Multilevel BDDC

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Development of Multilevel BDDC Method

- BDDC (Dohrmann 2002): method from the Balancing Domain Decomposition (BDD) family (Mandel 1993), related to the Additive Schwarz framework of Neumann Neumann type (Dryja, Widlund 1995). BDD is Neumann-Neumann (De Roeck, Le Tallec, Vidrascu 1992, Glowinski, Wheeler 1988) with a coarse space; BDDC has a different coarse space. BDD is an approach dual to FETI (Farhat, Roux 1991). The eigenvalues of BDDC and FETI-DP are same except possibly for 0 and 1 (Mandel, Dohrmann, Tezaur 2005).

- **Motivation:** Large coarse problem is a bottleneck (case of many substructures) $\Longrightarrow$ **necessity of a multilevel algorithm.**

- Three-level BDDC (BDDC with two coarse levels) in two and three dimensions (Tu 2004, 2005).

New Here

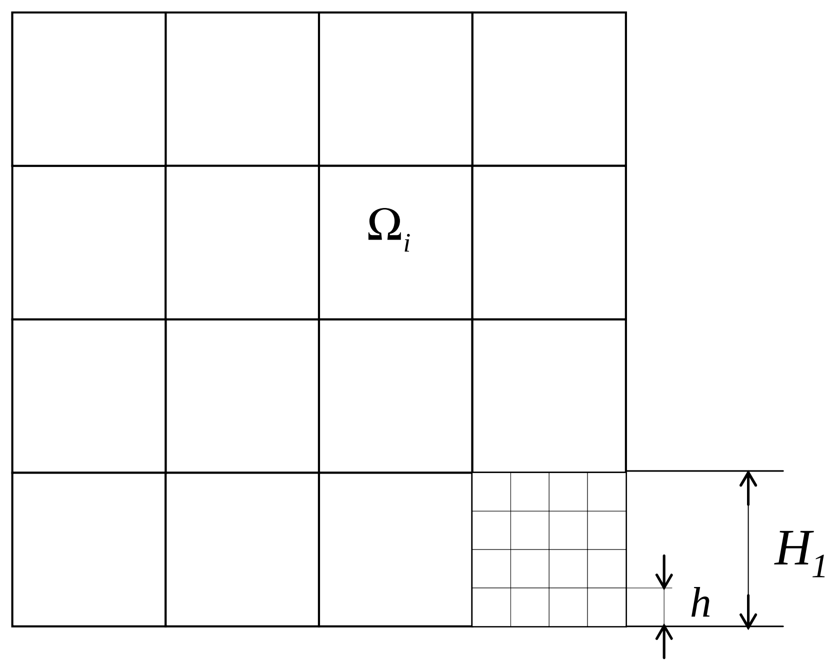
- **Abstract** multilevel/multispace BDDC formulation.

- **Algebraic** multilevel condition number estimate.

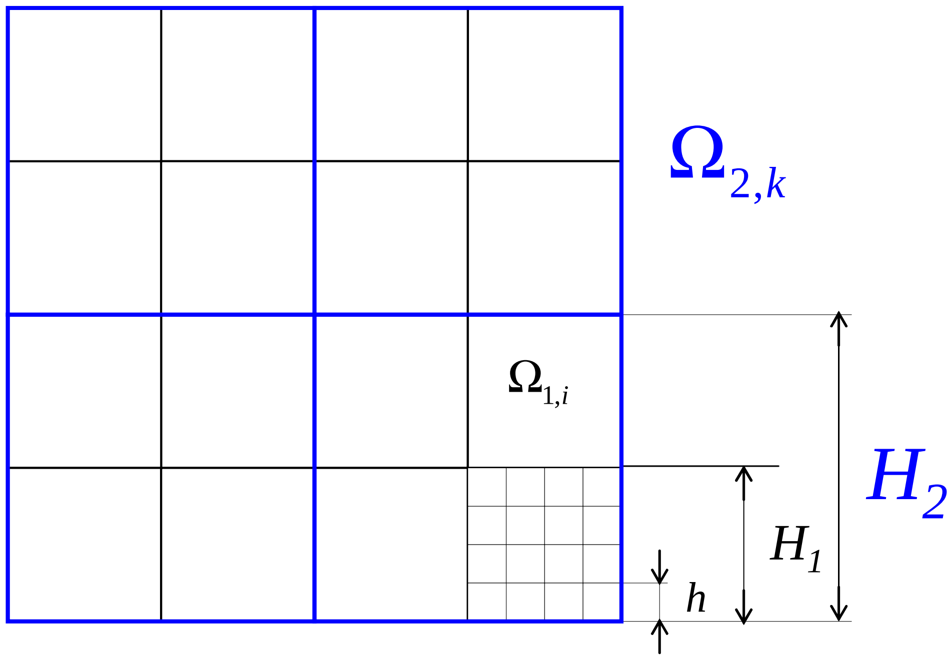
- **Polylogarithmic** multilevel condition number estimate for any fixed number of levels.

