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Talk: Time-Decomposition Methods for Parabolic Problems : Convergence results of Iterative Splitting methods.

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Outline of the talk

1.) Introduction

2.) Decomposition-methods

3.) Time-Decomposition methods

3.1) Sequential Splitting methods

3.2) Iterative Splitting method

4.) Numerical experiments

5.) Future Works

Motivation and Ideas

Design of fast solvers with high accuracy

Efficient solver by decoupling in simpler equations or domains for solving multi-physics problems

Parallelization and accelerating the solver-process

Physical correct splitting and analytical Decomposition method : preservation of physics

Fast computations for complicate and decoupable problems

Model-Equation

Systems of parabolic-differential equations with first order time-derivation and second order spatial-derivations

$$\frac{\partial c}{\partial t} = f(c) + Ac + Bc \; , \text{ in } \Omega \times (0,T) \; , \tag{1}$$
$$c(x,t) = g(x,t) \; , \text{ on } \partial\Omega \times (0,T) \text{ (Boundary-Condition)} \; ,$$
$$c(x,0) = c_0(x) \; , \text{ in } \Omega \text{ (Initial-Condition)} \; ,$$

where $c = (c_1, \ldots, c_n)^t$ and $f(c) = (f_1(c), \ldots, f_n(c))^t$,

$$A = \begin{pmatrix} -v_{11} \cdot \nabla & \cdots & -v_{n1} \cdot \nabla \\ \ldots & \cdots & \ldots \\ -v_{1n} \cdot \nabla & \cdots & -v_{nn} \cdot \nabla \end{pmatrix}, B = \begin{pmatrix} \nabla D_{11} \cdot \nabla & \ldots & \nabla D_{n1} \cdot \nabla \\ \ldots & \ldots & \ldots \\ \nabla D_{1n} \cdot \nabla & \ldots & \nabla D_{nn} \cdot \nabla \end{pmatrix} \; ,$$

Convection- and diffusion-operator with $A, B : X \to X$ and $X = I\!R^n$ a matrix-space.

sufficient smoothness $c_i \in C^{2,1}(\Omega, [0,T])$ for $i = 1, \ldots, n$

First Part : Decomposition Methods

Ideas :

Decoupling the time-scales, space-scales.

Decoupling the multi-physics.

Time-adaptivity, Space-adaptivity.

Parallelization in Time and Space.

Methods :

Operator-Splitting and Variational Splitting Methods (Time).

Iterative and extended Operator Splitting Methods (Time).

Waveform-Relaxation-Methods (Time).

Schwarz Wave form relaxation method (Space).

Additive and Multiplicative Schwarz method (Space).

Partition of Units combined with Splitting methods (Time and Space).

Time-Decomposition methods

History and Literature:

ADI-methods (Alternating direction implicit), see : Peaceman-Rachford (1955).

Strang-Marchuk-Splitting methods, see : Strang (1968).

Waveform-relaxation Methods, see : Vandewalle (1993).

Variational Splitting Methods, see : Lubich (2003).

Iterative Operator-Splitting Methods, see : Kanney, Miller, Kelly (2003), Farago, Geiser (2005).

Extended Iterative Operator Splitting Methods, see : Geiser (2006).

Decoupling methods as preservation of physics, see : Geiser (2006).

Introduction : Operator-Splitting-Method

Idea: Decoupling of complex equations in simpler equations, solving simpler equations and re-coupling the results over the initial-conditions.

Equations: $\partial_t c = Ac + Bc$,
where the initial-conditions are $c(t^n) = c^n$, (or Variational-formulation: $(\partial_t c, v) = (Ac, v) + (Bc, v)$.)

Splitting-method of first order

$$\partial_t c^* = Ac^* \quad \text{with} \quad c^*(t^n) = c^n \; ,$$
$$\partial_t c^{**} = Bc^{**} \quad \text{with} \quad c^{**}(t^n) = c^*(t^{n+1}) \; ,$$

where the results of the methods are $c(t^{n+1}) = c^{**}(t^{n+1})$,
and there are some splitting-errors for these methods,
Literature : [Strang 68], [Karlsen et al 2001].

Splitting-Errors of the Method

The error of the splitting-method of first order is

$$\partial_t c = (B + A)c \; ,$$
$$\tilde{c} = \exp(\tau(B + A))c(t^n) \; .$$

Local error for the decomposition and the full solution

$$\begin{aligned} e(c) &= \tilde{c}(t^n + \tau) - \exp(\tau B)\exp(\tau A)c(t^n) \; , \\ &= \exp(\tau(B + A))c(t^n) - \exp(\tau B)\exp(\tau A)c(t^n) \; , \\ e(c)/\tau &= \frac{1}{2}\tau(BA - AB)c(t^n) + O(\tau^2) \; , \end{aligned}$$

$O(\tau)$ for A, B not commuting, otherwise one get exact results, where $\tau = t^{n+1} - t^n$, [Strang 68].

Higher order splitting-methods

Strang or Strang-Marchuk-Splitting, cf. [Marchuk 68, Strang68]

$$\frac{\partial c^*(t)}{\partial t} = Ac^*(t), \text{ with } t^n \leq t \leq t^{n+1/2} \text{ and } c^*(t^n) = c_{\mathrm{sp}}^n, \tag{2}$$

$$\frac{\partial c^{**}(t)}{\partial t} = Bc^{**}(t), \text{ with } t^n \leq t \leq t^{n+1}\ , c^{**}(t^n) = c^*(t^{n+1/2}),$$

$$\frac{\partial c^{***}(t)}{\partial t} = Ac^{***}(t)\ , t^{n+1/2} \leq t \leq t^{n+1}\ , c^{***}(t^{n+1/2}) = c^{**}(t^{n+1}),$$

where $t^{n+1/2} = t^n + 0.5\tau_n$ and the approximation on the next time level t^{n+1} is defined as $c_{\mathrm{sp}}^{n+1} = c^{***}(t^{n+1})$.

The splitting error of the Strang splitting is

$$\rho_n = \frac{1}{24}\tau_n^2([B,[B,A]] - 2[A,[A,B]])\ c(t^n) + O(\tau_n^3)\ , \tag{3}$$

see, e.g.[Hundsdorfer, Verwer 2003].

Iterative splitting-Methods

$$\frac{\partial c_i(t)}{\partial t} = Ac_i(t) \; + \; Bc_{i-1}(t), \text{ with } \; c_i(t^n) = c_{\rm sp}^n, \qquad (4)$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = Ac_i(t) \; + \; Bc_{i+1}(t), \text{ with } \; c_{i+1}(t^n) = c_{\rm sp}^n, \qquad (5)$$

where $c_0(t)$ is any fixed function for each iteration. (Here, as before, $c_{\rm sp}^n$ denotes the known split approximation at the time level $t = t^n$.) The split approximation at the time-level $t = t^{n+1}$ is defined as $c_{\rm sp}^{n+1} = c_{2m+1}(t^{n+1})$. (Clearly, the functions $c_k(t)$ ($k = i-1, i, i+1$) depend on the interval $[t^n, t^{n+1}]$, too, but, for the sake of simplicity, in our notation we omit the dependence on n.)

Error for the Iterative splitting-method

Theorem 1. *The error for the splitting methods is given as :*

$$||e_i|| = K||B||\tau_n||e_{i-1}|| + O(\tau_n^2) \tag{6}$$

and hence

$$||e_{2m+1}|| = K_m||e_0||\tau_n^{2m} + O(\tau_n^{2m+1}), \tag{7}$$

where τ_n is the time-step, e_0 the initial error $e_0(t) = c(t) - c_0(t)$ and m the number of iteration-steps, K and K_m are constants, $||B||$ is the maximum norm of operator B and A and B are bounded, monotone operators.

Proof : Taylor-expansion and estimation of $\exp$-functions. See the work Geiser,Farago (2005).

Nonlinear Iterative splitting-Methods

$$\frac{\partial c_i(t)}{\partial t} = A(c_i(t)) \; + \; B(c_{i-1}(t)), \text{ with } \; c_i(t^n) = c_{\rm sp}^n, \tag{8}$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = A(c_i(t)) \; + \; B(c_{i+1}(t)), \text{ with } \; c_{i+1}(t^n) = c_{\rm sp}^n, \tag{9}$$

where $c_0(t)$ is any fixed function for each iteration. (Here, as before, $c_{\rm sp}^n$ denotes the known split approximation at the time level $t = t^n$.) The split approximation at the time-level $t = t^{n+1}$ is defined as $c_{\rm sp}^{n+1} = c_{2m+1}(t^{n+1})$. (Clearly, the functions $c_k(t)$ $(k = i-1, i, i+1)$ depend on the interval $[t^n, t^{n+1}]$, too, but, for the sake of simplicity, in our notation we omit the dependence on n.)

