Optimal and optimized domain decomposition methods for three-dimensional partial differential equations

L. Laayouni

Department of Mathematics and Statistics
McGill University
Montreal,Canada

Outline

- Brief introduction
- Problem setting
- Classical Schwarz method
- Optimal Schwarz methods
- Optimized Schwarz methods
- Numerical experiments

Brief introduction

- Classical Schwarz methods
  - Slow convergence in particular for lower frequencies
  - No convergence if subdomains don′t overlap
- Optimal Schwarz methods
  - Convergence in finite number of iterations
  - Converge for overlapping and non-overlapping subdomains
  - Involve non-local transmission operators
  - Expensive to implement
- Optimized Schwarz methods
  - Local transmission conditions
  - Converge for overlapping and non-overlapping subdomains

Problem setting

We consider the model problem

$$\mathcal{L}u := (\eta - \Delta)u = f, \qquad \text{in} \quad \Omega = \mathbb{R}^3$$

We require the solution to decay at infinity. We decompose $\Omega = \Omega_1 \cup \Omega_2$, in two subdomains Ω_1 and Ω_2, where $\Omega_1 = (-\infty, L) \times \mathbb{R}^2$ and $\Omega_2 = (0, \infty) \times \mathbb{R}^2$. The Jacobi Schwarz method:

$$\begin{aligned} \mathcal{L}u_1^n &= f \quad \text{in} \quad \Omega_1 \\ u_1^n(L,y,z) &= u_2^{n-1}(L,y,z) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{L}u_2^n &= f \quad \text{in} \quad \Omega_2 \\ u_2^n(0,y,z) &= u_1^{n-1}(0,y,z) \end{aligned}$$

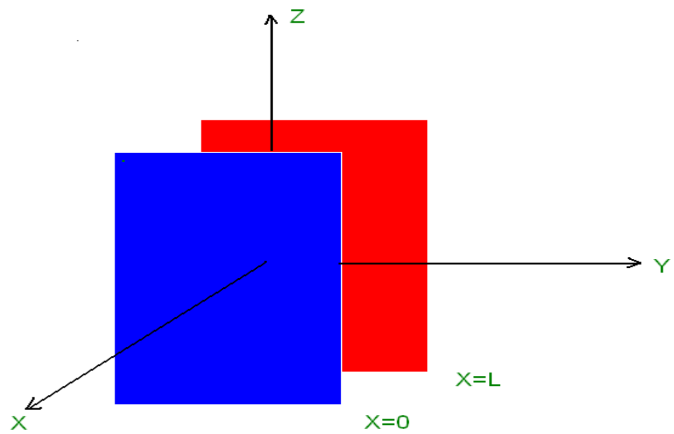


Figure 1: Decomposition of Ω in two overlapping subdomains.

Classical Schwarz

Setting $f = 0$ and taking a Fourier transform in y and z directions, we obtain

$$\begin{array}{lll} (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_1^n = 0 & x < L,\ k \in \mathbb{R},\ m \in \mathbb{R} & \hat{u}_1^n(L,k,m) = \hat{u}_2^{n-1}(L,k,m), \\ (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_2^n = 0 & x > 0,\ k \in \mathbb{R},\ m \in \mathbb{R} & \hat{u}_2^n(0,k,m) = \hat{u}_1^{n-1}(0,k,m), \end{array}$$

The solutions have the form

$$\hat{u}_j^n(x,k,m) = A_j(k,m)e^{\lambda_1(k,m)x} + B_j(k,m)e^{\lambda_2(k,m)x}, \qquad j = 1,2,$$

where $\lambda_1(k,m) = \sqrt{k^2 + m^2 + \eta}$ and $\lambda_2(k,m) = -\sqrt{k^2 + m^2 + \eta}$. Using the transmission conditions, we obtain

$$\hat{u}_1^n(x,k,m) = \hat{u}_2^{n-1}(L,k,m)e^{\lambda_1(k,m)(x-L)}, \quad \hat{u}_2^n(x,k,m) = \hat{u}_1^{n-1}(0,k,m)e^{\lambda_2(k,m)x}.$$

Evaluating the second equation at $x = L$ for iteration index $n-1$ and inserting it into the first equation, we get over a double step the following relation

$$\hat{u}_1^n(x,k,m) = e^{-\sqrt{k^2+m^2+\eta}L} e^{\sqrt{k^2+m^2+\eta}(x-L)} \hat{u}_1^{n-2}(0,k,m)$$

Evaluating this equation at $x = 0$, we get

$$\hat{u}_1^n(0,k,m) = e^{-2\sqrt{k^2+m^2+\eta}L} \hat{u}_1^{n-2}(0,k,m)$$

.

Classical Schwarz (continue)

The convergence factor $\rho(\eta, k, m, L)$ of the classical Schwarz algorithm is given by

$$\rho_{cla} = \rho_{cla}(\eta, k, m, L) := e^{-2\sqrt{k^2+m^2+\eta}L} \leq 1, \; \forall k \in \mathbb{R}, \quad \forall m \in \mathbb{R}$$

By induction, we obtain

$$\hat{u}_1^{2n}(0, k, m) = \rho_{cla}^n \hat{u}_1^0(0, k, m), \; \hat{u}_2^{2n}(L, k, m) = \rho_{cla}^n \hat{u}_2^0(L, k, m).$$