- Multilevel BDDC numerical experiments.

Substructuring for a Problem with H/h=4



Substructuring for a Three-level Model Problem



Abstract BDDC (Two Levels): Variational Setting of the Problem and Algorithm Components

$$u \in U : a(u,v) = (f,v)\,, \quad \forall v \in U$$

a symmetric, **positive definite** on U and **positive semidefinite** on $W \supset U$.

Example:
$W = W_1 \times \cdots \times W_N$ (spaces on substructures)
$U =$ functions continuous across interfaces

Choose preconditioner components:

space $\widetilde{W}$, $U \subset \widetilde{W} \subset W$, such that a is positive definite on $\widetilde{W}$.
Example: functions with continuous coarse dofs, such as values at substructure corners

projection $E : \widetilde{W} \to U, \quad \text{range}\, E = U$.
Example: averaging across substructure interfaces

Abstract BDDC Preconditioner

Given a on $W \supset U$, $A : U \to U$, $a(v,w) = (Av,w)\ \forall v,w \in U$
Choose $\widetilde{W}$, $U \subset \widetilde{W} \subset W$ and projection $E : \widetilde{W} \to U$

Theorem 1 **The abstract BDDC preconditioner** $M : U \longrightarrow U$,

$$M : r \longmapsto u = Ew, \quad w \in \widetilde{W} : \quad a\left(w,z\right) = \left(r, Ez\right), \quad \forall z \in \widetilde{W},$$

satisfies

$$\kappa = \frac{\lambda_{\text{max}}(MA)}{\lambda_{\text{min}}(MA)} \leq \omega = \sup_{w \in \widetilde{W}} \frac{\|Ew\|_a^2}{\|w\|_a^2}.$$

This framework is similar to partial subassembly by Li and Widlund (2005).

In implementation, $\widetilde{W}$ is decomposed into

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi$$

$\widetilde{W}_\Delta$ = functions with zero coarse dofs $\Rightarrow$ local problems on substructures
$\widetilde{W}_\Pi$ = functions given by coarse dofs & energy minimal
$\Rightarrow$ global coarse problem

Abstract Multi-Space (Multilevel) BDDC

Decompose the space $\widetilde{W}$ and choose **projections** Q_k as

$$\boxed{U \subset \widetilde{W} = \sum_{k=1}^{N} V_k \subset W, \quad Q_k : V_k \to U}$$

V_k energy orthogonal: $V_k \perp_a V_\ell, k \neq \ell$,

so $\forall u \in \widetilde{W} : u = \displaystyle\sum_{k=1}^{N} v_k, \; v_k \in V_k$

assume $\forall u \in U : u = \displaystyle\sum_{k=1}^{N} Q_k v_k$

Equivalently, assume $\Pi_k : \widetilde{W} \to V_k$ are orthogonal projections, and

$$I = \sum_{k=1}^{N} \Pi_k \text{ on } \widetilde{W}, \quad I = \sum_{k=1}^{N} Q_k \Pi_k \text{ on } U$$