Consistency Theory for the nonlinear iterative splitting method

Theorem 2. *Let us consider the nonlinear operator-equation in a Banach space* **X**

$$\begin{aligned} \partial_t c(t) &= A(c(t)) + B(c(t)), \quad 0 < t \leq T \\ c(0) &= c_0 \end{aligned} \tag{10}$$

We linearised the nonlinear operators and obtain the linearised equation

$$\begin{aligned} \partial_t c(t) = \tilde{A}c(t) + \tilde{B}c(t) + R(\tilde{c}), \quad 0 < t \leq T\};, \\ \tilde{A} = \frac{\partial A}{\partial c}(\tilde{c}) \\ \tilde{B} = \frac{\partial B}{\partial c}(\tilde{c}) \\ R(\tilde{c}) = A(\tilde{c}) + B(\tilde{c}) - \tilde{c}(\frac{\partial A}{\partial c}(\tilde{c}) + \frac{\partial B}{\partial c}(\tilde{c})) \\ c(0) = c_0 \ , \end{aligned} \tag{11}$$

where $\tilde{A}, \tilde{B}, \tilde{A}+\tilde{B} : \mathbf{X} \rightarrow \mathbf{X}$ are given linear operators being generators of the C_0-semigroup and $c_0 \in \mathbf{X}$ is a given element. Then the iteration process (8)–(9) is convergent and the and the rate of the convergence is of second order.

We obtain the iterative result :

$$\|e_i\| = K\tau_n \|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{12}$$

and hence

$$\|e_{2m+1}\| = K_1 \tau_n^{2m+1} \|e_0\| + \mathcal{O}(\tau_n^{2m+1}), \tag{13}$$

where $e_i(t) = c(t) - c_i(t)$ and $2m+1$ are the number of iterates.

Proof 3. *See [Geiser & Kravvaritis 2006, Preprint]*
Let us consider the iteration (8)–(9) on the sub-interval $[t^n, t^{n+1}]$. *For the error function* $e_i(t) = c(t) - c_i(t)$ *we have the relations*

$$\begin{aligned} \partial_t e_i(t) &= A(e_i(t)) + B(e_{i-1}(t)), \quad t \in (t^n, t^{n+1}], \\ e_i(t^n) &= 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \partial_t e_{i+1}(t) &= A(e_i(t)) + B(e_{i+1}(t)), \quad t \in (t^n, t^{n+1}], \\ e_{i+1}(t^n) &= 0 \end{aligned} \tag{15}$$

for $m = 0, 2, 4, \ldots,$ *with* $e_0(0) = 0$ *and* $e_{-1}(t) = c(t)$.

We obtain the linearised equations :

In the following we use the notations $\mathbf{X}^2$ *for the product space* $\mathbf{X} \times \mathbf{X}$ *enabled with the norm* $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ *(*$u, v \in \mathbf{X}$*). The elements* $\mathcal{E}_i(t)$*,* $\mathcal{F}_i(t) \in \mathbf{X}^2$ *and the linear operator* $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ *are defined as follows*

$$\mathcal{E}_i(t) = \left[\begin{array}{c} e_i(t) \\ e_{i+1}(t) \end{array} \right]; \quad \mathcal{A} = \left[\begin{array}{cc} \frac{\partial A(c_{i-1})}{\partial c} & 0 \\ \frac{\partial A(c_{i-1})}{\partial c} & \frac{\partial B(c_{i-1})}{\partial c} \end{array} \right]. \tag{16}$$

$$\mathcal{F}_i(t) = \left[\begin{array}{c} A(e_{i-1}(t)) + B(e_{i-1}(t)) - e_{i-1}\frac{\partial A(e_{i-1})}{\partial c} \\ A(e_{i-1}(t)) + B(e_{i-1}(t)) - e_{i-1}\frac{\partial A(e_{i-1})}{\partial c} - e_{i-1}\frac{\partial B(e_{i-1})}{\partial c} \end{array} \right]; \tag{17}$$

The relation can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_i(t) &= \mathcal{A}\mathcal{E}_i(t) + \mathcal{F}_i(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_i(t^n) &= 0. \end{aligned} \tag{18}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semigroup $(\mathcal{A}(t))_{t \geq 0}$. We have to estimate the 2 terms : $\mathcal{F}_i(t)$ and $\exp(\mathcal{A}(t))$ *.*

We could estimate the right hand side $\mathcal{F}_i(t)$ in the following lemma

Lemma 4. *Let us consider the the bounded Jacobians of $A(u)$ and $B(u)$. We could then estimate the $\mathcal{F}_i(t)$ as*

$$||\mathcal{F}_i(t)|| \leq C||e_{i-1}|| \tag{19}$$

Proof see [Geiser & Kravvaritis 2006]

We estimate our abstract Cauchy problem (18) that be solved as

$$\mathcal{E}_i(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s))\mathcal{F}_i(s)ds, \quad t \in [t^n, t^{n+1}]. \tag{20}$$

Hence, using the denotation

$$\|\mathcal{E}_i\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_i(t)\| \tag{21}$$

we have

$$\begin{aligned} \|\mathcal{E}_i\|(t) &\leq \|\mathcal{F}_i\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds = \\ &= C \, \|e_{i-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds, \quad t \in [t^n, t^{n+1}]. \end{aligned} \tag{22}$$

We have estimate $||\mathcal{F}_i|| \leq C||e_{i-1}||$.

Since $(\mathcal{A}(t))_{t\geq 0}$ is a semigroup therefore the so called $growth$ $estimation$

$$\|\exp(\mathcal{A}t)\| \leq K\exp(\omega t); \quad t \geq 0 \tag{23}$$

holds with some numbers $K \geq 0$ and $\omega \in I\!R$.

Assume that $(\mathcal{A}(t))_{t\geq 0}$ is a bounded or exponentially stable semi-group, i.e. (23) holds with some $\omega \leq 0$. Then obviously the estimate

$$\|\exp(\mathcal{A}t)\| \leq K; \quad t \geq 0 \tag{24}$$

holds, and, hence on base of (22), we have the relation

$$\|\mathcal{E}_i\|(t) \leq K\tau_n\|e_{i-1}\|, \quad t \in (0, \tau_n). \tag{25}$$

Assume that $(\mathcal{A}(t))_{t\geq 0}$ has an exponential growth with some $\omega > 0$.

Using (22) we have

$$\int_{t^n}^{t^{n+1}} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{26}$$

where

$$K_\omega(t) = \frac{K}{\omega} \left(\exp(\omega(t-t^n)) - 1\right), \quad t \in [t^n, t^{n+1}]. \tag{27}$$

Hence

$$K_\omega(t) \leq \frac{K}{\omega} \left(\exp(\omega\tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2) \tag{28}$$

The estimations (25) and (28) result in that

$$\|\mathcal{E}_i\|_\infty = K\tau_n \|e_{i-1}\| + \mathcal{O}(\tau_n^2). \tag{29}$$

and we obtain result by using definition of $\mathcal{E}_i$

$$\|e_i\| = K\tau_n \|e_{i-1}\| + \mathcal{O}(\tau_n^2). \tag{30}$$

Extended Iterative splitting methods

For the extended iterative splitting methods with weighting factors

$$\frac{\partial c_i(t)}{\partial t} = Ac_i(t) \; + \; \omega \; Bc_{i-1}(t), \text{ with } \; c_i(t^n) = c^n \tag{31}$$

$$\text{and } c_0(t^n) = c^n \; , \; c_{-1} = 0.0,$$

$$\text{with } \; c_i(t^n) = \omega \; c^n + (1 - \omega) \; c_i(t^{n+1}) \; ,$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = \omega \; Ac_i(t) \; + \; Bc_{i+1}(t), \tag{32}$$

$$\text{with } \; c_{i+1}(t^n) = \omega \; c^n + (1 - \omega) \; c_i(t^{n+1}) \; ,$$

where c^n is the known split approximation at the time level $t = t^n$. The split approximation at the time-level $t = t^{n+1}$ is defined as $c^{n+1} = c_{2m+1}(t^{n+1})$. Our parameter $\omega \in [0, 1]$. For $\omega = 0$ we have the sequential-splitting and for $\omega = 1$ we have the iterative splitting method.

Stability Theory

We concentrate on the stability theory for the linear ordinary differential equations with commutative operators. First we apply the recursion for the general case and obtain the commutative case.

The stability for the extended iterative splitting method (31) and (32) is studied. We treat the special case for the initial-values with $c_i(t^n) = c_n$ and $c_{i+1}(t^n) = c_n$ for an overview. The general case $c_{i+1}(t^n) = \omega c_n + (1-\omega) c_i(t^{n+1})$ could be treated in the same manner.

We consider the suitable vector norm $|| \cdot ||$ on $I\!R^M$, together with its induced operator norm. We assume that

$$|| \exp(\tau \ A) || \leq 1 \ \text{ and } \ || \exp(\tau \ B) || \leq 1 \ \text{ for all } \ \tau > 0.$$

and also implies $|| \exp(\tau \ (A + B)) || \leq 1$.

For the linear problem (31) and (32) it follows by integration that

$$c_i(t) = \exp((t - t^n)A)c^n + \int_{t^n}^{t} \exp((t-s)A)\ \omega\ Bc_{i-1}(s)\ ds\ , \quad (33)$$

$$c_{i+1}(t) = \exp((t - t^n)B)c^n + \int_{t^n}^{t} \exp((t-s)B)\ \omega\ Ac_{i}(s)\ ds\ . \quad (34)$$

With elimination of c_i we get

$$c_{i+1}(t) = \exp((t - t^n)B)c^n\ +\ \omega\ \int_{t^n}^{t} \exp((t-s)B)\ A\ \exp((s-t^n)A)$$
$$+\omega^2\ \int_{s=t^n}^{t}\ \int_{s'=t^n}^{s} \exp((t-s)B)\ A\ \exp((s-s')A)\ B\ c_{i-1}(s')\ ds'$$

For the following commuting case we could evaluate the double integral $\int_{s=t^n}^{t}\ \int_{s'=t^n}^{s}$ as $\int_{s'=t^n}^{t}\ \int_{s=s'}^{t}$ and could derive the weighted stability-theory.