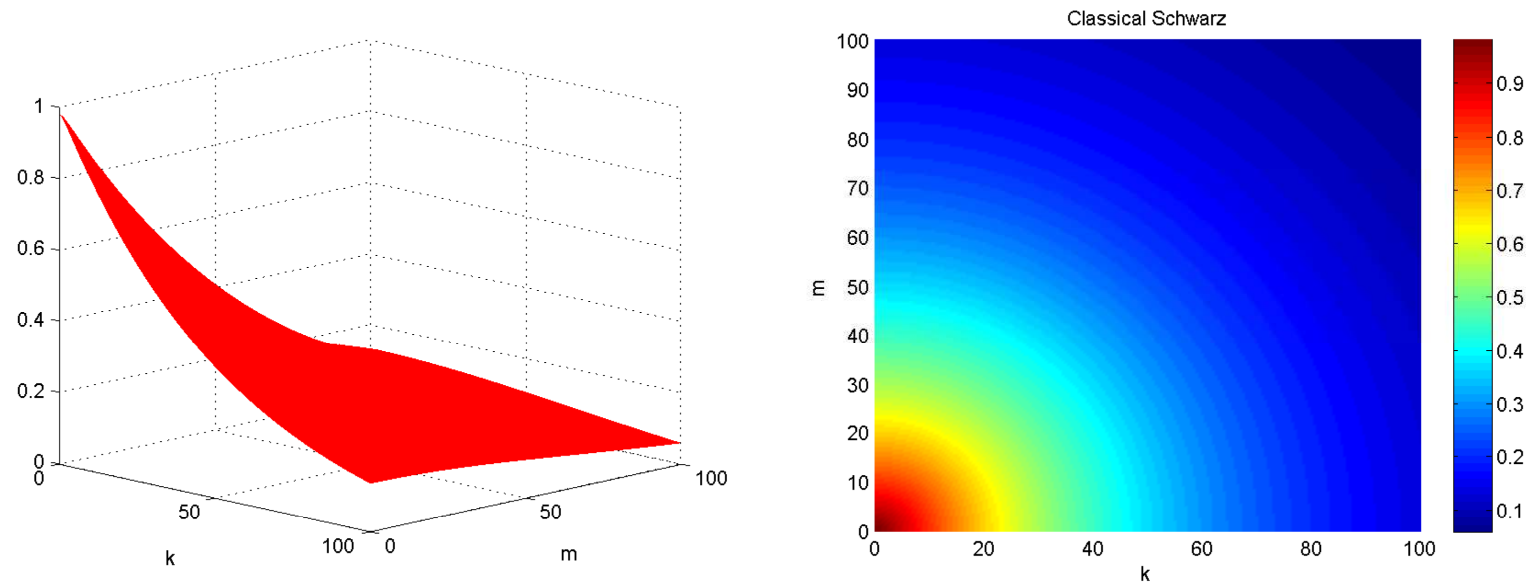


Figure 2: Dependence of the convergence factor ρ_{cla} on the frequencies k and m for a fixed size of the overlap $L = \frac{1}{100}$ and problem parameter $\eta = 1$.

Optimal schwarz methods

Introducing the following modified algorithm with new transmission conditions

$$
\begin{array}{ll}
\mathcal{L}(u_1^n) = f, \text{ in } \Omega_1 & (\boxed{S_1} + \partial_x)(u_1^n)(L,.,.) = (\boxed{S_1} + \partial_x)(u_2^{n-1})(L,.,.) \\
\mathcal{L}(u_2^n) = f, \text{ in } \Omega_2 & (\boxed{S_2} + \partial_x)(u_2^n)(0,.,.) = (\boxed{S_2} + \partial_x)(u_1^{n-1})(0,.,.)
\end{array}
$$

where S_j, $j = 1, 2$, are operators along the interface that depend on y and z. Taking a Fourier transform, we obtain

$$
\begin{array}{rcl}
(\eta + k^2 + m^2 - \partial_{xx})\hat{u}_1^n & = & 0, \quad x < L, \; k \in \mathbb{R}, \; m \in \mathbb{R} \\
(\boxed{\sigma_1(k,m)} + \partial_x)(\hat{u}_1^n)(L,k,m) & = & (\boxed{\sigma_1(k,m)} + \partial_x)(\hat{u}_2^{n-1})(L,k,m),
\end{array}
$$

and

$$
\begin{array}{rcl}
(\eta + k^2 + m^2 - \partial_{xx})\hat{u}_2^n & = & 0, \quad x > 0, \; k \in \mathbb{R}, \; m \in \mathbb{R} \\
(\boxed{\sigma_2(k,m)} + \partial_x)(\hat{u}_2^n)(0,k,m) & = & (\boxed{\sigma_2(k,m)} + \partial_x)(\hat{u}_1^{n-1})(0,k,m),
\end{array}
$$

where $\sigma_j(k,m)$, $j = 1, 2$, denotes the symbol of the operator $S_j(y,z)$, $j = 1, 2$, respectively.

Optimal Schwarz methods (continue)

Taking again Fourier transform and using the transmission conditions and the fact that

$$\frac{\partial \hat{u}_1^n}{\partial x} = \sqrt{\eta + k^2 + m^2}\,\hat{u}_1^n, \qquad \frac{\partial \hat{u}_2^n}{\partial x} = -\sqrt{\eta + k^2 + m^2}\,\hat{u}_2^n,$$

we find that

$$\hat{u}_1^n(0,k,m) = \underbrace{\frac{\sigma_1(k,m) - \sqrt{\eta + k^2 + m^2}}{\sigma_1(k,m) + \sqrt{\eta + k^2 + m^2}}\,\frac{\sigma_2(k,m) + \sqrt{\eta + k^2 + m^2}}{\sigma_2(k,m) - \sqrt{\eta + k^2 + m^2}}\,\rho_{cla}}\hat{u}_1^{n-2}(0,k,m)$$

Defining the new convergence factor ρ_{opt} by

$$\rho_{opt} = \rho(\eta,k,m,L,\sigma_1,\sigma_2) := \frac{\sigma_1(k,m) - \sqrt{\eta + k^2 + m^2}}{\sigma_1(k,m) + \sqrt{\eta + k^2 + m^2}}\,\frac{\sigma_2(k,m) + \sqrt{\eta + k^2 + m^2}}{\sigma_2(k,m) - \sqrt{\eta + k^2 + m^2}}\,\rho_{cla}$$

If we choose

$$\sigma_1(m) := \sqrt{\eta + k^2 + m^2}, \quad \text{and} \quad \sigma_2(m) := -\sqrt{\eta + k^2 + m^2},$$

then, $\rho_{opt} \equiv 0$, and the algorithm converges in two iterations, independently of the initial guess, the overlap size L and the problem parameter η.

Optimized Schwarz methods

Now, we approximate the symbols σ_j by polynomials in ik and im as follow

$$\sigma_1^{app}(k,m) = p_1 + q_1(k^2+m^2), \qquad \sigma_2^{app}(k,m) = -p_2 - q_2(k^2+m^2). \qquad (1)$$

The convergence factor of the optimized Schwarz methods becomes

$$\rho = \rho(\eta,k,m,L,p_1,q_1,p_2,q_2) = \frac{\sqrt{\eta+k^2+m^2} - p_1 - q_1(k^2+m^2)}{\sqrt{\eta+k^2+m^2} + p_1 + q_1(k^2+m^2)} \times \frac{\sqrt{\eta+k^2+m^2} - p_2 - q_2(k^2+m^2)}{\sqrt{\eta+k^2+m^2} + p_2 + q_2(k^2+m^2)}\, \rho_{cla}$$

Theorem

The optimized Schwarz method with transmission conditions defined by the symbols (1) converges for $p_j > 0$, $q_j \geq 0$, $j = 1, 2$, faster than the classical Schwarz method, $|\rho(k,m)| < |\rho_{cla}(k,m)|$ for all k and m.