Abstract Multi-Space (Multilevel) BDDC

Theorem 2 **The abstract multi-space BDDC preconditioner** $M : U \longrightarrow U$,

$$M : r \mapsto u, \quad u = \sum_{k=1}^{N} Q_k v_k, \quad v_k \in V_k : \quad a\left(v_k, z_k\right) = \left(r, Q_k z_k\right), \quad \forall z_k \in V_k,$$

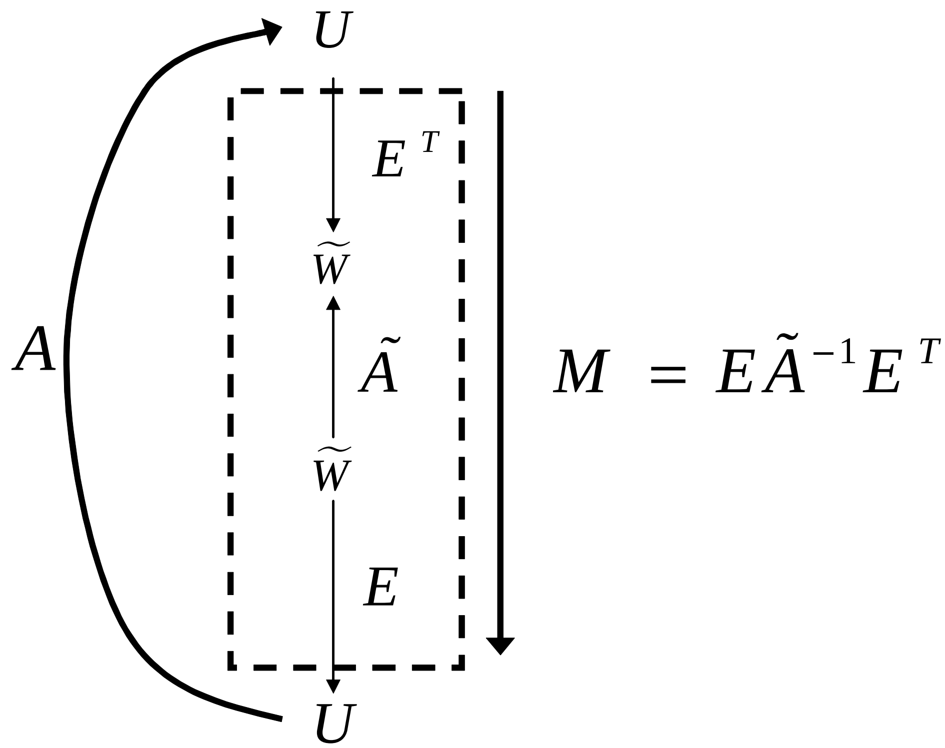
satisfies

$$\kappa = \frac{\lambda_{\max}(MA)}{\lambda_{\min}(MA)} \leq \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2}.$$

For $N = 1$ we recover the abstract BDDC algorithm and condition number bound. Proved from generalized Schwarz theory (Dryja and Widlund, 1995). Unlike in the Schwarz theory, we decompose $\widetilde{W} \supset U$, not U.

In a sense, the projections $Q_k : V_k \rightarrow U$ decompose the projection $E : \widetilde{W} \rightarrow U$.