Commuting operators

For more transparency of the formula (35) we consider a well-conditioned system of eigenvectors and the eigenvalues λ_1 of A and λ_2 of B instead of the operators A, B themselves. Replacing the operators A and B by λ_1 and λ_2 respectively, we obtain after some calculations

$$\begin{aligned} c_{i+1}(t) &= c^n \frac{1}{\lambda_1 - \lambda_2} (\omega \lambda_1 \exp((t - t^n)\lambda_1) \\ &\quad + ((1-\omega)\lambda_1 - \lambda_2) \exp((t - t^n)\lambda_2)) \\ &+ c^n \, \omega^2 \, \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \int_{s=t^n}^{t} (\exp((t-s)\lambda_1) \\ &\quad - \exp((t-s)\lambda_2)) \, ds \; . \end{aligned} \tag{36}$$

Note that this relation is symmetric in λ_1 and λ_2.

Strong Stability

We define $z_k = \tau \lambda_k$, $k = 1, 2$. We start with $c_0(t) = u^n$ and we obtain

$$c_{2m}(t^{n+1}) \;\; = \;\; S_m(z_1, z_2) \; c^n \;, \qquad (37)$$

where S_m is the stability function of the scheme with m-iterations. We use (36) and obtain after some calculations

$$\begin{aligned} S_1(z_1, z_2) \;\; &= \;\; \omega^2 \; c^n \; + \; \frac{\omega \; z_1 + \omega^2 \; z_2}{z_1 - z_2} \; \exp(z_1) \; c^n \qquad (38) \\ &+ \;\; \frac{(1 - \omega - \omega^2) \; z_1 - z_2}{z_1 - z_2} \; \exp(z_2) \; c^n \end{aligned}$$

$$
\begin{aligned}
S_2(z_1, z_2) \;=\;& \omega^4 \; c^n \; + \; \frac{\omega \; z_1 + \omega^4 \; z_2}{z_1 - z_2} \; \exp(z_1) \; c^n \\
+ \;& \frac{(1 - \omega - \omega^4) \; z_1 - z_2}{z_1 - z_2} \; \exp(z_2) \; c^n \\
+ \;& \frac{\omega^2 \; z_1 \; z_2}{(z_1 - z_2)^2} \, ((\omega z_1 + \omega^2 z_2) \exp(z_1) \\
& + (-(1 - \omega - \omega^2) z_1 + z_2) \exp(z_2)) \; c^n \\
+ \;& \frac{\omega^2 \; z_1 \; z_2}{(z_1 - z_2)^3} \, ((-\omega z_1 - \omega^2 z_2)(\exp(z_1) - \exp(z_2)) \\
& + ((1 - \omega - \omega^2) z_1 - z_2)(\exp(z_1) - \exp(z_2))) \; c^n
\end{aligned} \tag{39}
$$

Let us consider the stability given by the following eigenvalues in a wedge

$$\mathcal{W} = \{\zeta \in \mathbb{C} : |\arg(\zeta) \leq \alpha\}$$

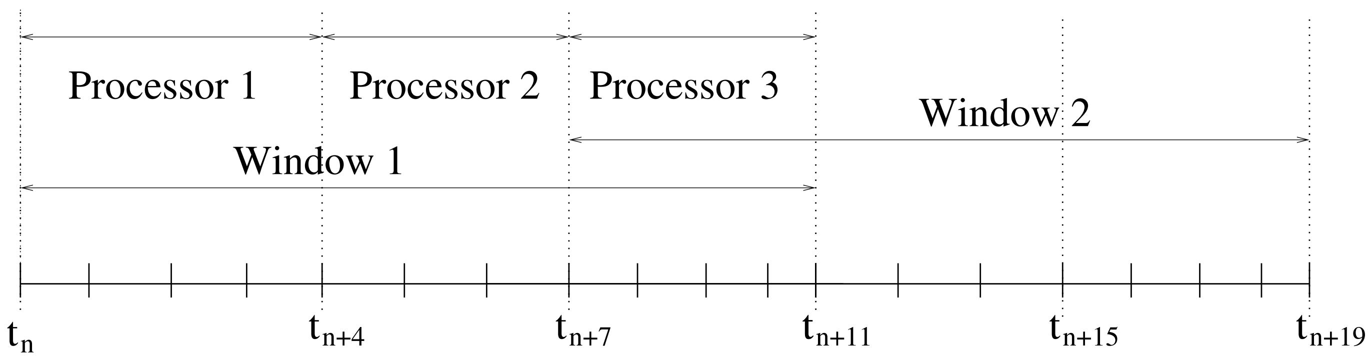
For the stability we have $|S_m(z_1, z_2)| \leq 1$ whenever $z_1, z_2 \in \mathcal{W}_{\pi/2}$. The stability of the two iterations is given in the following theorem with respect to the stability.

Theorem 5. *We have the following stability :*
For S_1 we have a strong stability with
$\max_{z_1 \leq 0, z_2 \in W_\alpha} |S_1(z_1, z_2)| \leq 1$, $\forall\, \alpha \in [0, \pi/2]$ *with* $\omega = \frac{1}{\sqrt[4]{3}}$
For S_2 we have a strong stability with
$\max_{z_1 \leq 0, z_2 \in W_\alpha} |S_2(z_1, z_2)| \leq 1$, $\forall\, \alpha \in [0, \pi/2]$ *with* $\omega \leq \left(\frac{1}{8\ \tan^2(\alpha)+1}\right)^{1/8}$

Proof see [Geiser 2006, Preprint]

Parallelization of the Time-Decomposition method : Windowing

The idea for parallelization in time are the windowing, that the processors has an amount of time-steps to compute and to share the end-result of the computation as an initial-condition for the next processor.



Numerical Experiments

First example : 2D Diffusion-Reaction equation

We deal with the time dependent 2-D equation:

$$\begin{aligned} \partial_t u(x,y,t) &= u_{xx} + u_{yy} - 4(1+y^2)e^{-t}e^{x+y^2} \\ u(x,y,0) &= e^{x+y^2} \text{ in } \Omega = [-1,1] \times [-1,1] \\ u(x,y,t) &= e^{-t}e^{x+y^2} \text{ on } \partial\Omega \end{aligned}$$