Optimized Schwarz methods (continue)

Low frequency approximations

Expanding the symbols $\sigma_j(k,m)$, $j=1,2$, we find

$$\sigma_1(k,m) = \sqrt{\eta} + \frac{1}{2\sqrt{\eta}}k^2 + \frac{1}{2\sqrt{\eta}}m^2 + \mathcal{O}_1(k^4,m^4),$$

and

$$\sigma_2(k,m) = -\sqrt{\eta} - \frac{1}{2\sqrt{\eta}}k^2 - \frac{1}{2\sqrt{\eta}}m^2 + \mathcal{O}_2(k^4,m^4).$$

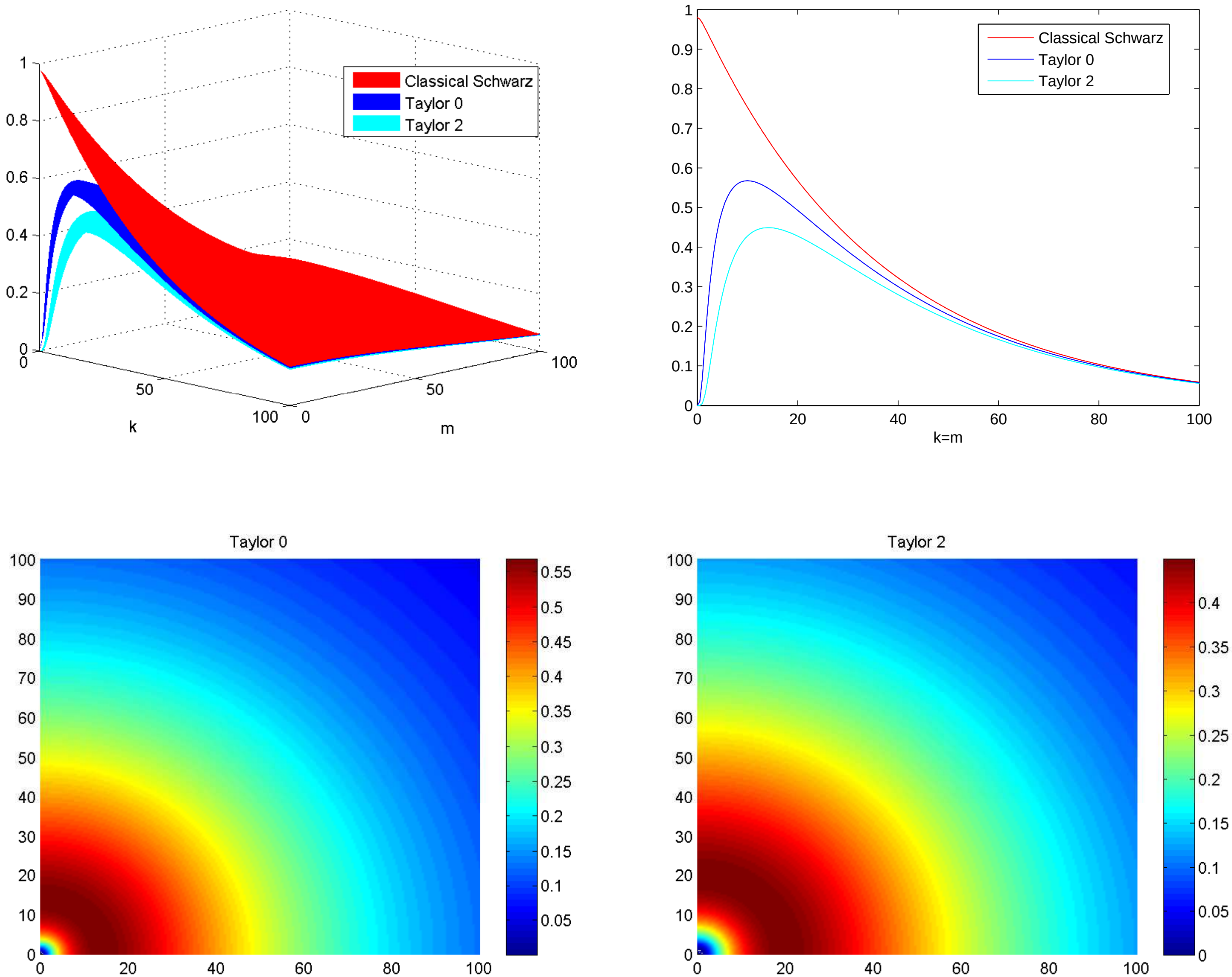
Zeroth order Taylor approximation ($p_1 = p_2 = \sqrt{\eta}$; $q_1 = q_2 = 0$)

$$\rho_{T0}(\eta,k,m,L) = \left(\frac{\sqrt{\eta+k^2+m^2}-\sqrt{\eta}}{\sqrt{\eta+k^2+m^2}+\sqrt{\eta}}\right)^2 \boxed{\rho_{cla}}.$$

Second order Taylor approximation ($p_1 = p_2 = \sqrt{\eta}$; $q_1 = q_2 = \frac{1}{2\sqrt{\eta}}$)

$$\rho_{T2}(\eta,k,m,L) = \left(\frac{\sqrt{\eta+k^2+m^2}-\sqrt{\eta}-\frac{1}{2\sqrt{\eta}}k^2-\frac{1}{2\sqrt{\eta}}m^2}{\sqrt{\eta+k^2+m^2}+\sqrt{\eta}+\frac{1}{2\sqrt{\eta}}k^2+\frac{1}{2\sqrt{\eta}}m^2}\right)^2 \boxed{\rho_{cla}}$$

Optimized Schwarz methods (continue)



Optimized Schwarz methods (continue)

Theorem The optimized Schwarz methods with Taylor transmission conditions and overlap $L = h$ have an asymptotically superior performance than the classical Schwarz method with the same overlap. As h goes to zero, we have

$$\max_{|k|\le\frac{\pi}{h},|m|\le\frac{\pi}{h}} \rho_{cla}(\eta,k,m,h)| = 1 - 2\sqrt{\eta}\, h + \mathcal{O}(h^2)$$

$$\max_{|k|\le\frac{\pi}{h},|m|\le\frac{\pi}{h}} |\rho_{T0}(\eta,k,m,h)| = 1 - 4\sqrt{2}\eta^{1/4}\sqrt{h} + \mathcal{O}(h)$$

$$\max_{|k|\le\frac{\pi}{h},|m|\le\frac{\pi}{h}} |\rho_{T2}(\eta,k,m,h)| = 1 - 8\eta^{1/4}\sqrt{h} + \mathcal{O}(h).$$

Without overlap, the optimized Schwarz methods with Taylor transmission conditions are asymptotically comparable to the classical Schwarz method with overlap $L = h$. As h goes to zero, we have

$$\max_{|k|\le\frac{\pi}{h},|m|\le\frac{\pi}{h}} |\rho_{T0}(\eta,k,m,0)| = 1 - 4\frac{\sqrt{\eta}}{\pi}\, h + \mathcal{O}(h^2)$$

$$\max_{|k|\le\frac{\pi}{h},|m|\le\frac{\pi}{h}} |\rho_{T2}(\eta,k,m,0)| = 1 - 8\frac{\sqrt{\eta}}{\pi}\, h + \mathcal{O}(h^2).$$

Optimized Schwarz methods (continue)

Uniformly optimized approximations

For the zeroth order optimized Schwarz method we have the min-max problem

$$\min_{p_j\ge 0}\left(\max_{k,m}\left|\frac{\sqrt{k^2+m^2+\eta}-p_1}{\sqrt{k^2+m^2+\eta}+p_1}\right|\left|\frac{\sqrt{k^2+m^2+\eta}-p_2}{\sqrt{k^2+m^2+\eta}+p_2}\right|e^{-2\sqrt{k^2+m^2+\eta}L}\right).$$

