Coupled FETI/BETI Solvers for Nonlinear Potential Problems in Unbounded Domains

Clemens Pechstein[1] Ulrich Langer[1,2]

[1]Special Research Programm SFB F013 on
Numerical and Symbolic Scientific Computing

[2]Institute of Computational Mathematics
Johannes Kepler University Linz

DD 17, Strobl/Wolfgangsee, 2006

Outline

Outline

1. FETI/BETI Introduction
   - Motivation and Overview
   - Coupled FETI/BETI Formulation
   - Preconditioners
2. Generalizations
   - Unbounded Domains
   - Nonlinear Problems
3. Conclusion

Motivation – Electromagnetic Field Computations

Nonlinear Magnetostatics in 2D:

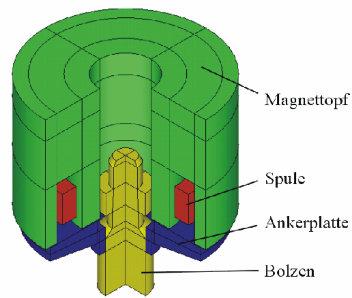
$$-\nabla \cdot \left[\nu(|\nabla u|)\nabla u\right] = f$$

+ boundary or radiation conditions

+ transmission conditions

Characteristics:

- Nonlinear in ferromagnetic materials
- Linear behavior in surrounding air / air gaps
- Typically large jumps over material interfaces



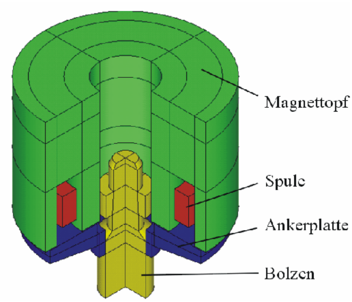
Motivation – Electromagnetic Field Computations

Nonlinear Magnetostatics in 2D:

$$-\nabla \cdot \left[\nu(|\nabla u|)\nabla u\right] = f$$

+ boundary or radiation conditions

+ transmission conditions



Characteristics:

- Nonlinear in ferromagnetic materials
- Linear behavior in surrounding air / air gaps
- Typically large jumps over material interfaces

Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity $\longrightarrow$ Lagrange multipliers
- Elimination $\longrightarrow$ dual problem
- PCG sub-space iteration

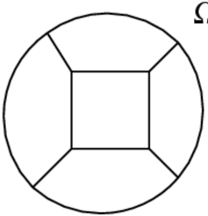
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity $\rightarrow$ Lagrange multipliers
- Elimination $\rightarrow$ dual problem
- PCG sub-space iteration



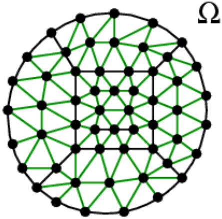
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity → Lagrange multipliers
- Elimination → dual problem
- PCG sub-space iteration



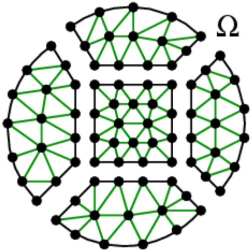
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity → Lagrange multipliers
- Elimination → dual problem
- PCG sub-space iteration



Tearing

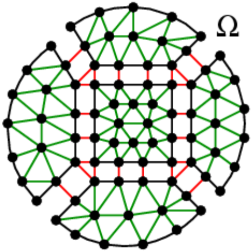
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity → Lagrange multipliers
- Elimination → dual problem
- PCG sub-space iteration



Interconnecting

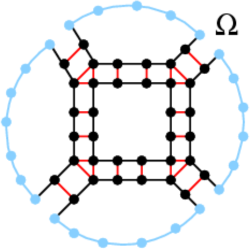
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity $\longrightarrow$ Lagrange multipliers
- Elimination $\longrightarrow$ dual problem
- PCG sub-space iteration



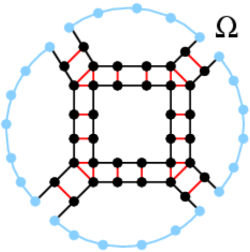
Finite Element Tearing and Interconnecting – Overview

