Non Uniform Discrete Fourier Transform for adaptive acceleration of the Aitken-Schwarz DDM

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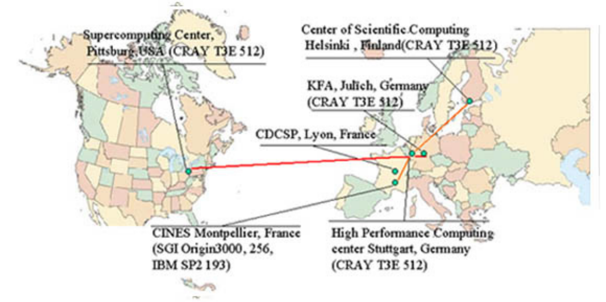

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  - 3D Poisson Pb 762Mdof/60s 5Mbit/s
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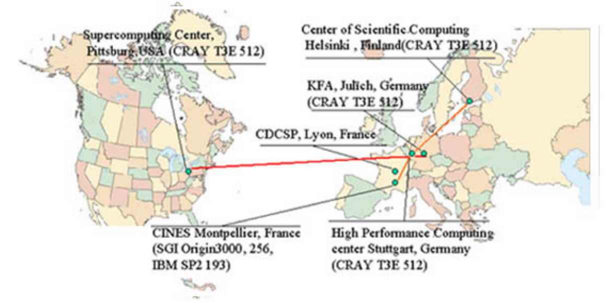
  - Aitken acceleration of Fourier modes
  - *Barberou, Garbey, Hess, Resch, Rossi, Toivanen and Tromeur-Dervout, J. of Parallel and Distributed Computing, special issue on Grid computing, 63(5) :564-577, 2003*
- Aim : extension of this method to non uniform meshes

Introduction



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Outline

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M.Garbey and D.Tromeur-Dervout : *On some Aitken like acceleration of the Schwarz method*,
Int. J. for Numerical Methods in Fluids, 40(12) :1493-1513,2002

- 1D additive Schwarz algorithm for linear differential operators :
  - $L[u_1^{n+1}] = f \; in \; \Omega_1, \; u_{1|\Gamma_1}^{n+1} = u_{2|\Gamma_1}^{n},$
  - $L[u_2^{n+1}] = f \; in \; \Omega_2, \; u_{2|\Gamma_2}^{n+1} = u_{1|\Gamma_2}^{n}.$
- the interface error operator T is **linear**, i.e
  - $u_{1|\Gamma_2}^{n+1} - U_{|\Gamma_2} = \delta_1 (u_{2|\Gamma_1}^{n} - U_{|\Gamma_1}),$
  - $u_{2|\Gamma_1}^{n+1} - U_{|\Gamma_1} = \delta_2 (u_{1|\Gamma_2}^{n} - U_{|\Gamma_2}).$
- Consequently
  - $u_{1|\Gamma_2}^{2} - u_{1|\Gamma_2}^{1} = \delta_1 (u_{2|\Gamma_1}^{1} - u_{2|\Gamma_1}^{0}),$
  - $u_{2|\Gamma_1}^{2} - u_{2|\Gamma_1}^{1} = \delta_2 (u_{1|\Gamma_2}^{1} - u_{1|\Gamma_2}^{0}),$
- Computation of $\delta_{1/2}$:
  $L[v_{1/2}] = 0 \; in \; \Omega_{1/2}, \; v_{\Gamma_{1/2}} = 1$, thus $\delta_{1/2} = v_{\Gamma_{2/1}}$.
- iff $\delta_1 \delta_2 \neq 1$ **Aitken-Schwarz** gives the solution with exactly 3 iterations and possibly 2 in the analytical case.

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Acceleration of Schwarz Method for Elliptic Problems

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The algorithm in 2D or 3D writes :

- step1 : reconstruct P from datas given by two Schwarz iterates
- step2 : apply one additive Schwarz iterate to the Poisson problem with block solver of choice i.e multigrids, FFT etc...
- step3 :
  - compute the Fourier expansion $\hat{u}^n_{j|\Gamma_i}, n = 0, 1$ of **the traces on the artificial interface** $\Gamma_i, i = 1..nd$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the Schwarz iterate solution $u^1_{|\Gamma_i}$.
  - apply generalized Aitken acceleration based on

$$\hat{u}^\infty = (Id - P)^{-1}(\hat{u}^1 - P\hat{u}^0)$$

    in order to get $\hat{u}^\infty_{|\Gamma_i}$.
  - recompose the trace $u^\infty_{|\Gamma_i}$ in physical space.
- step4 : compute in parallel the solution in each subdomains Ω_i, with new inner BCs and blocksolver of choice.