For the second order optimized Schwarz method we have the min-max problem

$$\min_{p_j,q_j,r_j\ge 0}\left(\max_{k,m}\left|\frac{\sqrt{k^2+m^2+\eta}-p_1-q_1k^2-r_1m^2}{\sqrt{k^2+m^2+\eta}+p_1+q_1k^2+r_1m^2}\right|\right.$$
$$\left.\left|\frac{\sqrt{k^2+m^2+\eta}-p_2-q_2k^2-r_2m^2}{\sqrt{k^2+m^2+\eta}+p_2+q_2k^2+r_2m^2}\right|e^{-2\sqrt{k^2+m^2+\eta}L}\right)$$

Optimized Schwarz methods (continue)

Zeroth order optimized tarnsmission conditions

Setting $p_1 = p_2 = p$, the convergence factor becomes

$$\rho_{OO0}(k,m,L,\eta,p) := \left(\frac{\sqrt{k^2+m^2+\eta}-p}{\sqrt{k^2+m^2+\eta}+p}\right)^2 e^{-2\sqrt{k^2+m^2+\eta}L}.$$

and the corresponding min-max problem is given by

$$\min_{p\geq 0}(\max_{k,m}\rho_{OO0}(k,m,L,\eta,p)) = \min_{p\geq 0}\left(\max_{k,m}\left(\frac{\sqrt{k^2+m^2+\eta}-p}{\sqrt{k^2+m^2+\eta}+p}\right)^2 e^{-2\sqrt{k^2+m^2+\eta}L}\right)$$

Let $f_{\min}$ and $f_{\max}$ be the minimal and the maximal frequency, respectively.

Optimized Schwarz methods (continue)

Theorem (Optimal Robin parameter). For $L > 0$ and $f_{max} = \infty$, the unique solution p^* of the min-max problem (2) is given by

$$\rho_{OO0}(\bar{k}_{\min}, \bar{m}_{\min}, L, \eta, p^*) = \rho_{OO0}(\bar{k}(p^*), \bar{m}(p^*), L, \eta, p^*),$$

where

$$\bar{k}^2(p, L, \eta) + \bar{m}^2(p, L, \eta) = \bar{f}^2(p, L, \eta) \quad \text{and} \quad \bar{f}(p, L, \eta) = \frac{\sqrt{L(2p + L(p^2 - \eta))}}{L},$$

and

$$\bar{k}^2_{\min} + \bar{m}^2_{\min} = f^2_{\min}.$$

For $L = 0$ and $f_{\max}$, finite, the optimal parameter p^* is given by the closed form

$$p^* = ((f^2_{\min} + \eta)(f^2_{\max} + \eta))^{1/4},$$

Optimized Schwarz methods (continue)

Proof The partial derivative of $\rho OO0$ w.r.t. p is

$$\frac{\partial \rho_{OO0}}{\partial p} = 4\frac{(p-\sqrt{k^2+m^2+\eta})\sqrt{k^2+m^2+\eta}e^{-2\sqrt{k^2+m^2+\eta}L}}{(\sqrt{k^2+m^2+\eta}+p)^3}$$

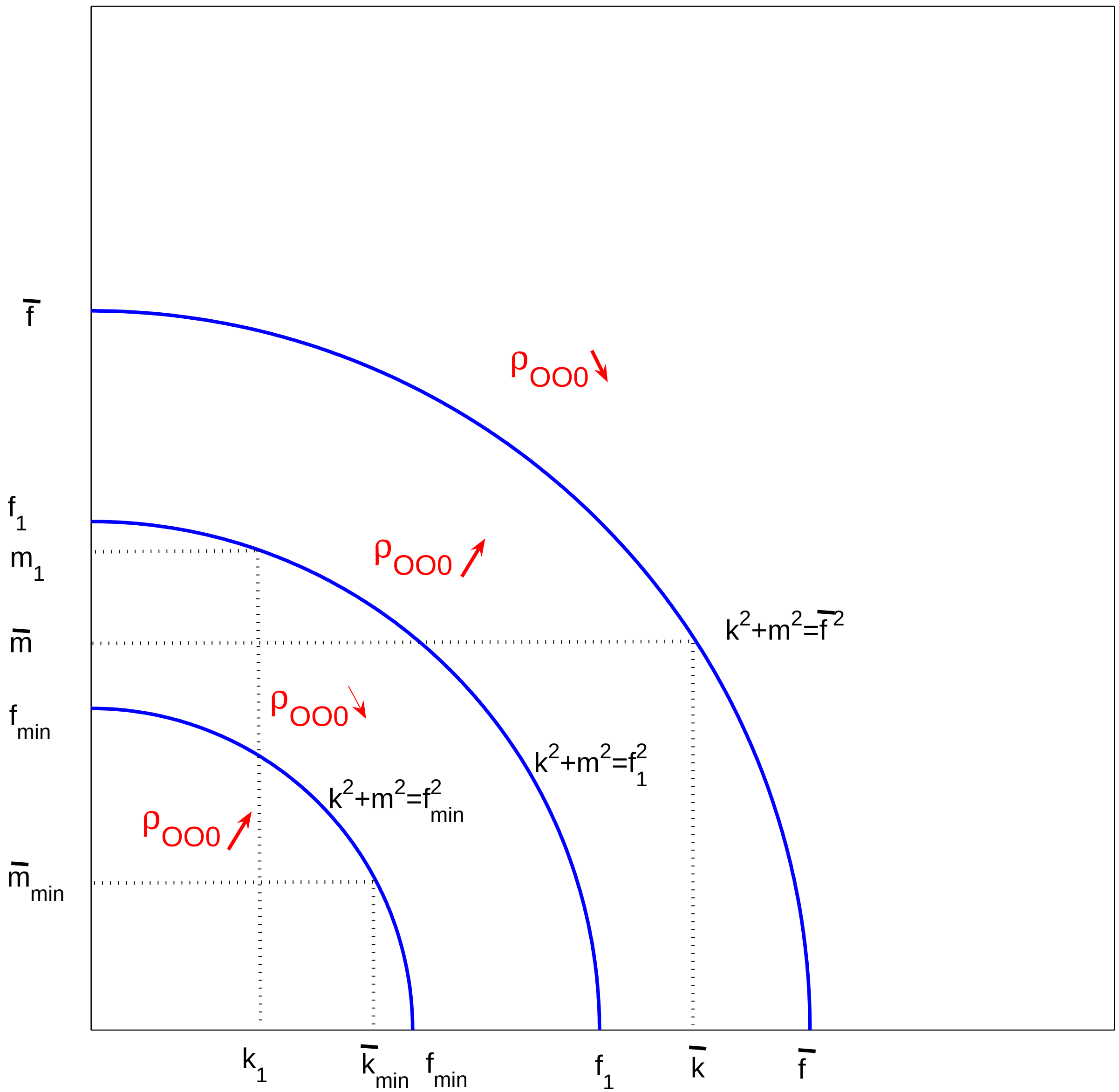
If $p < \sqrt{f_{\min}^2+\eta}$, increasing p decreases ρ_{OO0} for all k and m such that $k^2+m^2 > f_{\min}^2$. We can restrict the range for p to $p \geq \sqrt{f_{\min}^2+\eta}$. Hence ρ_{OO0} has roots at $k=k_1$ and $m=m_1$ such that $k_1^2+m_1^2 = p^2-\eta$, then we define the function

$$R(k,m,L,\eta,k_1,m_1) := \frac{\sqrt{k^2+m^2+\eta}-\sqrt{k_1^2+m_1^2+\eta}}{\sqrt{k^2+m^2+\eta}+\sqrt{k_1^2+m_1^2+\eta}}e^{-\sqrt{k^2+m^2+\eta}L}$$