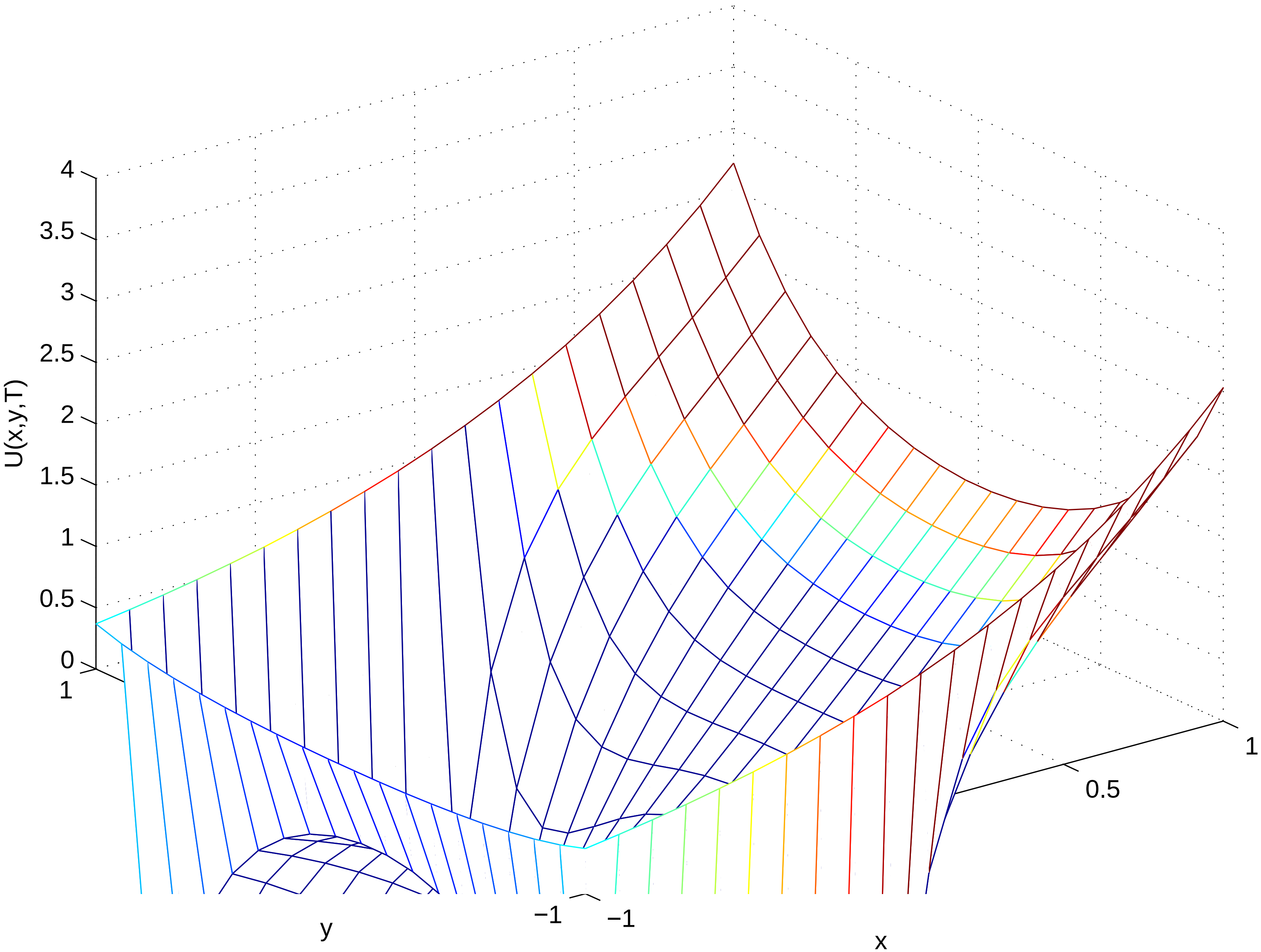
with exact solution

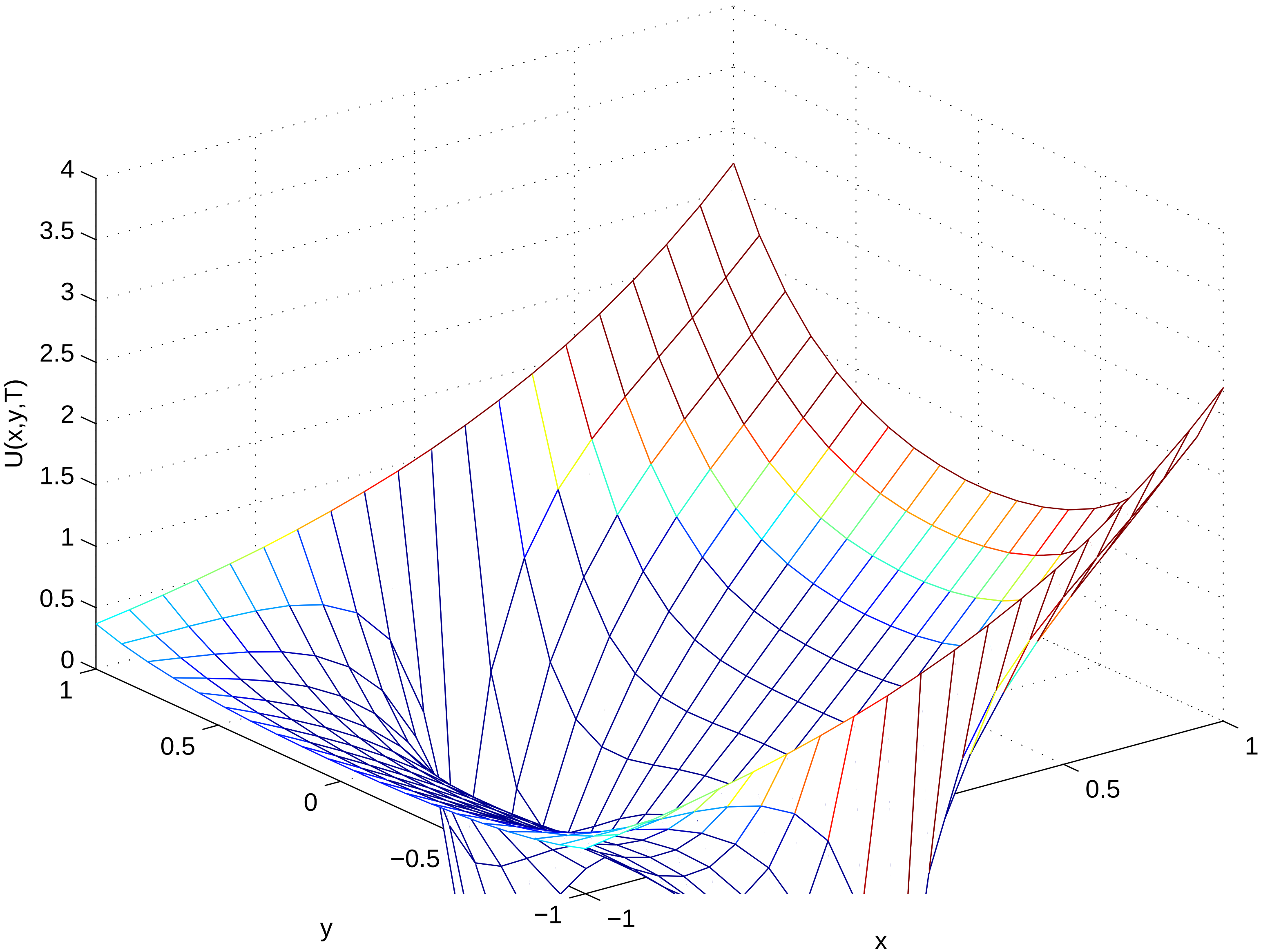
$$u(x,y,t) = e^{-t}e^{x+y^2}$$

We choose the time Itervall [0,1] and again use Finite Differences for the space with $\Delta x = 2/19$.
We define our operators by splitting the plane into two halfs.
We choose one splitting intervall.

Iterative Steps	Number of splitting-partitions	Max-error
1	1	2.7183e+000
2	1	8.2836e+000
3	1	3.8714e+000
4	1	2.5147e+000
5	1	1.8295e+000
10	1	6.8750e-001
15	1	2.5764e-001
20	1	8.7259e-002
25	1	2.5816e-002
30	1	5.3147e-003
35	1	2.8774e-003

Table 1: Numerical results for the first example with the Iterative Operator Splitting method and BDF3 with $h = 10^{-1}$.