Farhat and Roux, 1991

Non-overlapping Domain Decomposition

b.v.p. for Poisson Problem, Structural Mechanics

- Domain Decomposition
- Conformal mesh
- Separate d.o.f.
- Continuity → Lagrange multipliers
- Elimination → dual problem
- PCG sub-space iteration



Finite Element Tearing and Interconnecting – Overview

FETI – Features

- PCG iteration and preconditioning via local Dirichlet and Neumann solvers
- Allows massive parallelization
- Condition number $\mathcal{O}((1 + \log(H/h))^2)$
- Robust w.r.t. coefficient jumps

Mandel/Tezaur, 1996
Klawonn/Widlund, 2001

Boundary Element Tearing and Interconnecting – Overview

- FETI-technique can be carried over to the BEM $\longrightarrow$ BETI

  Langer/Steinbach, 2003

- Coupling BETI and FETI:
  - Benefit from advantages of both techniques
  - air gaps / surrounding air $\longrightarrow$ BEM
  - nonlinear materials $\longrightarrow$ FEM

  Langer/Steinbach, 2004

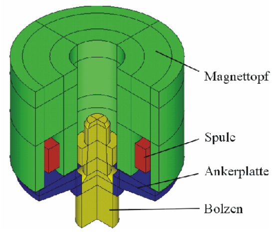
Boundary Element Tearing and Interconnecting – Overview

- FETI-technique can be carried over to the BEM $\longrightarrow$ BETI

  Langer/Steinbach, 2003

- Coupling BETI and FETI:
  - Benetit from advantages of both techniques
  - air gaps / surrounding air $\longrightarrow$ BEM
  - nonlinear materials $\longrightarrow$ FEM

  Langer/Steinbach, 2004



Outline

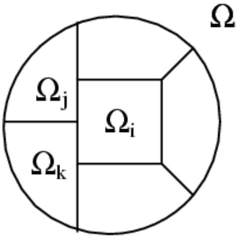
A Continuous Formulation

Quasi-regular non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^d$ bounded

$\Gamma = \partial\Omega$

$\overline{\Omega} = \bigcup_{i \in \mathcal{I}} \overline{\Omega}_i$



A Continuous Formulation

Quasi-regular non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^d$ bounded

$\Gamma = \partial\Omega$

$\overline{\Omega} = \bigcup_{i \in \mathcal{I}} \overline{\Omega}_i$

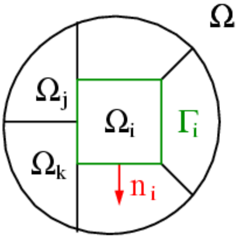
$\Gamma_i = \partial\Omega_i$

n_i : outward unit normal vector

$\overline{\Gamma}_{jk} = \overline{\Gamma}_j \cap \overline{\Gamma}_k$

$\Gamma_S = \bigcup_{i \in \mathcal{I}} \Gamma_i$

$H_i = \operatorname{diam} \Omega_i \simeq H$



A Continuous Formulation

Quasi-regular non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^d$ bounded

$\Gamma = \partial\Omega$

$\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \overline{\Omega}_i$

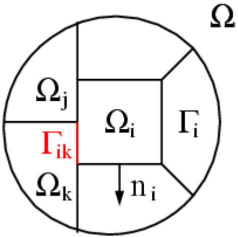
$\Gamma_i = \partial\Omega_i$

n_i : outward unit normal vector

$\overline{\Gamma}_{jk} = \overline{\Gamma}_j \cap \overline{\Gamma}_k$

$\Gamma_S = \bigcup_{i\in\mathcal{I}} \Gamma_i$

$H_i = \operatorname{diam}\Omega_i \simeq H$



A Continuous Formulation

Quasi-regular non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^d$ bounded

$\Gamma = \partial\Omega$

$\overline{\Omega} = \bigcup_{i \in \mathcal{I}} \overline{\Omega}_i$

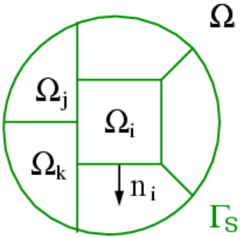
$\Gamma_i = \partial\Omega_i$

n_i : outward unit normal vector

$\overline{\Gamma}_{jk} = \overline{\Gamma}_j \cap \overline{\Gamma}_k$

$\Gamma_S = \bigcup_{i \in \mathcal{I}} \Gamma_i$

$H_i = \operatorname{diam} \Omega_i \simeq H$



A Continuous Formulation

Quasi-regular non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^d$ bounded