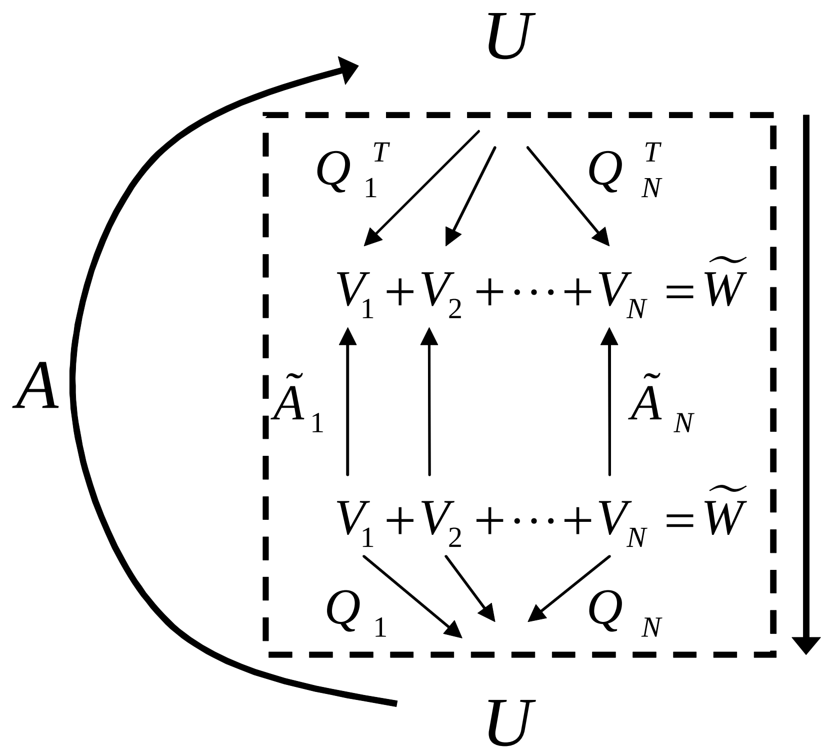
Algebraic View of the Abstract BDDC preconditioner



The same bilinear form a defines $A : U \rightarrow U$ and $\widetilde{A} : \widetilde{W} \rightarrow \widetilde{W} \supset U$

The preconditioner M to A is obtained by solving a problem with the same bilinear form on the bigger space $\widetilde{W}$ and mapping back to U via the projection E and its transpose E^T.

Algebraic View of the Abstract Multi-Space BDDC Preconditioner



$$M = \sum_{i=1}^{N} Q_i \tilde{A}_i^{-1} Q_i^T$$

The same bilinear form a defines $A : U \to U$ and $\widetilde{A}_i : V_i \to V_i,\ \sum_{i=1}^{N} V_i \supset U$
The preconditioner M to A is obtained by solving problems with the same bilinear form on the bigger spaces V_i and mapping back to U via the projections Q_i and their transposes Q_i^T.

BDDC with Interiors as Multi-Space BDDC

Abstract BDDC was presented here as operating on the space of discrete harmonic functions. The original BDDC formulation had "interior correction".

$$\boxed{U_I \;\overset{P}{\underset{\subset}{\leftarrow}}\; U \;\overset{E}{\underset{\subset}{\leftarrow}}\; \widetilde{W}}$$

Lemma 3 *The original BDDC preconditioner is the abstract multi-space BDDC method with* $N = 2$ *and the spaces and operators*

$$V_1 = U_I, \quad V_2 = \widetilde{W}, \quad Q_1 = I, \quad Q_2 = (I - P)\,E.$$

The space $\widetilde{W}$ has an a-orthogonal decomposition

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi.$$

so the problem on $\widetilde{W}$ splits into independent problems on $\widetilde{W}_\Delta$ and $\widetilde{W}_\Pi$.

Example:
$\widetilde{W}_\Delta =$ functions zero on substructure corners
$\widetilde{W}_\Pi =$ given by values on substructure corners and energy minimal

BDDC with Interiors as Multi-Space BDDC

Abstract BDDC was presented here as operating on the space of discrete harmonic functions. The original BDDC formulation had "interior correction".

$$\begin{array}{ccccc} U_I & \overset{P}{\underset{\subset}{\leftarrow}} & U & \overset{E}{\underset{\subset}{\leftarrow}} & \widetilde{W} \\ & & & & \| \\ & & \widetilde{W}_\Pi & \oplus & \widetilde{W}_\Delta \end{array}$$

Lemma 4 *The original BDDC preconditioner is the abstract multi-space BDDC method with $N = 3$ and the spaces and operators*

$$V_1 = U_I, \quad V_2 = \widetilde{W}_\Pi, \quad V_3 = \widetilde{W}_\Delta, \quad Q_1 = I, \quad Q_2 = Q_3 = (I - P)\, E.$$