U(x,y,T)
4
3.5
3
2.5
2
1.5
1
0.5
0
1
y
−1
−1
x
0.5
1


4
3.5
3
2.5
2
1.5
1
0.5
0
U(x,y,T)
1
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−0.5
−1
y
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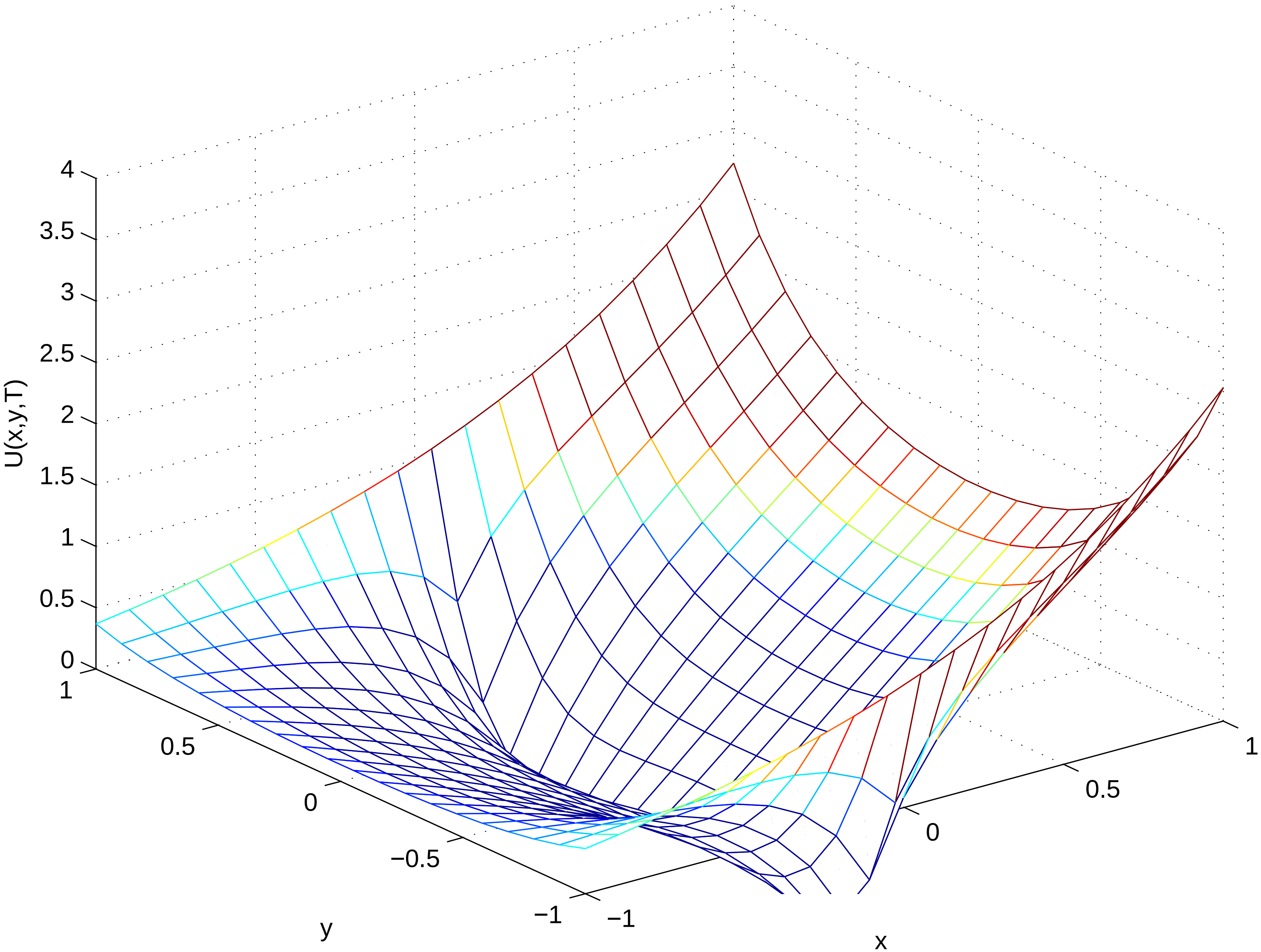
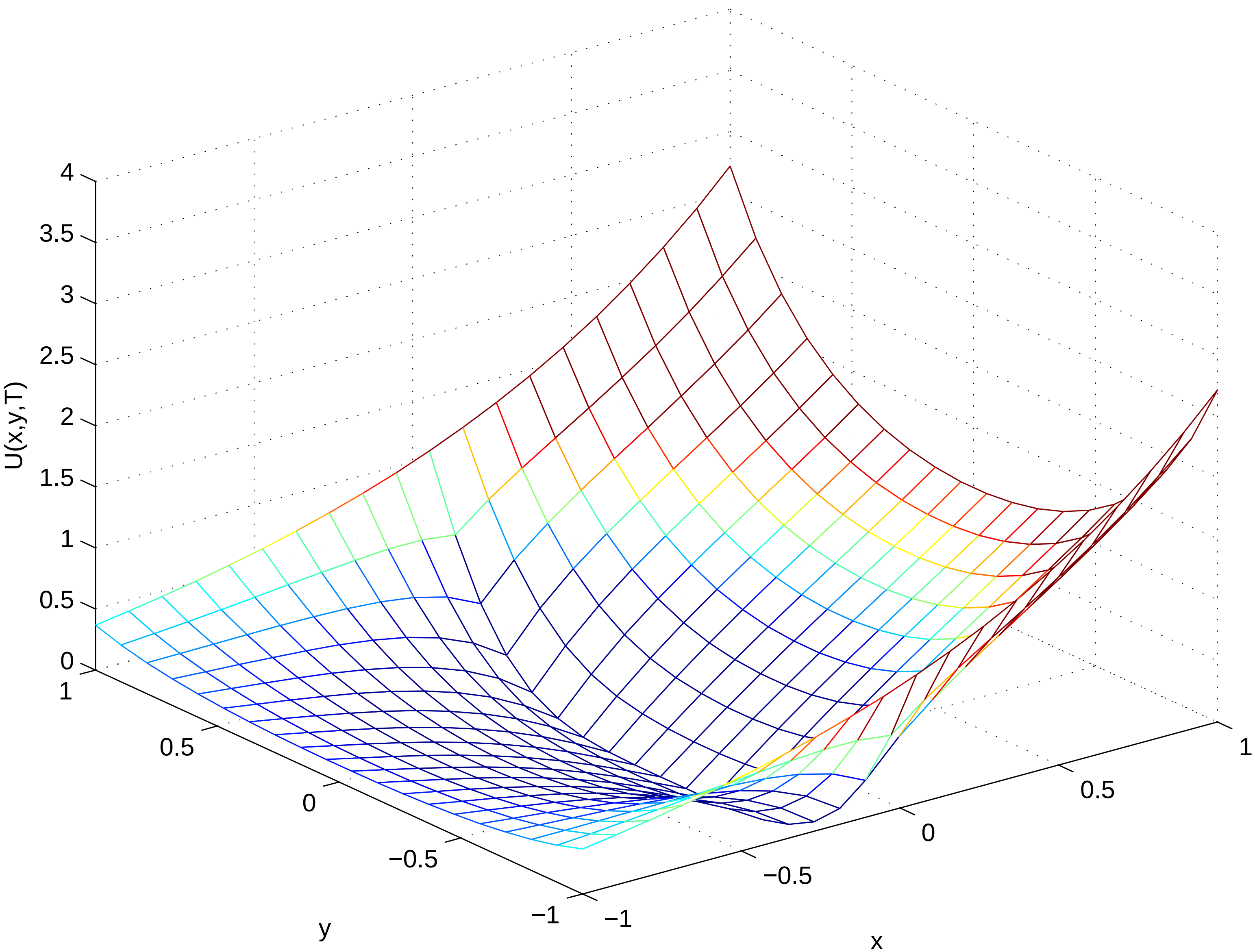
1
x



U(x,y,T)
4
3.5
3
2.5
2
1.5
1
0.5
0
1
0.5
0
−0.5
−1
y
−1
0
0.5
1
x



U(x,y,T)
4
3.5
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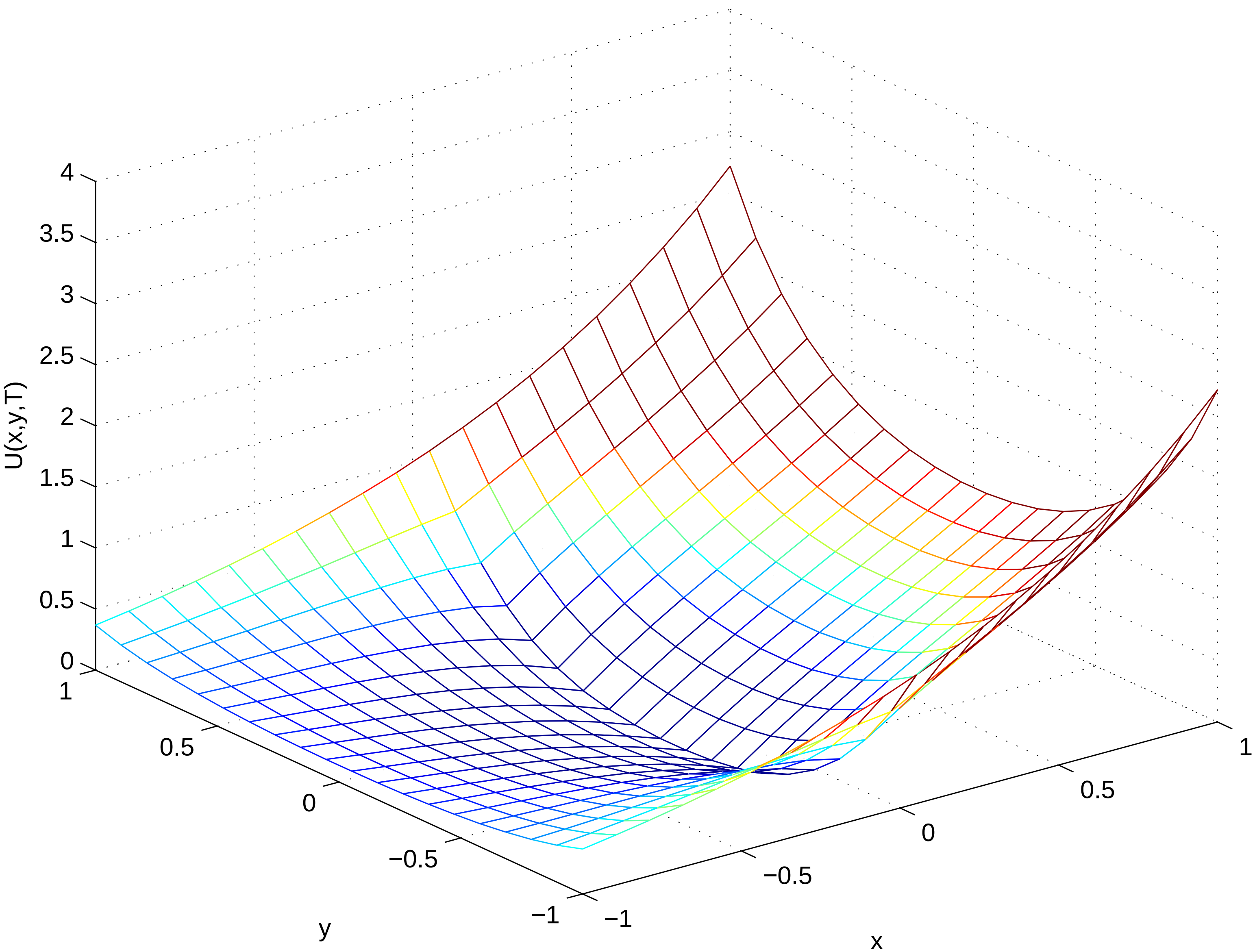