$\Gamma = \partial\Omega$

$\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \overline{\Omega}_i$

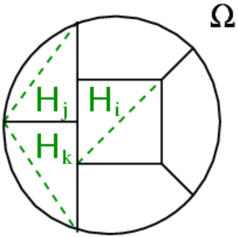
$\Gamma_i = \partial\Omega_i$

n_i : outward unit normal vector

$\overline{\Gamma}_{jk} = \overline{\Gamma}_j \cap \overline{\Gamma}_k$

$\Gamma_S = \bigcup_{i\in\mathcal{I}} \Gamma_i$

$H_i = \operatorname{diam}\Omega_i \simeq H$



A Continuous Formulation

Model Problem:

$$
\begin{aligned}
-\nabla \cdot [\alpha_i \nabla u] &= f \quad \text{in } \Omega_i \quad \forall i \in \mathcal{I} \\
u &= 0 \quad \text{on } \Gamma \\
\alpha_i \frac{\partial u}{\partial n_i} + \alpha_j \frac{\partial u}{\partial n_j} &= 0 \quad \text{on } \Gamma_{ij} \quad \forall i \neq j
\end{aligned}
$$

Solution fulfills

- Continuity: $[u]_{\Gamma_{ij}} = 0$
- Transmission: $t_i + t_j = 0 \quad \text{on } \Gamma_{ij}$

A Continuous Formulation

Model Problem:

$$
\begin{aligned}
-\nabla \cdot [\alpha_i \nabla u] &= f \quad \text{in } \Omega_i \quad \forall i \in \mathcal{I} \\
u &= 0 \quad \text{on } \Gamma \\
\underbrace{\alpha_i \frac{\partial u}{\partial n_i}}_{t_i} + \underbrace{\alpha_j \frac{\partial u}{\partial n_j}}_{t_j} &= 0 \quad \text{on } \Gamma_{ij} \quad \forall i \neq j
\end{aligned}
$$

Solution fulfills

- Continuity: $[u]_{\Gamma_{ij}} = 0$
- Transmission: $t_i + t_j = 0 \quad \text{on } \Gamma_{ij}$

A Continuous Formulation

Fixed sub-domain Ω_i: Solution of

$$\begin{aligned} -\nabla \cdot [\alpha_i \nabla u_i] &= f_i \quad \text{in } \Omega_i \\ u_i &= g_i \quad \text{on } \Gamma_i \end{aligned}$$

defines co-normal derivative $t_i = \alpha_i \frac{\partial u_i}{\partial n_i}$

Dirichlet-to-Neumann map:

$$g_i \;\mapsto\; t_i = S_i\, g_i - N_i\, f_i$$

Steklov-Poincaré operator: $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$
linear, bounded, injective

Newton potential: $N_i : H^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i)$

A Continuous Formulation

Fixed sub-domain Ω_i: Solution of

$$\begin{aligned} -\nabla\cdot[\alpha_i \nabla u_i] &= f_i \quad \text{in } \Omega_i \\ u_i &= g_i \quad \text{on } \Gamma_i \end{aligned}$$

defines co-normal derivative $t_i = \alpha_i \frac{\partial u_i}{\partial n_i}$

Dirichlet-to-Neumann map:

$$g_i \;\mapsto\; t_i = S_i\, g_i - N_i\, f_i$$

Steklov-Poincaré operator: $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$
linear, bounded, injective

Newton potential: $N_i : H^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i)$

A Continuous Formulation

Model Problem

$$
\begin{aligned}
-\nabla \cdot [\alpha_i \nabla u] &= f \quad \text{in } \Omega_i \quad \forall i \in \mathcal{I} \\
u &= 0 \quad \text{on } \Gamma \\
\underbrace{\alpha_i \frac{\partial u}{\partial n_i}}_{t_i} + \underbrace{\alpha_j \frac{\partial u}{\partial n_j}}_{t_j} &= 0 \quad \text{on } \Gamma_{ij} \quad \forall i \neq j
\end{aligned}
$$