The proof is based on the fact that the min-max problem is equivalent to the optimization problem

$$\min_{\substack{k_1,m_1\\ k_1^2+m_1^2\geq f_{\min}^2}}\left(\max_{\substack{k,m\\ f_{\min}^2\leq k^2+m^2\leq f_{\max}^2}} |R(k,m,L,\eta,k_1,m_1)|\right),$$

Optimized Schwarz methods (continue)



Optimized Schwarz methods (continue)

Theorem (Robin asymptotic performance)
The asymptotic performance of the Schwarz method with optimized Robin transmission conditions and overlap $L = h$, as h goes to zero, is given by

$$\max_{\substack{k,m \\ f_{\min} \le \sqrt{k^2+m^2} \le \frac{\pi}{h}}} |\rho_{OO0}(\eta, k, m, h, \eta, p^*)| = 1 - 4.2^{1/6}(f_{\min}^2 + \eta)^{1/6} \boxed{h^{1/3}} + \mathcal{O}(h^{2/3}).$$

The asymptotic performance without overlap, $L = 0$, is given by

$$\max_{\substack{k,m \\ f_{\min} \le \sqrt{k^2+m^2} \le \frac{\pi}{h}}} |\rho_{OO0}(\eta, k, m, 0, \eta, p^*)| = 1 - 4\frac{(f_{\min}^2 + \eta)^{1/4}}{\sqrt{\pi}} \boxed{\sqrt{h}} + \mathcal{O}(h).$$

Key of the proof, for $L > 0$, we make the ansatz $p^* = Ch^\alpha$ for $\alpha < 0$, we obtain

$$p^* = \frac{4(f_{\min}^2 + \eta)^{1/3}}{2} h^{-1/3}$$

Optimized Schwarz methods (continue)

Second Order Optimized Transmission conditions

Setting $p_1 = p_2 = p$, $q_1 = q_2 = q$, and $r_1 = r_2 = q$, the convergence factor becomes

$$\rho_{OO2}(k,m,L,\eta,p,q) = \left(\frac{\sqrt{k^2+m^2+\eta} - p - q(k^2+m^2)}{\sqrt{k^2+m^2+\eta} + p + q(k^2+m^2)}\right)^2 e^{-2\sqrt{k^2+m^2+\eta}L},$$

To determine the optimal parameters p and q for the Optimized Schwarz method of order 2, we need to solve the min-max problem

$$\min_{p,q\geq 0}\left(\max_{k,m}|\rho_{OO2}(k,m,L,\eta,p,q)|\right) =$$

$$\min_{p,q\geq 0}\left(\max_{k,m}\left(\frac{\sqrt{k^2+m^2+\eta} - p - q(k^2+m^2)}{\sqrt{k^2+m^2+\eta} + p + q(k^2+m^2)}\right)^2 e^{-2\sqrt{k^2+m^2+\eta}L}\right) \quad (3)$$

Optimized Schwarz methods (continue)

Theorem (Optimal Second Order Parameters) For $L > 0$ and $f_{\max} = \infty$ the solution p^*, q^* of the min-max problem (3) is given by the unique root of the system of equations

$$\rho_{OO2}(\bar{k}_{\min}, \bar{m}_{\min}, L, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_1, \bar{m}_1, L, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_2, \bar{m}_2, L, \eta, p^*, q^*),$$

where $\bar{k}_{\min}^2 + \bar{m}_{\min}^2 = f_{\min}^2$, and

$$\begin{aligned}
\bar{k}_1^2 + \bar{m}_1^2 &= \frac{L+2q-2Lpq-\sqrt{L^2+4Lq-4L^2pq+4q^2-16Lpq^2+16Lq^3\eta+4L^2q^2\eta}}{2Lq^2} \\
\bar{k}_2^2 + \bar{m}_2^2 &= \frac{L+2q-2Lpq+\sqrt{L^2+4Lq-4L^2pq+4q^2-16Lpq^2+16Lq^3\eta+4L^2q^2\eta}}{2Lq^2}
\end{aligned}$$

For $L = 0$ and $f_{\max}$ finite, the optimal parameters p^* and q^* are given by

$$\begin{aligned}
p^* &= \frac{\sqrt{f_{\min}^2+\eta}\sqrt{f_{\max}^2+\eta}+\eta}{\sqrt{2}(\sqrt{f_{\max}^2+\eta}+\sqrt{f_{\min}^2+\eta})^{1/2}(f_{\min}^2+\eta)^{1/8}(f_{\max}^2+\eta)^{1/8}} \\
q^* &= \frac{1}{\sqrt{2}(\sqrt{f_{\max}^2+\eta}+\sqrt{f_{\min}^2+\eta})^{1/2}(f_{\min}^2+\eta)^{1/8}(f_{\max}^2+\eta)^{1/8}}
\end{aligned} \qquad (4)$$

Optimized Schwarz methods (continue)

Theorem(Second order)
The asymptotic performance of the Schwarz method with optimized second order transmission conditions and overlap $L = h$, as h goes to zero, is given by

$$\max_{\substack{k,m \\ f_{\min}\le\sqrt{k^2+m^2}\le f_{\max}}} |\rho_{OO2}(\eta,k,m,h,\eta,p^*,q^*)| = 1 - 4.2^{3/5}(f_{\min}^2+\eta)^{1/10}\, h^{1/5} + \mathcal{O}(h^{2/5}$$

The asymptotic performance without overlap, $L = 0$, is given by

$$\max_{\substack{k,m \\ f_{\min}\le\sqrt{k^2+m^2}\le f_{\max}}} |\rho_{OO2}(\eta,k,m,0,\eta,p^*,q^*)| = 1 - 4\frac{\sqrt{2}(f_{\min}^2+\eta)^{1/8}}{\pi^{1/4}}\, h^{1/4} + \mathcal{O}(h^{1/2})$$

Proof The optimized parameters of the Schwarz method with second order and overlap $L = h$, are given by

$$p^* = 2^{-3/5}(f_{\min}^2+\eta)^{2/5}h^{-1/5}$$

$$q^* = (2(f_{\min}^2+\eta))^{-1/5}h^{3/5}$$

Optimized Schwarz methods (continue)

Two-sided Optimized Robin Transmission Condition

Theorem (Optimal two-sided Robin conditions) If there is overlap, $L > 0$, then the optimal two-sided Robin parameters are given by

$$p_1^* = \frac{1 - \sqrt{1 + 4\eta(q^*)^2 - 4p^*q^*}}{2q^*}, \qquad p_2^* = \frac{1 + \sqrt{1 + 4\eta(q^*)^2 - 4p^*q^*}}{2q^*}, \tag{5}$$

where p^* and q^* are solutions of (3) with L replaced by $2L$. If there is no overlap, $L = 0$, then the optimal two-sided Robin parameters are (5) where p^* and q^* are given by (4).