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Methods for non-uniform interface meshes (up to now) :

- Projection technique : spectral interpolation of the interface traces on a third regular grid + classical FFT

  Boursier,Tromeur-Dervout and Vassilevsky, *Parallel solution of Mixed Finite Element/ Spectral Element systems for convection-diffusion equations on non matching grids*,Preprint CDCSP-0300, 2004

- Analysis of the error operator, solving for eigenvalues and eigenvectors, chosen as generalized Fourier basis

  Baranger, Garbey and Oudin-Dardun *Generalized Aitken-like acceleration of the Schwarz method*, Lecture Notes in Computational Science and Engineering, pages 505-512, 2004. Based on an a priori approximation of the error operator P. No available tool to know how the eigenvalues of the approximate P are close to the eigenvalues of true P.

NUDFT formulation

- Define a set of basis functions $\Phi_l = (\phi_l(x_j))_{0 \le j \le N}$ strictly related to the nonuniform mesh and orthogonal with respect to a sesquilinear form $[[.,.]]$, i.e $[[\phi_l, \phi_k]] = 0, \text{ if } l \neq k$.
- Compute the associated interface operator $P_{[[.,.]]}$
- Approximate $P_{[[.,.]]}$ with $P^*_{[[.,.]]}$ through a posteriori estimates of Fourier coefficients behavior.

Instead of :

- Approximate in the physical space P with P^*.
- Compute eigenvalues and eigenvectors of matrix P^*.
- Take eigenvectors as basis functions for generalized Fourier decomposition.

Definition

Let $(x_i)_{0 \leq i \leq N}$ and $z_i = \frac{2\pi i}{N}$ such that $x_i = z_i + \epsilon_i$, and

$$\phi_l(x) = \begin{cases} \psi_l(x) = \exp(ilx), \ 0 \leq l \leq N/2 \\ D^{-N}\exp(i(N-l)x), \ N/2+1 \leq l \leq N, \\ D = diag(\epsilon_i)_{0 \leq i \leq N} \end{cases} \quad (1)$$

$\Rightarrow \phi_{N-l}(x) = \overline{\phi_l(x)}$.

Definition

Define sesquilinear form on S_N =span$\{\phi_l(x), 0 \leq l \leq N\}$, using Hermite integration formula :

$$[[f,g]] = \sum_{l=0}^{N} \gamma_l f(x_l)\overline{g(x_l)} + \sum_{l=0}^{N} \beta_l (f'(x_l)\overline{g(x_l)} + f(x_l)\overline{g'(x_l)})$$

$\{\gamma_l\}$ and $\{\beta_l\}$: $[[\phi_l, \phi_k]] = \delta_{lk} \Rightarrow$ solve one L.S. (size 2N)

H = $([[\phi_l, \phi_k]])_{l,k=0,\dots N} = Id \Rightarrow [[\,:,\,:]]$ hermitian

Definition

The discrete Fourier coefficients of f are given by :

$$\tilde{f}_k = [[f, \Phi_k]], \quad k = -N/2, \dots, N/2$$
$$\tilde{f} = M_1 f + M_2 f', \quad M_1, M_2 \in \mathcal{M}_{N+1}(\mathbb{C})$$
$$M_1(k,l) = \gamma_l \overline{\phi_k(x_l)} + \beta_l \overline{\phi'_k(x_l)}, \; M_2(k,l) = \beta_l \overline{\phi_k(x_l)}$$

Proposition

$$\Pi_N^F(f(x)) = \sum_{l=0}^{N} \tilde{f}_k \phi_k(x), \;\; \textit{is exact} \; \forall f \in \mathbb{T}^{N/2}([0, 2\pi[)$$