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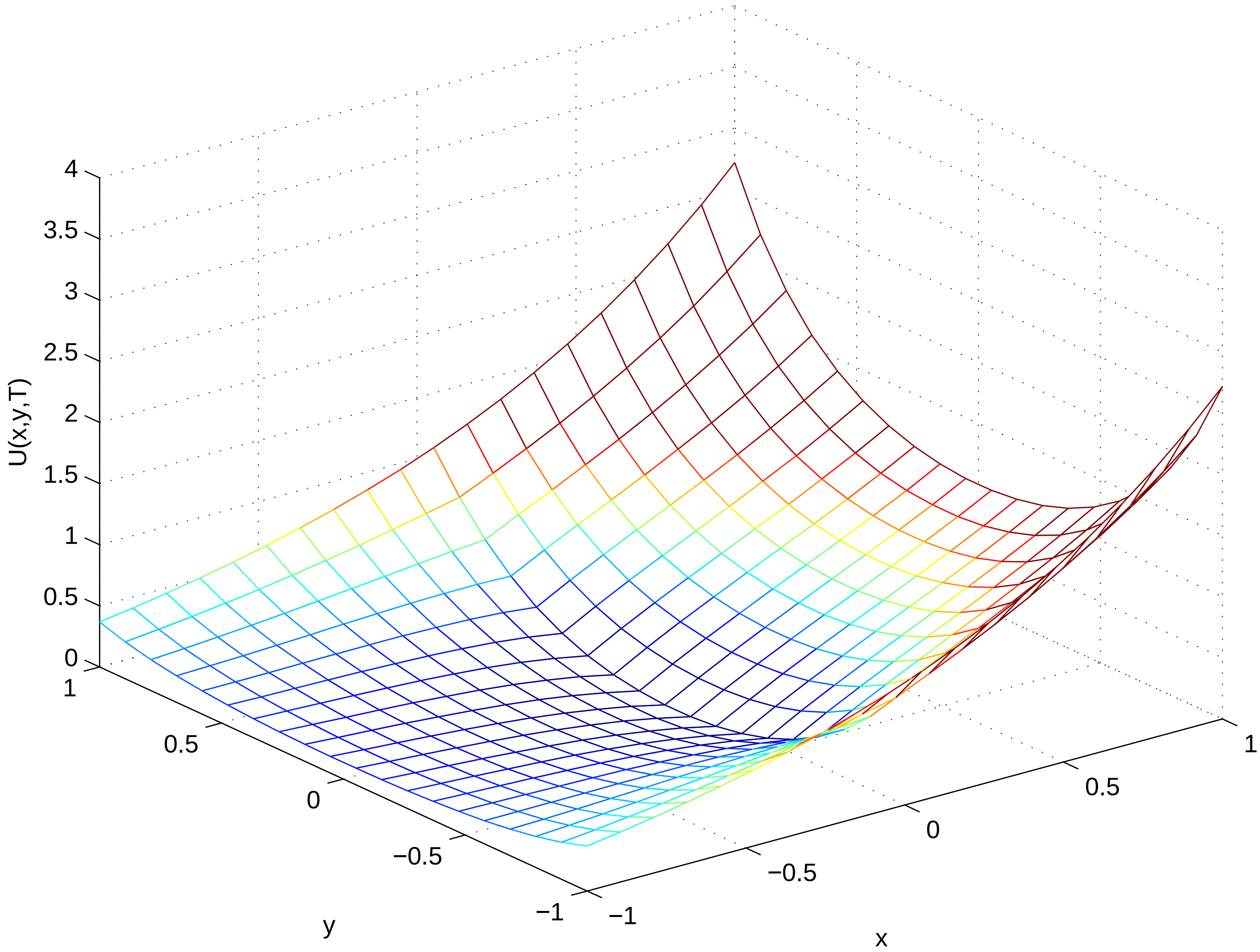

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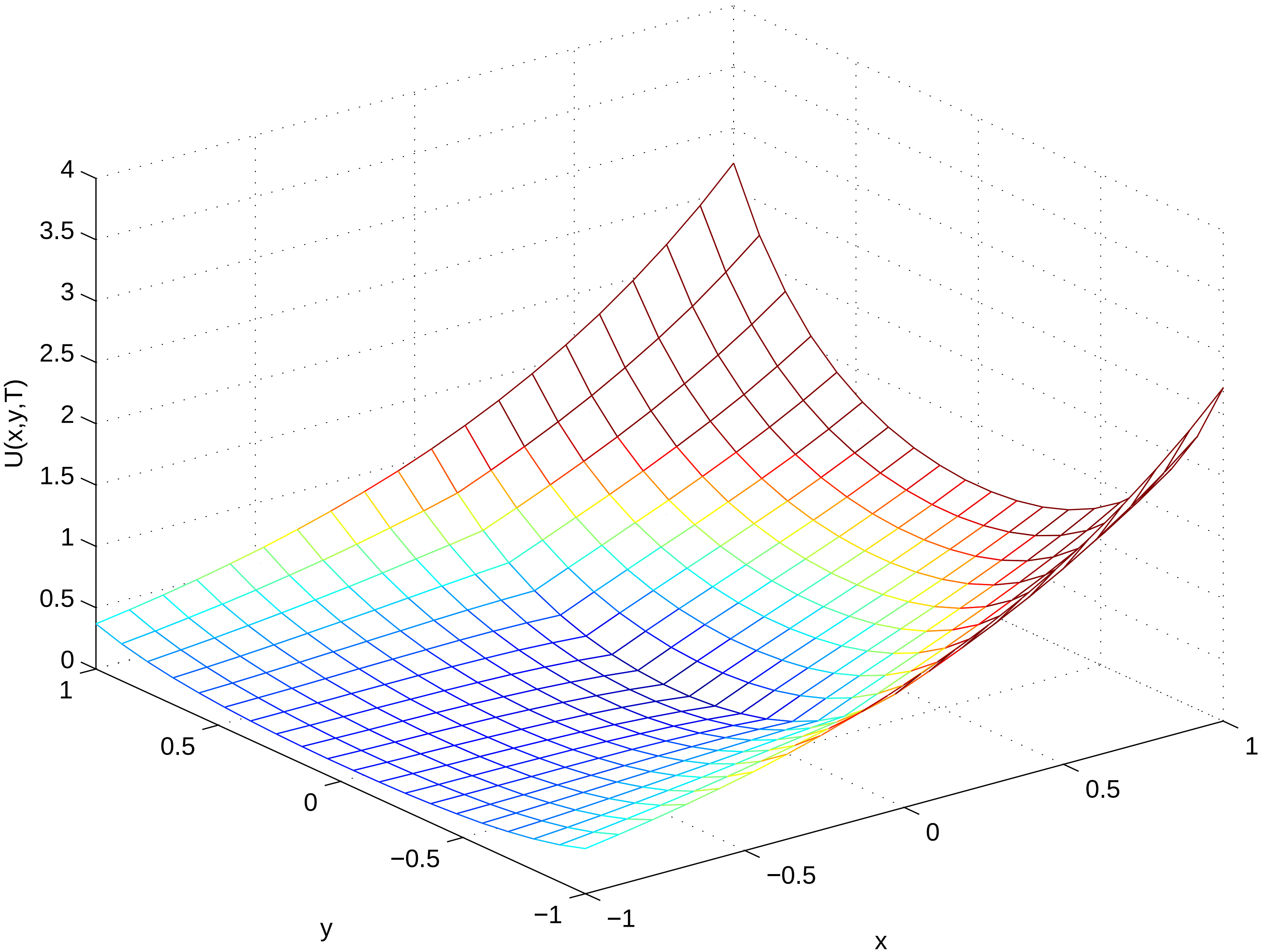