Solving on $U_I \Rightarrow$ independent Dirichet problems on substructures
Solving on $\widetilde{W}_\Delta \Rightarrow$ independent constrained Neumann problems on substructures + correction in U_I
Solving on $\widetilde{W}_\Pi \Rightarrow$ **Global coarse problem with substructures as coarse elements and energy minimal function as coarse shape functions.**

Three-Level BDDC

Coarse problem solved approximately by the BDDC preconditioner

$$
\begin{array}{ccccccc}
 & & U & & & & \\
 & & \parallel & & & & \\
U_{I1} & \overset{P_1}{\underset{\subset}{\leftarrow}} & U_1 & \overset{E_1}{\underset{\subset}{\leftarrow}} & \widetilde{W}_1 & & \\
 & & & & \parallel & & \\
 & & & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & \\
 & & & \parallel & & & \\
 & U_{I2} & \overset{P_2}{\underset{\subset}{\leftarrow}} & U_2 & \overset{E_2}{\underset{\subset}{\leftarrow}} & \widetilde{W}_2 & \\
 & & & & & \parallel & \\
 & & & & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2}
\end{array}
$$

Lemma 5 *The three-level BDDC preconditioner is the abstract multi-space BDDC method with $N = 5$ and the spaces and operators*

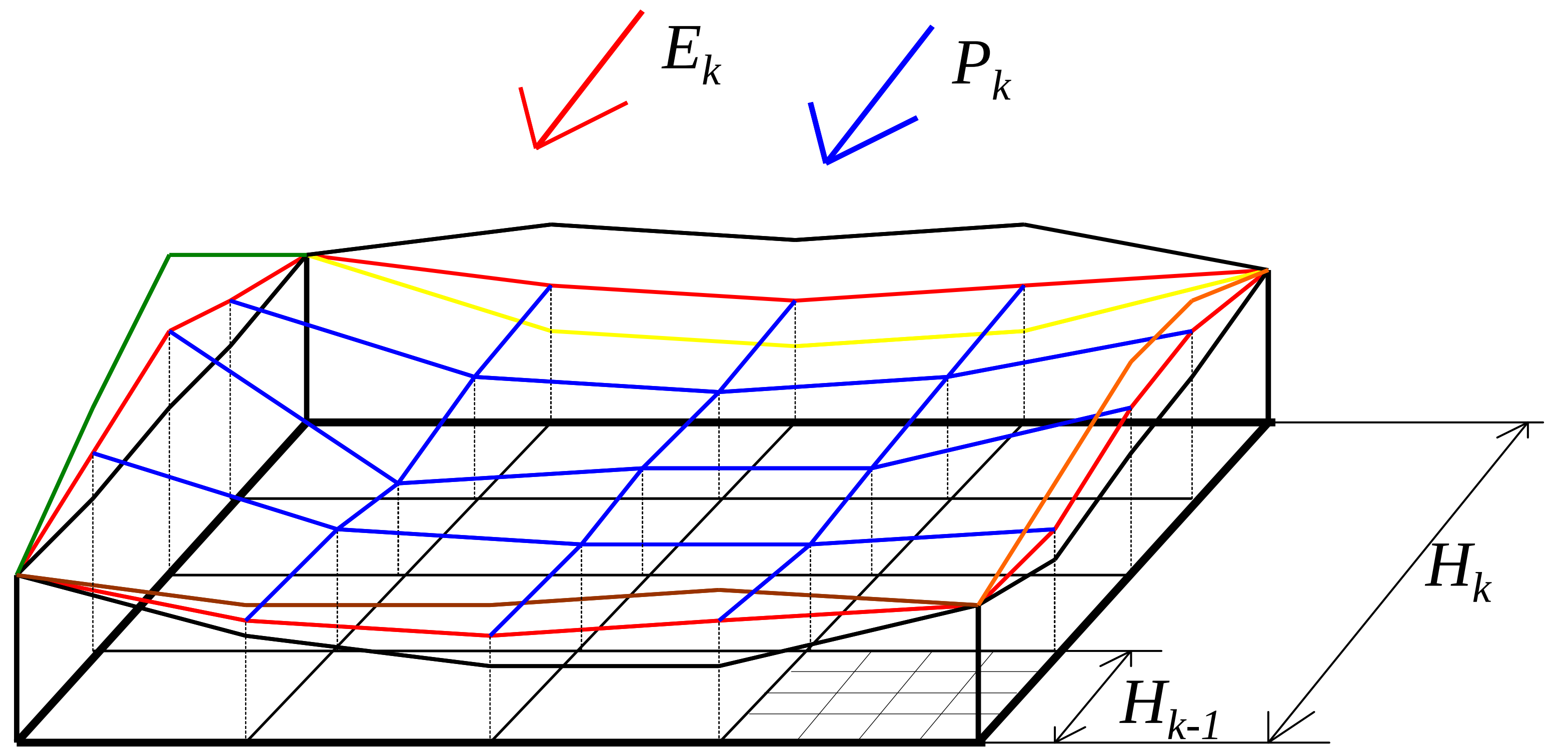
$$
\begin{gathered}
V_1 = U_{I1}, \quad V_2 = \widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \quad V_4 = \widetilde{W}_{\Pi 2}, \quad V_5 = \widetilde{W}_{\Delta 2}, \\
Q_1 = I, \quad Q_2 = Q_3 = (I - P_1)\, E_1, \quad Q_4 = Q_5 = (I - P_1)\, E_1\, (I - P_2)\, E_2.
\end{gathered}
$$