- Problem : in the applications one is given the vector f which represents the values of a function $f(x)$ on the points $(x_i)_{0\leq i\leq N}$. No information is given on the vector f' which is needed in definition 3.
- Solution : we determine the vector f' implicitly by imposing

$$\frac{d}{dx}(\Pi_N^F(f(x)))_{|x=x_l} = f'(x_l), \quad l = 0, ..., N-1$$

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NUDFT formulation

In an algebraic form, if we note M_ϕ the matrix whose elements are :

$$M_\phi(l, k) = \phi'_k(x_l)$$

then the vector f' is obtained by solving the algebraic system :

$$(id_{N+1} - M_\phi M_2)f' = M_\phi M_1 f$$

where id_N is the identity matrix in $\mathcal{M}_{N+1}(\mathbb{C})$.

- Given a nonuniform mesh $(x_i)_{0\leq i\leq N}$, define the basis functions and solve one L.S. (size 2N) to determine the two sets $\{\gamma_l\}$ and $\{\beta_l\}$.
- Solve the algebraic system (size N) :

$$(id_{N+1} - M_\phi M_2)f' = M_\phi M_1 f$$

to determine f' implicitly.

- Compute Fourier coefficients through matrix-vector products :

$$\tilde{f} = M_1 f + M_2 f'$$

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N	$\varepsilon = h_u/8$	$\varepsilon = h_u/4$	$\varepsilon = h_u/2$	$\varepsilon = h_u$
40	0.13E-14 6.17E+3	0.39E-15 1.21E+4	0.56E-13 1.26E+5	0.62E-7 4.24E+10
100	0.77E-14 8.40E+4	0.17E-14 1.82E+5	0.69E-12 1.25E+6	0.83E-7 2.07E+10
200	0.13E-13 6.02E+5	0.16E-13 1.27E+6	0.27E-12 5.75E+6	0.5E-6 9.35E+11
400	0.26E-13 5.18E+6	0.29E-13 1.24E+7	0.11E-10 2.96E+8	0.53E-8 7.30E+10

TAB.: $\|f - \Pi_N^F(f)\|_\infty$ and $cond_2([[.,.]])$ for $f(x) = \exp(-40(x-(2\pi/3))^2)$, with $h_u = 2\pi/N$.

- Given a nonuniform cartesian 2D mesh $\mathbf{x} \times \mathbf{y} := \{(x_i, y_j)_{0 \leq i,j \leq N}\} \subset \mathbb{R}^2$ define the basis functions, the sesquilinear form :

$$
\begin{aligned}
[[f,g]] \quad = \quad & \sum_{j=0}^{N} \gamma_j \Big(\sum_{l=0}^{N} \alpha_l (f\overline{g})(x_j, y_l) + \sum_{l=0}^{N} \eta_l \partial_y (f\overline{g})(x_j, y_l) \Big) + \\
& \sum_{j=0}^{N} \beta_j \Big(\sum_{l=0}^{N} \alpha_l \partial_x (f\overline{g})(x_j, y_l) + \sum_{l=0}^{N} \eta_l \partial_{xy} (f\overline{g})(x_j, y_l) \Big)
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- Fourier coefficients computed algebraically by previously solving implicitly for $\partial_x f, \partial_y f$ and $\partial_{xy} f$.

NUDFT algorithm 2D

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Numerical results 2D

N	$\epsilon = h_u/2$	$\epsilon = h_u$	$\epsilon = 2h_u$	$\epsilon = 4h_u$
2^7	1.1E-13 1.5E+3	3.7E-13 8E+3	9.5E-7 2.5E+6	2.09E+3 2.2E+12
2^8	2.62E-13 6E+3	1.48E-10 5E+5	8E-4 1.7E+10	3E+6 1E+14

TAB.: $\|f - \Pi_N^F(f)\|_\infty$ and $cond_2([[.,.]])$ for $f(x,y) = \cos^2(x)\cos(y)$, with $h_u = 2\pi/N$.