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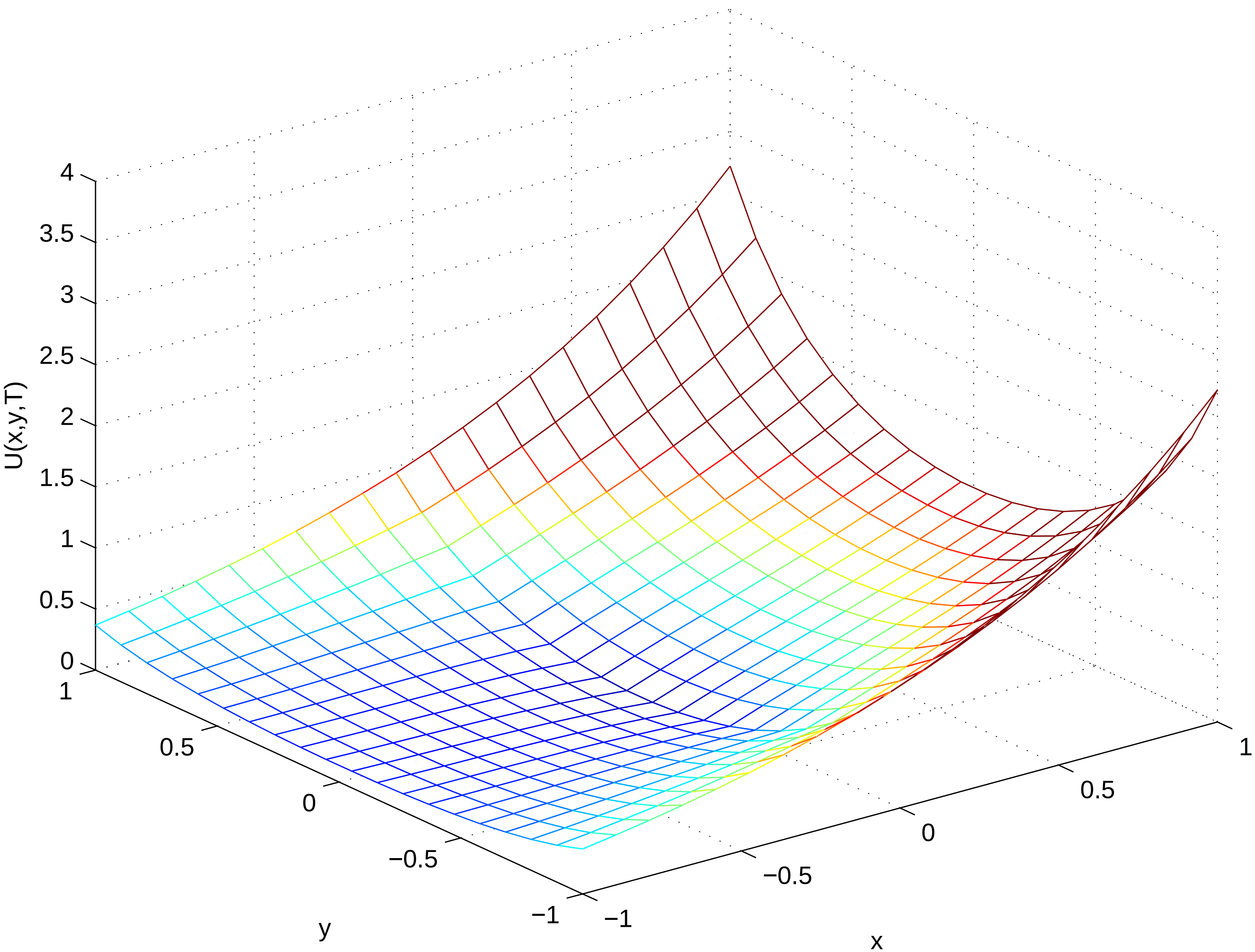
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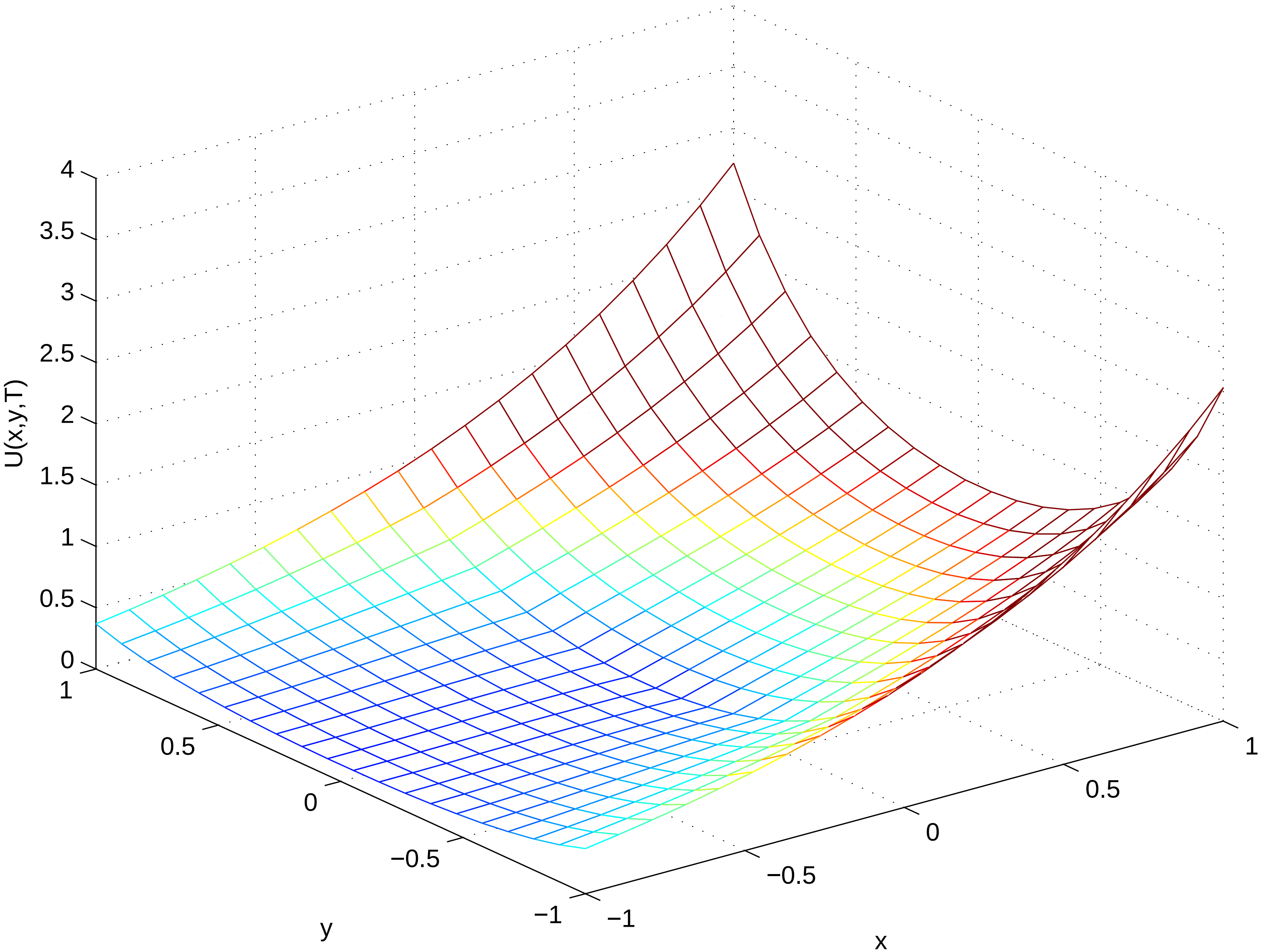