“equivalent” to Variational Formulation:

Find $u \in H^{1/2}_{0,\Gamma}(\Gamma_S)$:

$$
\sum_{i \in \mathcal{I}} \langle S_i\, u|_{\Gamma_i}, v|_{\Gamma_i} \rangle = \sum_{i \in \mathcal{I}} \langle N_i\, f|_{\Omega_i}, v|_{\Gamma_i} \rangle \qquad \forall v \in H^{1/2}_{0,\Gamma}(\Gamma_S)
$$

Dirichlet-to-Neumann map – FEM

Fixed sub-domain Ω_i, FEM-triangulation $\mathcal{T}_{i,h}$
Given: Dirichlet data $g_i \in V_h(\Gamma_i)$.
Find: Galerkin representation $t_i \in V_h^*(\Gamma_i)$ of Neumann trace.

via Schur complement:

$$\underline{t}_i = S_{i,h}^{\mathrm{FEM}}\, \underline{g}_i - N_{i,h}^{\mathrm{FEM}}\, \underline{f}_i$$

with

$$S_{i,h}^{\mathrm{FEM}} = K_{i,h}^{\Gamma\Gamma} - \left[K_{i,h}^{I\Gamma}\right]^\top \left[K_{i,h}^{II}\right]^{-1} K_{i,h}^{I\Gamma}$$
$$N_{i,h}^{\mathrm{FEM}} \underline{f}_i = \underline{f}_i^{\Gamma} - \left[K_{i,h}^{I\Gamma}\right]^\top \left[K_{i,h}^{II}\right]^{-1} \underline{f}_i^{I}$$

Dirichlet-to-Neumann map – BEM

Fixed sub-domain Ω_i, BEM-triangulation $\mathcal{T}_{i,h}$ of Γ_i
Homogeneous b.v.p. is characterized by the Caldéron projector

$$\begin{pmatrix} g_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} g_i \\ t_i \end{pmatrix}$$

Symmetric and stable approximation:

$$S_{i,h}^{\mathrm{BEM}} = D_{i,h} + \Big(\frac{1}{2}M_{i,h}^\top + K_{i,h}^\top\Big) V_{i,h}^{-1} \Big(\frac{1}{2}M_{i,h} + K_{i,h}\Big)$$

General Dirichlet-to-Neumann map:

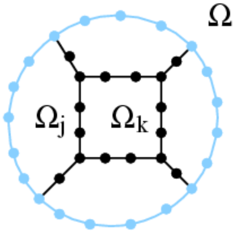
$$\underline{t}_i = S_{i,h}^{\mathrm{BEM}} \underline{g}_i - N_{i,h}^{\mathrm{BEM}} f_i$$

Minimization Problem

Variational Problem $\leftrightarrow$ Minimization Problem:

$$\min_{u \in V_{0,\Gamma,h}(\Gamma_S)} \sum_{i \in \mathcal{I}} \frac{1}{2} \left(S_{i,h}^{\mathrm{FEM/BEM}} A_i \underline{u}, A_i \underline{u} \right) - \sum_{i \in \mathcal{I}} \left(N_{i,h}^{\mathrm{FEM/BEM}} \underline{f}_i, A_i \underline{u} \right)$$

where A_i are connectivity matrices



Constrained Minimization Problem

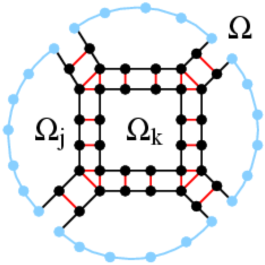
Variational Problem $\leftrightarrow$ Constrained Minimization Problem:

$$\min_{u_i \in V_{0,\Gamma,h}(\Gamma_i)} \sum_{i \in \mathcal{I}} \frac{1}{2} (S_{i,h}^{\text{FEM/BEM}} \underline{u}_i, \underline{u}_i) - \sum_{i \in \mathcal{I}} (N_{i,h}^{\text{FEM/BEM}} \underline{f}_i, \underline{u}_i)$$