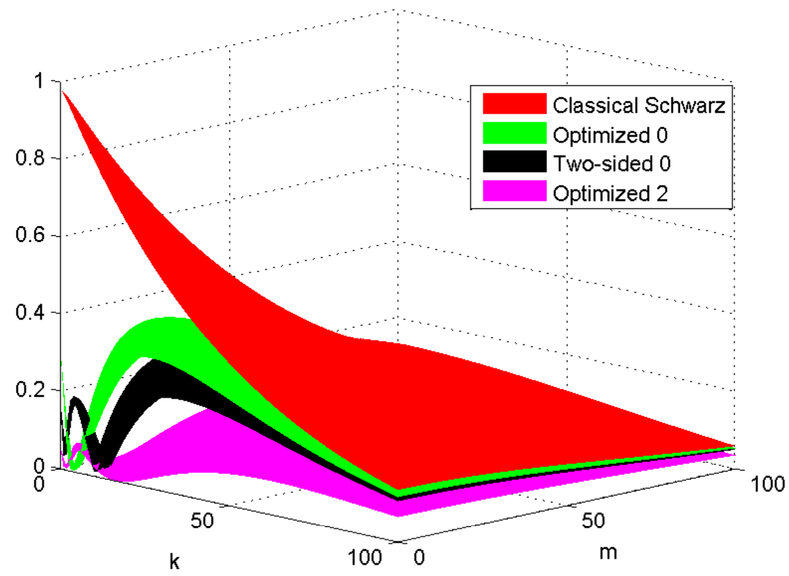
Corollary The asymptotic performance of the two-sided optimized Schwarz method with overlap $L = h$ is given by

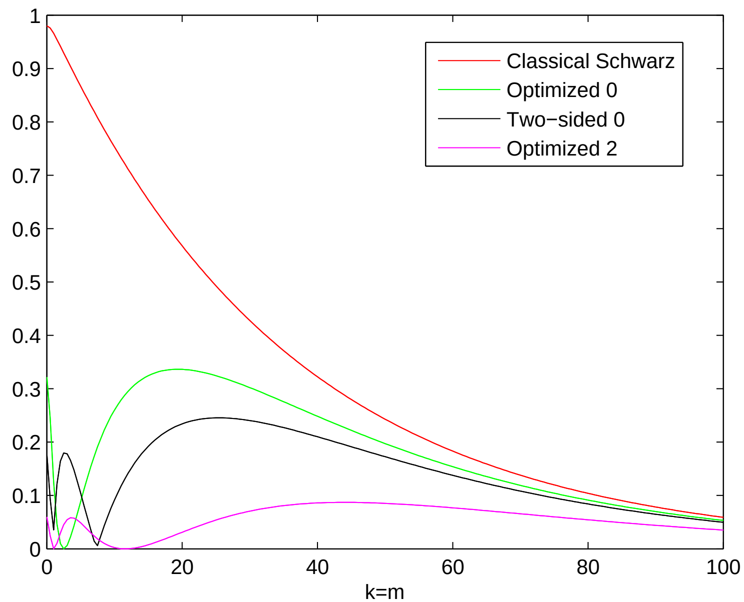
$$\max_{\substack{k,m \\ f_{\min} \leq \sqrt{k^2+m^2} \leq f_{\max}}} |\rho(\eta, k, m, h, \eta, p_1^*, p_2^*)| = 1 - 4.2^{3/5}(f_{\min}^2 + \eta)^{1/10} \boxed{h^{1/5}} + \mathcal{O}(h^{2/5}). \tag{6}$$

The asymptotic performance without overlap, $L = 0$, is given by

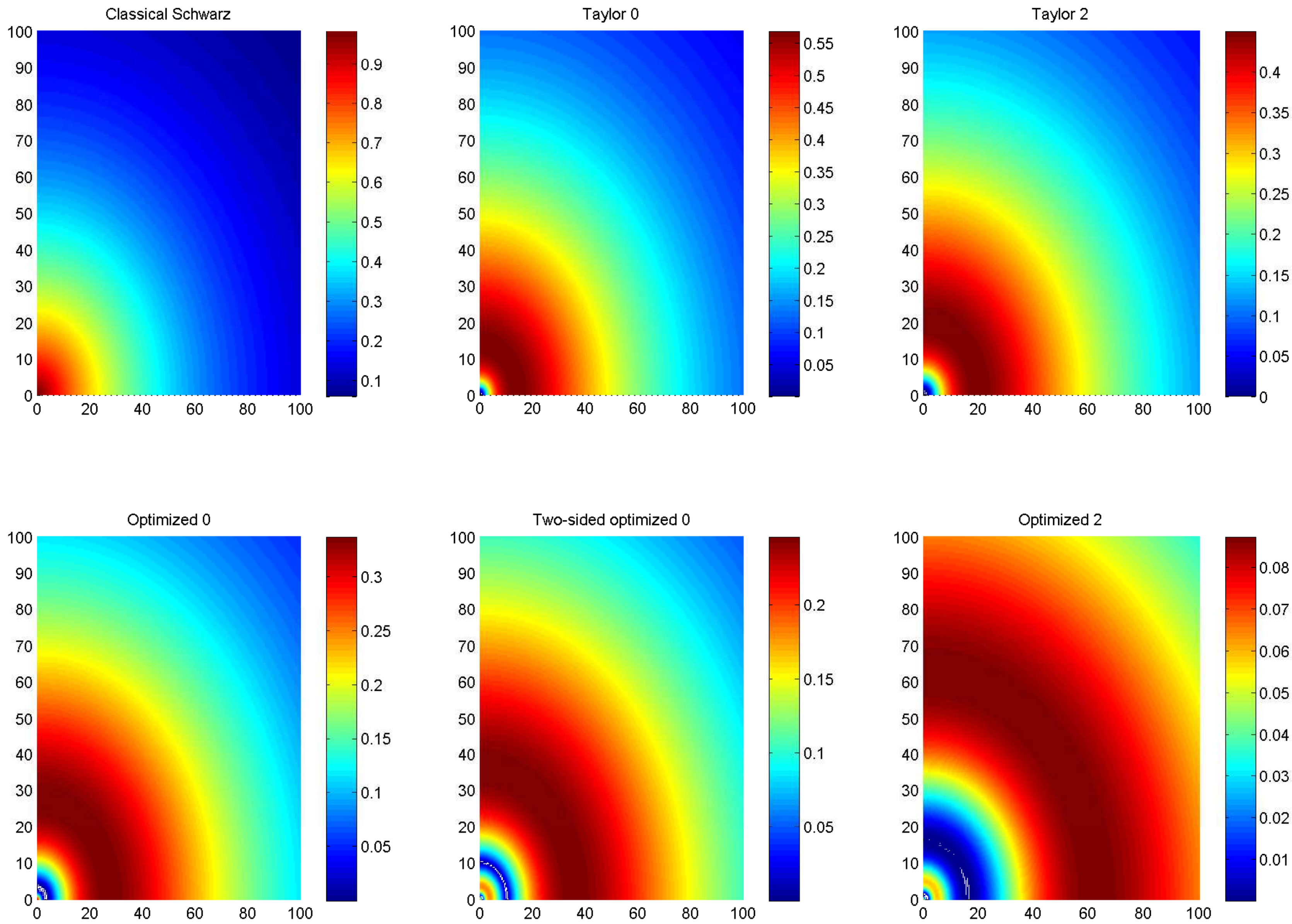
$$\max_{\substack{k,m \\ f_{\min} \leq \sqrt{k^2+m^2} \leq f_{\max}}} |\rho(\eta, k, m, 0, \eta, p_1^*, p_2^*)| = 1 - 4\frac{\sqrt{2}(f_{\min}^2 + \eta)^{1/8}}{\pi^{1/4}} \boxed{h^{1/4}} + \mathcal{O}(h^{1/2}). \tag{7}$$