- Advantages :
  - better performance than FFT on nonuniform meshes when applied to Aitken-Schwarz DDM
  - $O(N^2)$ operations $\rightarrow$ cheaper in time in comparison with the $O(N^3)$ operations to solve for the eigenvalues and eigenvectors of the full interface operator
  - Adaptive approximation of the trace transfer operator P, based on a posteriori error estimates of Fourier modes convergence
- Gridding : interpolation and use of the FFT on an oversampled grid Greengard and Lee, *Accelerating the Nonuniform Fast Fourier Transform*, SIAM REVIEW, vol.46, No.3, pp.443-454, 2004

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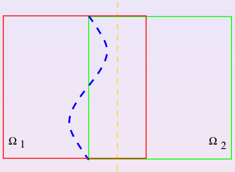
At interfaces Γ_1 and Γ_2, the Fourier coefficients of the error of additive Schwarz algorithm can be rearranged on the form :

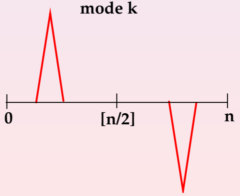
$$\hat{e}_1^{(n+2)}(\Gamma_1) = P_{[[.,.]]}\hat{e}_1^{(n)}(\Gamma_1)$$
$$\hat{e}_2^{(n+2)}(\Gamma_2) = P_{[[.,.]]}\hat{e}_2^{(n)}(\Gamma_2)$$

Numerically, $P_{[[.,.]]}$ is computed by applying two Schwarz iterates for each Fourier mode of the interface solution (computed through the NUDFT), as a relation between all the modes at the two iterates.

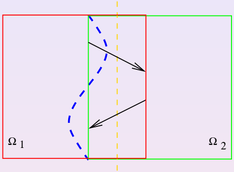
- Take one basis function on the interface (blue line) :



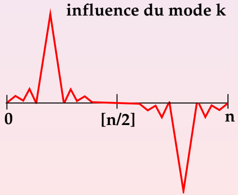
- Applying NUDFT to the basis function, obtain a symmetric decomposition :



- With 2 Schwarz iterates determine how this function is modified by the additif Schwarz algorithm :



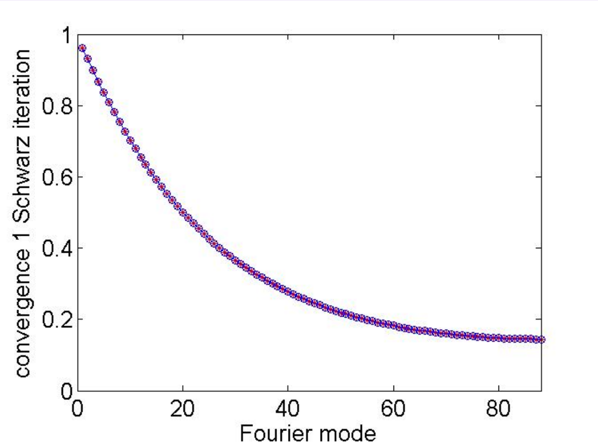
- Applying NUDFT, compute the influence of one Fourier mode on all modes :



- Fill $k-$column of matrix $P_{[[.,.]]}$, not symmetric.

Uniform grids : NUDFT $\rightarrow$ FFT

$P_{[[.,.]]}$ diagonal and $\|P_{[[.,.]]} - P_{an}\|_\infty = O(10^{-12})$



- Nonuniform cartesian grids and/or non separable differential operator
- P is no longer diagonal
- we can approximate $P_{[[.,.]]}$ using only the most important modes, then accelerate only these modes through the equation :

$$\tilde{v}^{\infty} = (Id - P^*_{[[.,.]]})^{-1}(\tilde{v}^{n+1} - P^*_{[[.,.]]}\tilde{v}^{n})$$

  where $\tilde{v}$ is the subset of $\tilde{u}$ used to approximate $P_{[[.,.]]}$ with $P^*_{[[.,.]]}$. Other modes are not accelerated.
- $P^*_{[[.,.]]}$ columns can be built in parallel and the number of columns computed during the Schwarz iterates can be set according to the computer architecture

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Outline

AS-DDM on a strongly non separable operator and irregular matching grids

Solution of 2D convection-diffusion equation with Aitken-Schwarz DDM : the trace of the iterate solutions on the irregular mesh are projected on a Fourier orthogonal basis. The Fourier modes are accelerated through the Aitken technique.

$$\begin{aligned} \nabla.(a(x,y)\nabla)u(x,y) &= f(x,y), \quad \text{on } \Omega =]0,1[^2 \\ u(x,y) &= 0, \quad (x,y) \in \partial\Omega \end{aligned}$$

$a(x,y) = a_0 + (1-a_0)(1+tanh((x-(3h*y+1/2-h))/\mu))/2,$
and $a_0 = 10^1, \mu = 10^{-2}$.