subject to the continuity constraints

$$\sum_{i \in \mathcal{I}} B_i \underline{u}_i = 0$$

with $(B_i)_{mn} \in \{0, -1, +1\}$



Lagrange Multipliers

$$
\begin{pmatrix}
S_{1,h}^{\mathrm{FEM}} & & & & & & B_1^\top \\
& \ddots & & & & & \vdots \\
& & S_{q,h}^{\mathrm{FEM}} & & & & B_q^\top \\
& & & S_{q+1,h}^{\mathrm{BEM}} & & & B_{q+1}^\top \\
& & & & \ddots & & \vdots \\
& & & & & S_{p,h}^{\mathrm{BEM}} & B_p^\top \\
B_1 & \cdots & B_q & B_{q+1} & \cdots & B_p & 0
\end{pmatrix}
\begin{pmatrix}
\underline{u}_1 \\ \vdots \\ \underline{u}_q \\ \underline{u}_{q+1} \\ \vdots \\ \underline{u}_p \\ \underline{\lambda}
\end{pmatrix}
=
\begin{pmatrix}
\underline{f}_1^{\mathrm{FEM}} \\ \vdots \\ \underline{f}_q^{\mathrm{FEM}} \\ \underline{f}_{q+1}^{\mathrm{BEM}} \\ \vdots \\ \underline{f}_p^{\mathrm{BEM}} \\ 0
\end{pmatrix}
$$

Elimination of primal variables

$$
\underline{u}_i = \left[S_{i,h}^{\mathrm{FEM/BEM}}\right]^\dagger \left[\underline{f}_i^{\mathrm{FEM/BEM}} - B_i^\top \underline{\lambda}\right] \quad \left(+ R_i \gamma_i \right)
$$

Lagrange Multipliers

$$\begin{pmatrix} S_{1,h}^{\mathrm{FEM}} & & & & & & B_1^\top \\ & \ddots & & & & & \vdots \\ & & S_{q,h}^{\mathrm{FEM}} & & & & B_q^\top \\ & & & S_{q+1,h}^{\mathrm{BEM}} & & & B_{q+1}^\top \\ & & & & \ddots & & \vdots \\ & & & & & S_{p,h}^{\mathrm{BEM}} & B_p^\top \\ B_1 & \dots & B_q & B_{q+1} & \dots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_q \\ \underline{u}_{q+1} \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1^{\mathrm{FEM}} \\ \vdots \\ \underline{f}_q^{\mathrm{FEM}} \\ \underline{f}_{q+1}^{\mathrm{BEM}} \\ \vdots \\ \underline{f}_p^{\mathrm{BEM}} \\ 0 \end{pmatrix}$$

Elimination of primal variables

$$\underline{u}_i = \left[S_{i,h}^{\mathrm{FEM/BEM}}\right]^\dagger \left[\underline{f}_i^{\mathrm{FEM/BEM}} - B_i^\top \underline{\lambda}\right] \quad \left(+ R_i \gamma_i \right)$$

Dual Problem

Reduced system:

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix}$$

with

$$F = \sum_{i \in \mathcal{I}} B_i \left[S_{i,h}^{\text{FEM/BEM}} \right]^\dagger B_i^\top, \qquad G = (B_i R_i)_{i : \Gamma_i \cap \Gamma = \emptyset}$$

P : suitable projection onto $(\ker G)^\perp$ $\longrightarrow$

spd Problem

Find $\underline{\tilde{\lambda}} \in (\ker G)^\perp$: $\quad P^\top F P \underline{\tilde{\lambda}} = \underline{\tilde{d}}$

$\longrightarrow$ PCG sub-space iteration on $(\ker G)^\perp$

Dual Problem

Reduced system:

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix}$$

with

$$F = \sum_{i \in \mathcal{I}} B_i \left[S_{i,h}^{\text{FEM/BEM}} \right]^\dagger B_i^\top, \qquad G = (B_i R_i)_{i : \Gamma_i \cap \Gamma = \emptyset}$$

P : suitable projection onto $(\ker G)^\perp$ $\quad \rightarrow$

spd Problem

Find $\underline{\tilde{\lambda}} \in (\ker G)^\perp$ *:* $\quad P^\top F P \, \underline{\tilde{\lambda}} = \underline{\tilde{d}}$