Optimized Schwarz methods (continue)





Optimized Schwarz methods (continue)



Numerical experiments

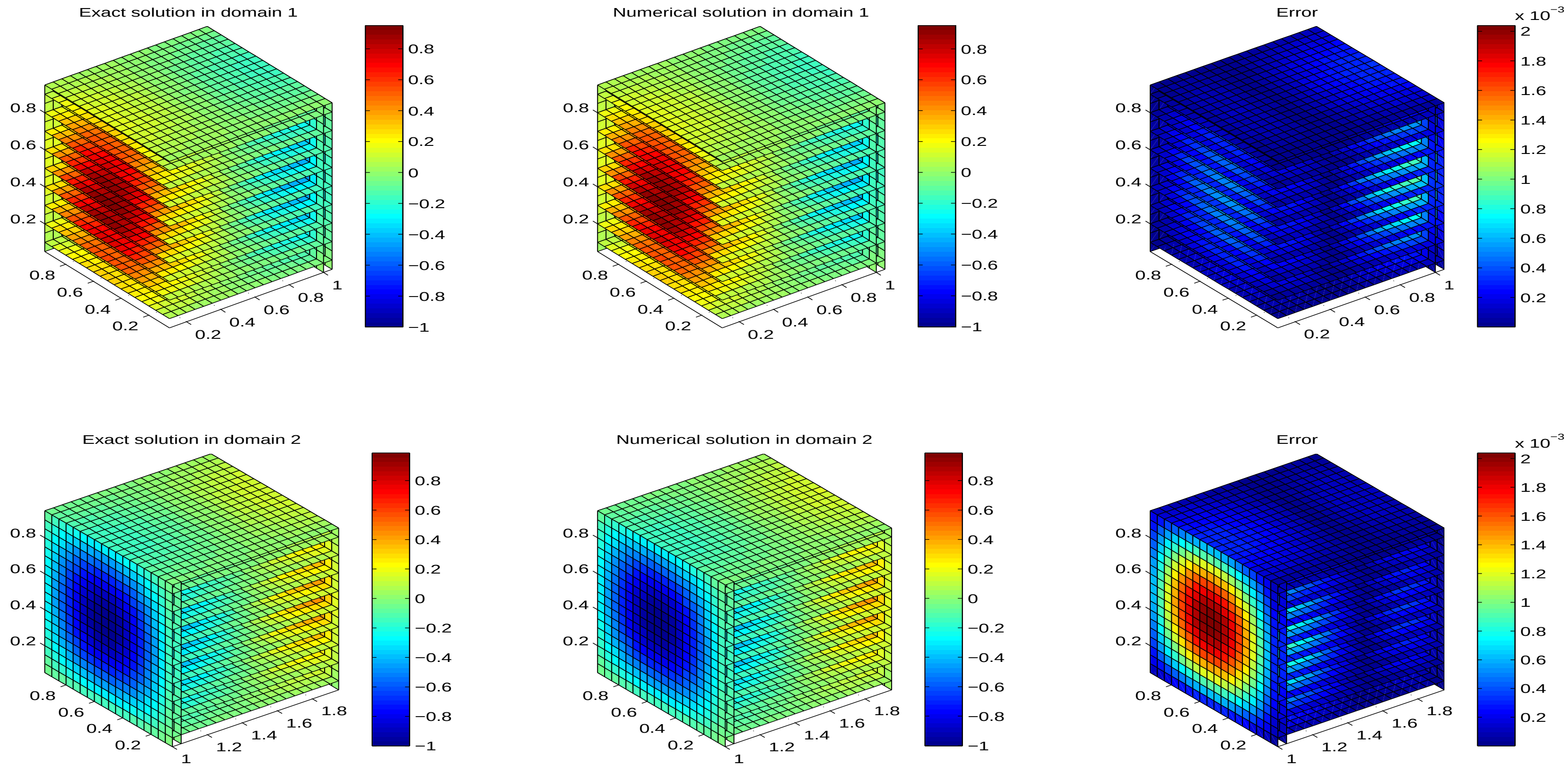


Figure 3: Screen shoots of the solutions and difference between exact and numerical solutions, with n=20 and overlap $h = 1/20$.

Numerical experiments

	Schwarz	Taylor 0	Taylor 2	Optimized 0	Two-sided Optimized 0	Optimized 2
h	Schwarz as an iterative solver					
$1/10$	52	26	20	9	8	6
$1/20$	63	26	20	9	8	6
$1/40$	75	27	20	9	8	6
$1/80$	86	26	20	9	8	6
$1/160$	93	26	20	9	8	6
	Schwarz used as a preconditioner					
$1/10$	11	8	7	6	6	4
$1/20$	11	8	7	6	6	4
$1/40$	12	8	7	6	6	4
$1/80$	11	8	7	6	6	4
$1/160$	12	8	7	6	6	4

Table 1: Number of iterations of the classical Schwarz method compared to the different optimized methods with fixed overlap $L = \frac{1}{10}$ between subdomains.

Numerical experiments

	Classical	Taylor 0	Taylor 2	Optimized 0	Two-sided Optimized 0	Optimized 2
h	Schwarz as an iterative solver					
$1/10$	52	26	20	9	8	6
$1/20$	106	37	26	11	9	6
$1/40$	214	53	36	14	11	7
$1/80$	425	76	52	17	13	9
$1/160$	852	108	75	22	16	10
	Schwarz used as a preconditioner					
$1/10$	11	8	7	6	6	4
$1/20$	15	9	8	6	6	4
$1/40$	20	10	9	7	7	5
$1/80$	29	12	11	8	8	5
$1/160$	40	14	13	9	9	5

Table 2: Number of iterations of the classical Schwarz method compared to the different optimized methods with variable overlap $L = h$ between subdomains.