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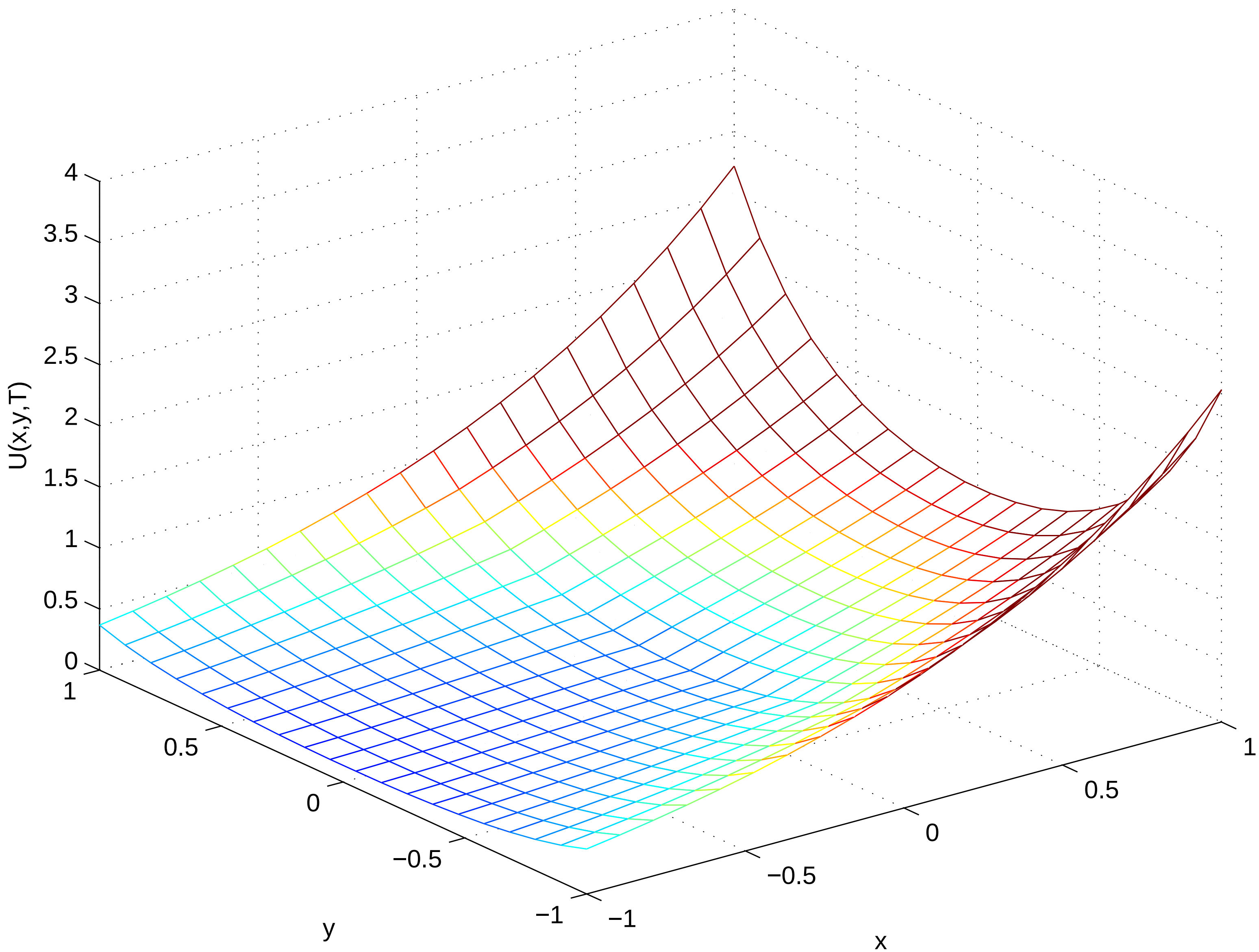
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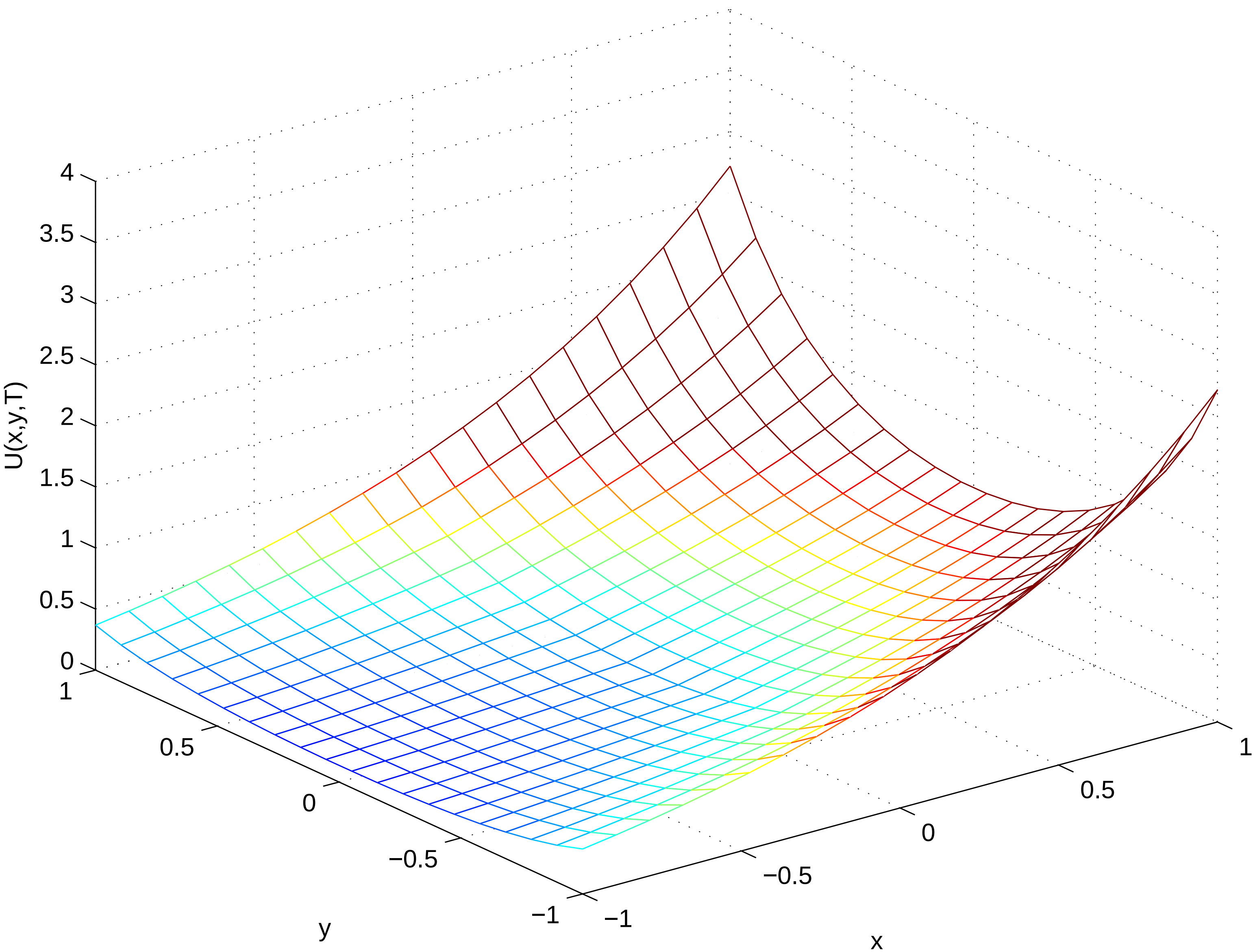
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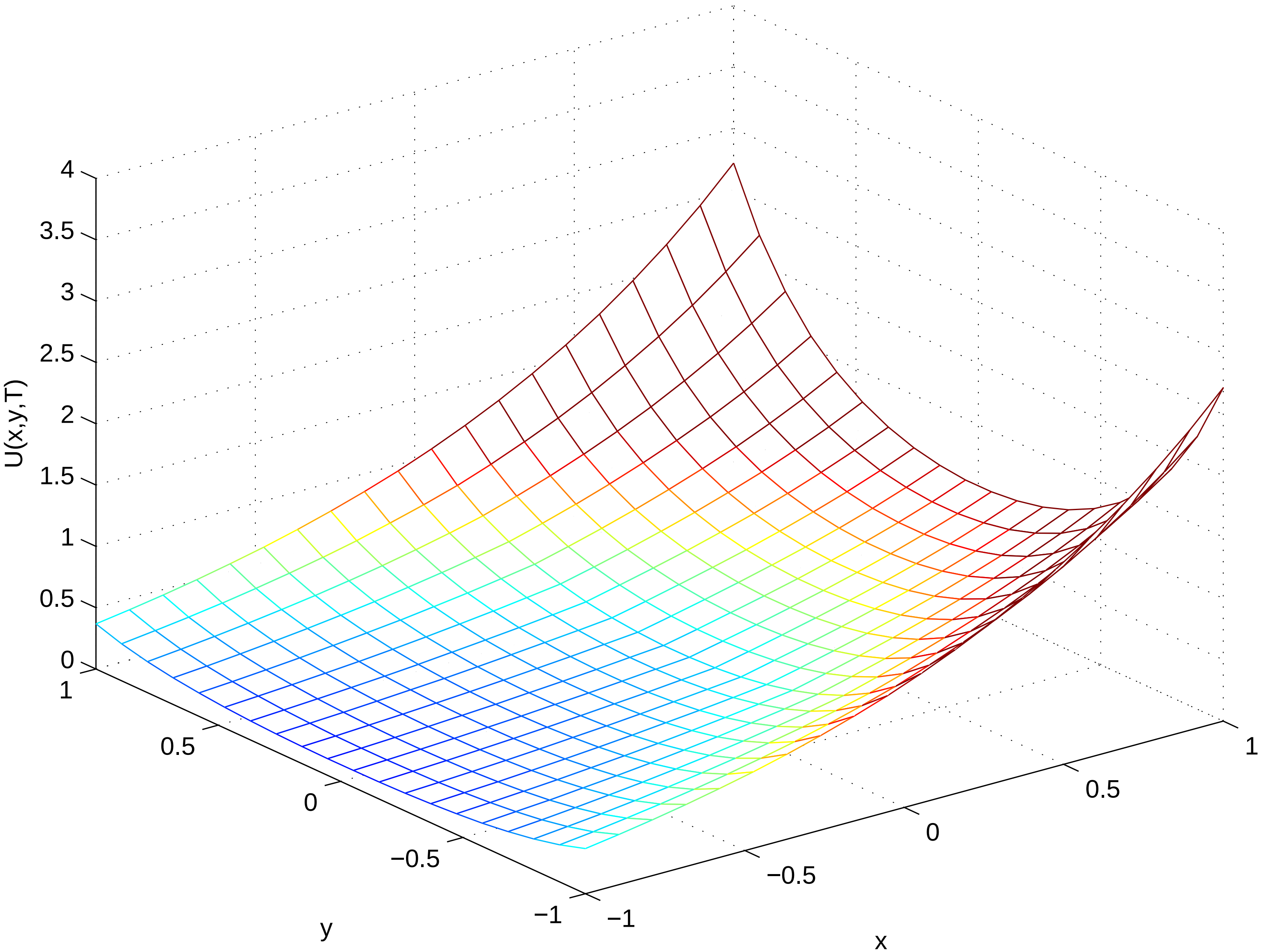
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0
0.5
1
x

Second Example : Bernoulli-Equation

We deal with the non linear Bernoulli-Equation:

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \lambda_1 u(t) + \lambda_2 u^n(t) \\ u(0) &= 1 \end{aligned}$$

with solution

$$u(t) = \left[(1 + \frac{\lambda_2}{\lambda_1}) \exp(\lambda_1 t(1-n)) - \frac{\lambda_2}{\lambda_1}) \right]^{-\frac{1}{1-n}}$$

We choose $n = 2$, $\lambda_1 = -1$, $\lambda_2 = -100$ and $h = 10^{-2}$

Iterative Steps	Number of splitting-partitions	error
2	1	7.3724e-001
2	2	2.7910e-002
2	5	2.1306e-003
10	1	1.0578e-001
10	2	3.9777e-004
20	1	1.2081e-004
20	2	3.9782e-004

Table 2: Numerical results for the Bernoulli-Equation with the Iterative Operator Splitting method and BDF3.

Future Work

1. Theory for the Stability of the iterative splitting methods.
2. Commutative, non-commutative theory : How to decouple
3. Dense coupling via full iterative coupling
4. Numerical examples