$\rightarrow$ PCG sub-space iteration on $(\ker G)^\perp$

Outline

1. FETI/BETI Introduction
   - Motivation and Overview
   - Coupled FETI/BETI Formulation
   - Preconditioners
2. Generalizations
   - Unbounded Domains
   - Nonlinear Problems
3. Conclusion

FETI/BETI Preconditioners

$$F = \sum_{i \in \mathcal{I}} B_i \left[S_{i,h}^{\text{FEM/BEM}}\right]^{\dagger} B_i^{\top}$$

Construct Preconditioner

$$M^{-1} = D_B^{-1} \left[\sum_{i \in \mathcal{I}} B_i \, D_{\alpha,i}^{-1} \underbrace{S_{i,h}^{\text{FEM/BEM}}}_{\text{or } D_{i,h}^{\text{BEM}}} D_{\alpha,i}^{-1} \, B_i^{\top} \right] D_B^{-1}$$

with suitable scaling matrices $D_{\alpha,i}$ and $[D_B]_i = B_i D_{\alpha,i}^{-1} B_i^{\top}$

Theorem

If projection P and scaling $D_{\alpha,i}$ suitably chosen then

$$\kappa(P\, M^{-1} P^{\top} P^{\top} F\, P) \leq C \left(1 + \log\left(\tfrac{H}{h}\right)\right)^2,$$

C independent of h, H and coefficient jumps.

FETI/BETI Preconditioners

$$F = \sum_{i\in\mathcal{I}} B_i \left[S_{i,h}^{\mathrm{FEM/BEM}}\right]^\dagger B_i^\top$$

Construct Preconditioner

$$M^{-1} = D_B^{-1}\Big[\sum_{i\in\mathcal{I}} B_i\, D_{\alpha,i}^{-1}\, \underbrace{S_{i,h}^{\mathrm{FEM/BEM}}}_{\text{or } D_{i,h}^{\mathrm{BEM}}}\, D_{\alpha,i}^{-1}\, B_i^\top\Big] D_B^{-1}$$

with suitable scaling matrices $D_{\alpha,i}$ and $[D_B]_i = B_i D_{\alpha,i}^{-1} B_i^\top$

Theorem

If projection P and scaling $D_{\alpha,i}$ suitably chosen then

$$\kappa\big(P\, M^{-1} P^\top P^\top F\, P\big) \le C\,\Big(1 + \log\big(\tfrac{H}{h}\big)\Big)^2,$$

C independent of h, H and coefficient jumps.

FETI/BETI Preconditioners

Proof.

- FETI: Klawonn/Widlund 2001
  Preconditioner based on scaling $D_{\alpha,i}(x) = \frac{\alpha_i}{\sum_{j \in \mathcal{N}_x} \alpha_j}$.

  Proof based on partition of unity into face/edge/vertex terms and inequalities in $H^{1/2}(\Gamma_i)$-norms.
- BETI and Coupling: Langer/Steinbach 2003/2004
  based on spectral equivalences $S_{i,h}^{\mathrm{FEM}} \simeq S_{i,h}^{\mathrm{BEM}} \simeq D_{i,h}^{\mathrm{BEM}}$ and other properties of $S_{i,h}^{\mathrm{BEM}}$.

□

FETI/BETI Preconditioners

Proof.

- FETI: Klawonn/Widlund 2001
  Preconditioner based on scaling $D_{\alpha,i}(x) = \frac{\alpha_i}{\sum_{j \in \mathcal{N}_x} \alpha_j}$.

  Proof based on partition of unity into face/edge/vertex terms and inequalities in $H^{1/2}(\Gamma_i)$-norms.
- BETI and Coupling: Langer/Steinbach 2003/2004
  based on spectral equivalences $S_{i,h}^{\text{FEM}} \simeq S_{i,h}^{\text{BEM}} \simeq D_{i,h}^{\text{BEM}}$ and other properties of $S_{i,h}^{\text{BEM}}$.