Numerical experiments

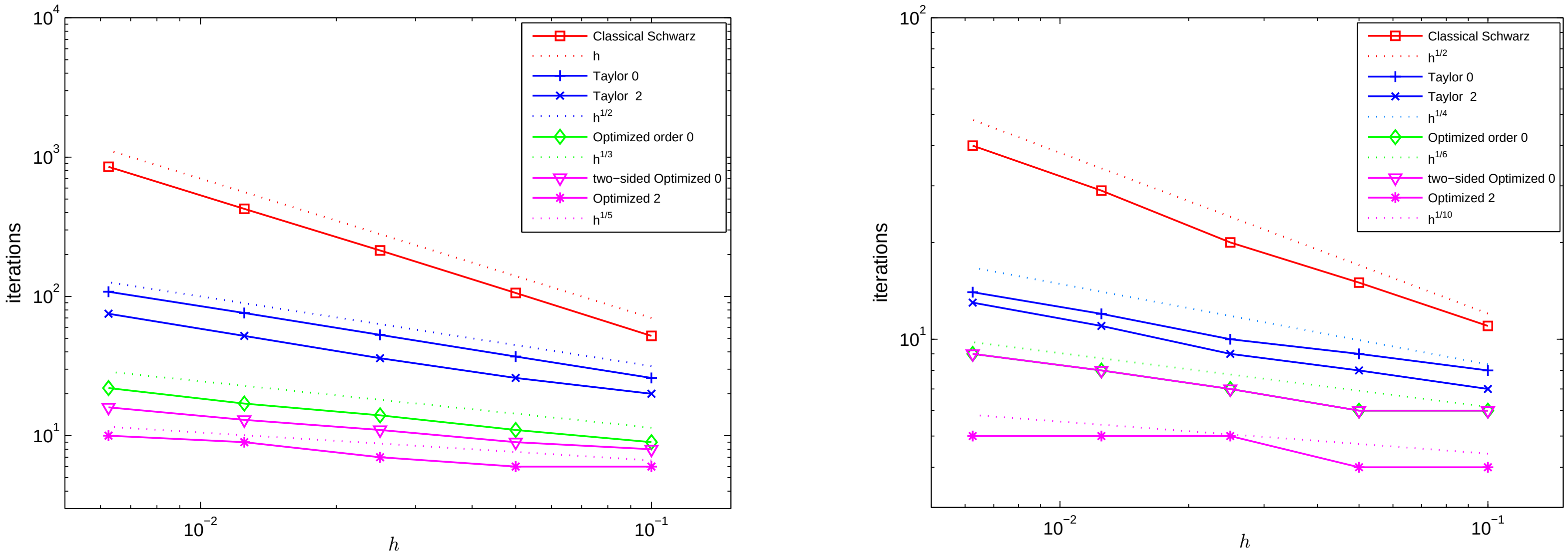


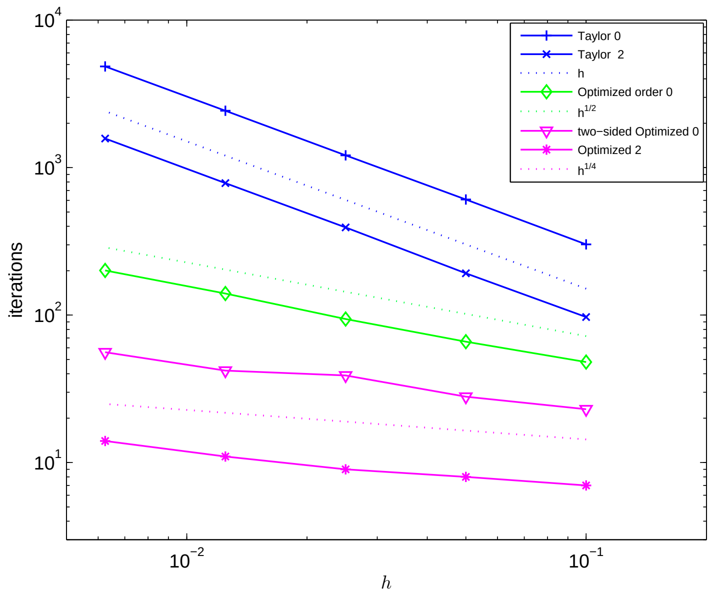
Figure 4: Number of iterations required by the classical and the optimized Schwarz methods, with overlap $L = h$. On the left the methods are used as iterative solvers, and on the right as preconditioners for a Krylov method.

Numerical experiments

	Taylor 0	Taylor 2	Optimized 0	Two-sided Optimized 0	Optimized 2
h	Optimized Schwarz as an iterative solver				
1/10	302	97	48	23	7
1/20	608	192	66	28	8
1/40	1211	393	94	39	9
1/80	2433	785	140	42	11
1/160	4855	1576	201	56	14
	Optimized Schwarz used as a preconditioner				
1/10	26	16	12	10	6
1/20	37	22	14	11	6
1/40	54	34	18	12	7
1/80	78	47	21	13	8
1/160	107	65	25	14	8

Table 3: Number of iterations of different optimized Schwarz methods without overlap between subdomains.

Numerical experiments



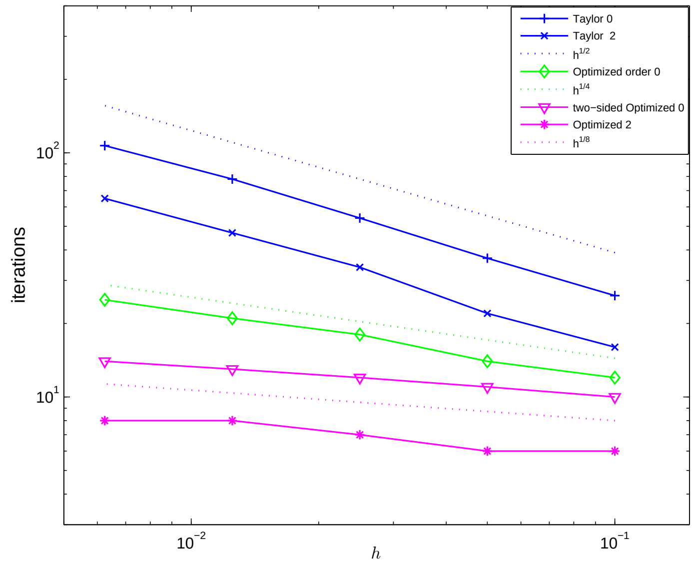


Figure 5: Number of iterations required by the optimized Schwarz methods without overlap between subdomains. On the left the methods are used as iterative solvers, and on the right as preconditioners for a Krylov method.

Concluding remarks

- We analyzed for three-dimensional positive definite model the influence of transmission conditions on the convergence factor of the classical Schwarz method.
- We showed analytically and numerically the great performance using optimized methods.
- The achieved performances are obtained without increasing the cost of computations.
- The analysis of three-dimensional problems is more involved technically but leads to similar results compared to two-dimensional case.