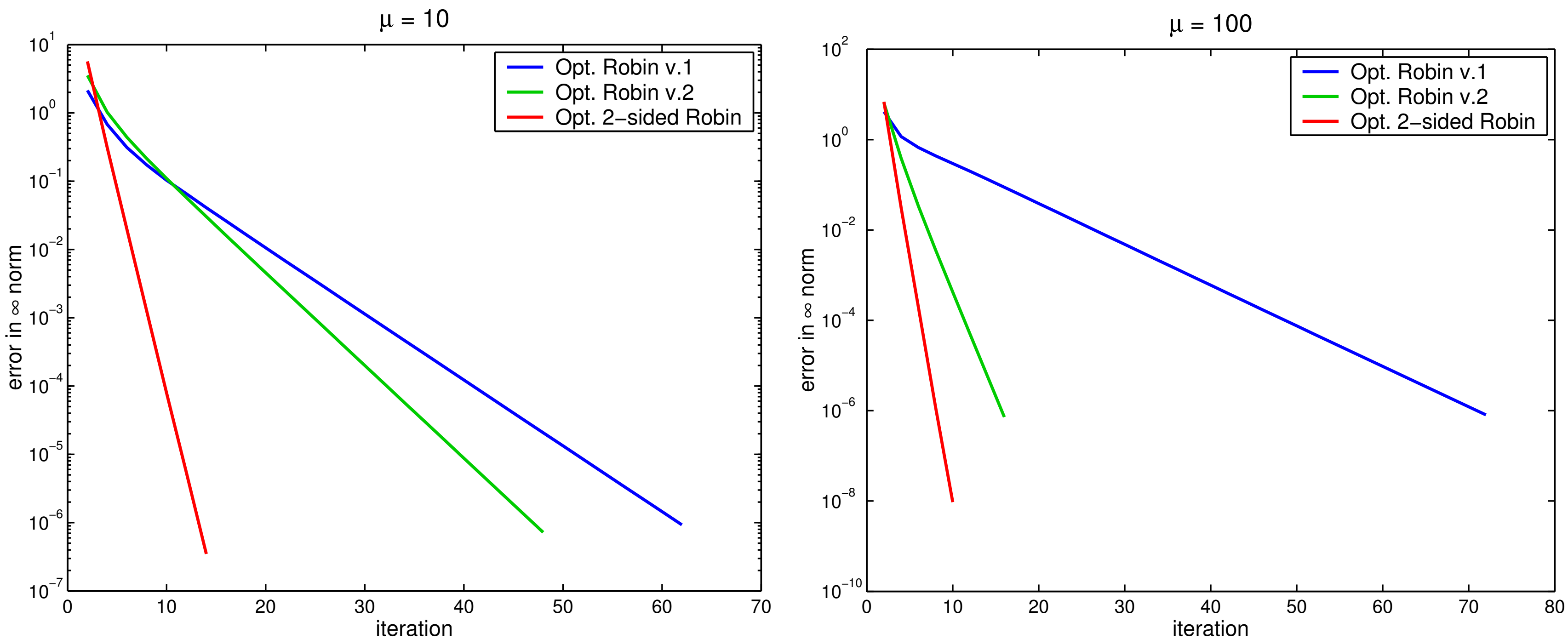
- Optimized second order conditions

$$\max_{k_1 \le k \le \pi/h} |\rho(k, p^*, q^*)| = \left(\frac{\pi}{4k_1 h}\right)^{\frac{1}{4}} \left(\frac{1}{\mu}\right) - O\left(\frac{1}{\mu}\right).$$

Numerical experiments

$$\begin{cases} -\nabla \cdot (\nu(x)\nabla u(x)) &= 1 \quad \text{on } \Omega = (0,\pi)^2, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

The domain is divided into two symmetric subdomains and a finite volume discretization is used. For $h = \frac{\pi}{300}$:



Numerical experiments

Here, we take $\mu = 10$ and vary the grid size h. The table shows the number of iterations to reach a tolerance of 10^{-6}.

h	Opt. Robin v.1	Opt. Robin v.2	Opt. 2-sided Robin
$\frac{\pi}{50}$	24	24	12
$\frac{\pi}{100}$	30	26	12
$\frac{\pi}{200}$	54	42	14
$\frac{\pi}{300}$	62	48	14

Numerical experiments

Now, let us fix $h = \frac{\pi}{300}$ and vary the heterogeneity ratio μ.

μ	Opt. Robin v.1	Opt. Robin v.2	Opt. 2-sided Robin
10^1	62	48	14
10^2	72	16	10
10^3	180	10	8
10^4	204	8	6
10^5	202	6	6

Conclusions

- We completely solved the min-max problem for
  - optimized one-sided Robin conditions (2 versions),
  - optimized two-sided Robin conditions.
- For one-sided Robin conditions, we showed that the second version leads to much better performance, particularly when the jump in the diffusion coefficient is large.
- For two-sided Robin conditions, we obtain an optimal asymptotic performance of $\rho = O(1/\mu)$, independent of h.
- For almost all optimized transmission conditions we considered, the convergence is improved as we increase the jump in the diffusion coefficient.

Work in progress

- Design of an efficient coarse space correction when using many subdomains.
- Optimized conditions for the advection-diffusion equation in 2D,

$$-\nabla \cdot (\nu(x)\nabla u(x)) + \mathbf{a}(x) \cdot \nabla u(x) + c(x)u(x) = f(x),$$

with discontinuous coefficients.