Multilevel BDDC

Coarse problem solved by the BDDC preconditioner, recursive

$$
\begin{array}{cccccccc}
 & & U & & & & & \\
 & & \parallel & & & & & \\
U_{I1} & \overset{P_1}{\underset{\subset}{\leftarrow}} & U_1 & \overset{E_1}{\underset{\subset}{\leftarrow}} & \widetilde{W}_1 & & & \\
 & & & & \parallel & & & \\
 & & & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & & \\
 & & & \parallel & & & & \\
 & U_{I2} & \overset{P_2}{\underset{\subset}{\leftarrow}} & U_2 & \overset{E_2}{\underset{\subset}{\leftarrow}} & \widetilde{W}_2 & & \\
 & & & & & \parallel & & \\
 & & & & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2} & \\
 & & & & \parallel & & & \\
 & & & & \vdots & & & \\
 & & & & \parallel & & & \\
 & & U_{IL-1} & \overset{P_{L-1}}{\underset{\subset}{\leftarrow}} & U_{L-1} & \overset{E_{L-1}}{\underset{\subset}{\leftarrow}} & \widetilde{W}_{L-1} & \\
 & & & & & & \parallel & \\
 & & & & & \widetilde{W}_{\Pi L-1} & \oplus & \widetilde{W}_{\Delta L-1}
\end{array}
$$

An Example of Action of Operators E_k and P_k



Values on this substructure and neighbors are averaged E_k then extended as "discrete harmonic" by P_k.
Basis functions on level k are given by dofs on level k + **energy minimal** w.r.t. basis functions on level $k-1$. Discrete harmonics on level k are given by boundary values + **energy minimal** w.r.t. basis functions on level $k-1$.

Multilevel BDDC

Coarse problem solved by the BDDC preconditioner, recursive

Lemma 6 *The multilevel BDDC preconditioner is the abstract multi-space BDDC preconditioner with N=2L-2 and*

$$
\begin{aligned}
V_1 &= U_{I1}, \quad V_2 = \widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \\
V_4 &= \widetilde{W}_{\Delta 2}, \quad V_5 = U_{I3}, \quad \ldots \\
V_{2L-4} &= \widetilde{W}_{\Delta L-2}, \quad V_{2L-3} = U_{IL-1,}, \\
V_{2L-2} &= \widetilde{W}_{L-1} \\
Q_1 &= I, \quad Q_2 = Q_3 = (I - P_1)\, E_1, \\
Q_4 &= Q_5 = (I - P_1)\, E_1\, (I - P_2)\, E_2, \quad \ldots \\
Q_{2L-4} &= Q_{2L-3} = (I - P_1)\, E_1 \cdots (I - P_{L-2})\, E_{L-2}. \\
Q_{2L-2} &= (I - P_1)\, E_1 \cdots (I - P_{L-1})\, E_{L-1}
\end{aligned}
$$