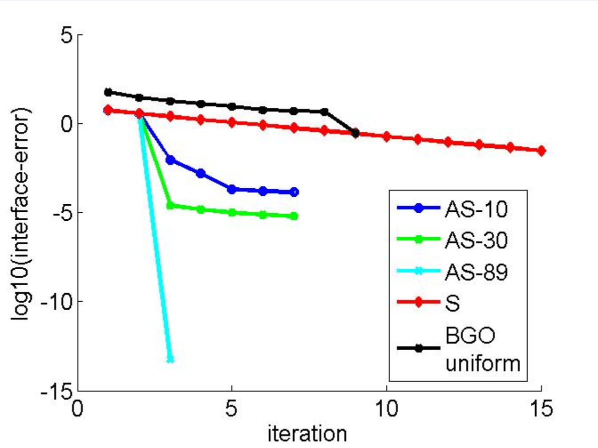


FIG.: acceleration using sub-blocks of $P_{[[\cdot,\cdot]]}$ with 90 points on the interface, overlap$=5$ and $\epsilon = h_u/2$. Black line refers to results in Baranger, Garbey and Oudin-Dardun *The Aitken-Like Acceleration of the Schwarz Method on Non-Uniform Cartesian Grids*, Technical Report Number UH-CS-05-18, 2005.

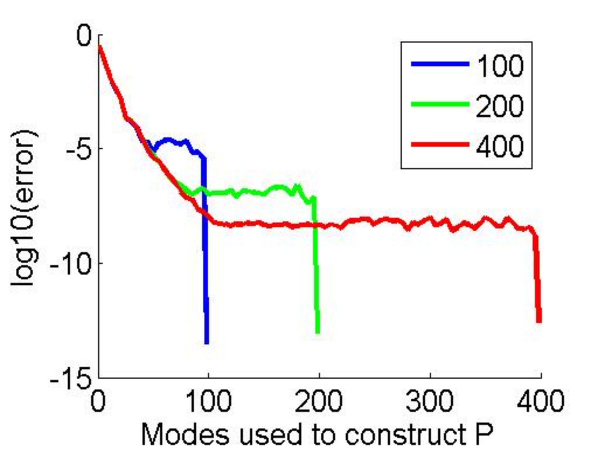
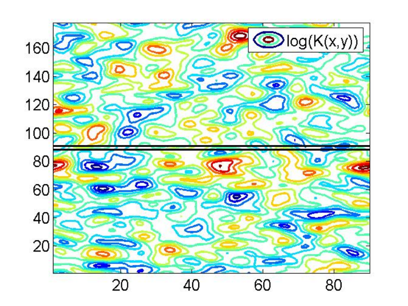


FIG.: influence of the approximation of the interface operator $P_{[[.,.]]}$ on the convergence of the interface error

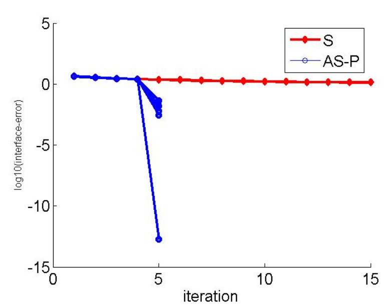
Convergence of AS in random porous media

K follows a log-normal random process

$$\nabla.(K(x,y)\nabla u) = f, \ \text{on}\Omega$$
$$u = 0, \ \text{on } \partial\Omega$$



$K(x,y) \in [0.0091, 242.66]$



Convergence of AS

Work under progress in collaboration with J-R De Dreuzy and J. Erhel SAGE/IRISA

Outline

- Extend ASDDM to nonuniform cartesian meshes by means of the NUDFT technique
- Reduce the numerical complexity by adaptively approximating the trace transfer operator P
- Validate the technique in the 2D case and DD in stripes

- Works also for Nonuniform non matching cartesian grids
- Under investigation : NUDFT $\rightarrow$ NUFFT