Recall condition number bound:

$$
\kappa = \frac{\lambda_{\max}(MA)}{\lambda_{\min}(MA)} \le \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2}.
$$

Algebraic Condition Estimate of Multilevel BDDC

Lemma 7 *If*

$$\begin{aligned}
\|(I-P_1)E_1 w_1\|_a^2 &\le \omega_1 \|w_1\|_a^2 \quad \forall w_1 \in \widetilde{W}_1, \\
\|(I-P_2)E_2 w_2\|_a^2 &\le \omega_2 \|w_2\|_a^2 \quad \forall w_2 \in \widetilde{W}_2, \\
&\vdots \\
\|(I-P_{L-1})E_{L-1} w_{L-1}\|_a^2 &\le \omega_{L-1} \|w_{L-1}\|_a^2 \quad \forall w_{L-1} \in \widetilde{W}_{L-1}.
\end{aligned}$$

then the multilevel BDDC preconditioner satisfies $\kappa \le \prod_{i=1}^{L-1} \omega_i$.

- All spaces are subspaces of the single space W.

- The functions $(I-P_i)E_i w_i$ are discrete harmonic functions on level i with energy minimal extension into the interior after averaging on level i, such that w_i has continuous coarse dofs (such as values at corners) at the decomposition level $i-1$.

Condition Number Estimate with Corner Contraints

Theorem 8 *The multilevel BDDC preconditioner with corner constraints only satisfies*

$$\kappa \leq C_1 \left(1 + \log \frac{H_1}{h}\right)^2 C_2 \left(1 + \log \frac{H_2}{H_1}\right)^2 \cdots C_{L-1} \left(1 + \log \frac{H_{L-1}}{H_{L-2}}\right)^2 .$$

For $L = 3$ we recover the estimate by Tu (2004).
This bound implies at most polylogarithmic growth of the condition number in the ratios of mesh sizes for a fixed number of levels L
For fixed H_i/H_{i-1} the growth of the condition number can be exponential in L and this is indeed seen in numerical experiments
With additional constraints, such as side averages, the condition number will be less but the bound is still principally the same, though possibly with (much) smaller constants. For small enough constants, the exponential growth of the condition number may no longer be apparent.

Numerical Examples

A multilevel BDDC implemented in Matlab for the 2D Laplace equation on a square domain with periodic boundary conditions.

2D Laplace equation results for $H/h = 2$.

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	2	1.5625	1	1	16
3	8	1.8002	5	1.1433	64
4	11	2.4046	7	1.2703	256
5	14	3.4234	8	1.3949	1,024
6	17	4.9657	9	1.5199	4,096
7	20	7.2428	9	1.6435	16,384
8	25	10.5886	10	1.7696	65,536

2D Laplace equation results for $H/h = 4$

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	9	2.1997	6	1.1431	256
3	14	4.0220	8	1.5114	4,096
4	21	7.7736	10	1.8971	65,536
5	30	15.1699	12	2.2721	1,048,576

2D Laplace equation results for $H/h = 8$

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	14	3.1348	7	1.3235	4,096
3	23	7.8439	10	2.0174	262,144
4	36	19.9648	13	2.7450	16,777,216

Conclusion

- Described an algorithm for multilevel BDDC preconditioning and derived a condition number estimate for case of corner constraints.

- Method tested on Laplace equation in 2D. Numerical results confirm the theory.

- The new concept of multi-space BDDC and algebraic estimate of its condition number could be of independent interest.

Future developments

- 3D condition number bounds.
- Other types of constraints - why does the condition number grow so much less when side averages are added in 2D?
- Lower bounds.
- Additive estimates like in the hierarchical basis method?
- Extensions of the adaptive approach (Mandel, Sousedík 2005) to the multilevel case $\Longrightarrow$ solution of problems that are both very large and numerically difficult.