$\square$

FETI/BETI Preconditioners

$$F = \sum_{i \in \mathcal{I}} B_i \left[S_{i,h}^{\text{FEM/BEM}} \right]^{\dagger} B_i^\top$$

Preconditioner

$$M^{-1} = D_B^{-1} \left[\sum_{i \in \mathcal{I}} B_i D_{\alpha,i}^{-1} \underbrace{S_{i,h}^{\text{FEM/BEM}}}_{\text{or } D_{i,h}^{\text{BEM}}} D_{\alpha,i}^{-1} B_i^\top \right] D_B^{-1}$$

Crucial steps in PCG can be done in parallel

FEM: local Dirichlet and Neumann solvers

BEM: application of $S_{i,h}^{\text{BEM}}$ and its pseudo-inverse

FETI/BETI Preconditioners

$$F = \sum_{i \in \mathcal{I}} B_i \left[S_{i,h}^{\text{FEM/BEM}} \right]^{\dagger} B_i^{\top}$$

Preconditioner

$$M^{-1} = D_B^{-1} \left[\sum_{i \in \mathcal{I}} B_i D_{\alpha,i}^{-1} \underbrace{S_{i,h}^{\text{FEM/BEM}}}_{\text{or } D_{i,h}^{\text{BEM}}} D_{\alpha,i}^{-1} B_i^{\top} \right] D_B^{-1}$$

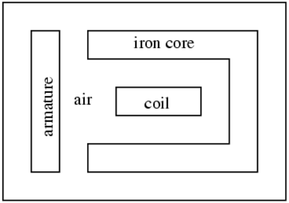
Crucial steps in PCG can be done in parallel

FEM: local Dirichlet and Neumann solvers
BEM: application of $S_{i,h}^{\text{BEM}}$ and its pseudo-inverse

FETI/BETI – Numerical Results

2D magnetic valve

glob. dof				PCG it.	
FETI	FETI/BETI	Lagr.	H/h	FETI	FETI/BETI
806	484	408	3	12	11
3539	1612	777	6	14	13
14357	5662	1515	12	17	16
57833	20983	2991	24	19	18
232145	80194	5943	48	21	20

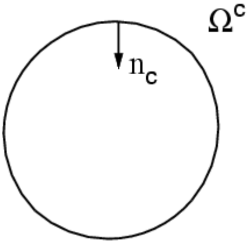


Outline

Unbounded Domains

Exterior Domain $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$

$$
\begin{aligned}
-\nabla \cdot [\alpha_{\text{ext}} \nabla u] &= 0 \quad \text{in } \Omega^c \\
u_i &= g \quad \text{on } \Gamma \\
|u(x) - u_0| &= \mathcal{O}(|x|^{-1}) \quad \text{for } |x| \to \infty
\end{aligned}
$$



Dirichlet-to-Neumann map: $g \mapsto t := \alpha_{\text{ext}} \frac{\partial u}{\partial n_c}$

Unbounded Domains

Interior Domain Ω:

$$\begin{pmatrix} g \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} g \\ t \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}$$

Dirichlet-to-Neumann map:

$$t = S_{\text{int}}\, g - N_{\text{int}}\, f \qquad \ker S_{\text{int}} = \operatorname{span}\{1\}$$

Symmetric and stable approximation:

$$S_{\text{int},h}^{\text{BEM}} = D_h + \Big(\frac{1}{2}M_h^\top + K_h^\top\Big) V_h^{-1} \Big(\frac{1}{2}M_h + K_h\Big)$$

Unbounded Domains

Exterior Domain Ω^c:

$$\begin{pmatrix} g \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & V \\ D & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} g \\ t \end{pmatrix} + \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$$

Dirichlet-to-Neumann map:

$$t = S_{\text{ext}}\, g - V^{-1} u_0 \qquad\qquad \ker S_{\text{ext}} = \{0\}$$

Symmetric and stable approximation:

$$S_{\text{ext},h}^{\text{BEM}} = D_h + \Big(\frac{1}{2} M_h^\top - K_h^\top\Big) V_h^{-1} \Big(\frac{1}{2} M_h - K_h\Big)$$

Standard FETI/BETI theory

Core part of the condition number estimate:

$$|P_D u|_S^2 \leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2 |u|_S^2,$$

where

$$|u|_S^2 := \sum_{i \in \mathcal{I}} |u_i|^2_{S_{i,h}^{\text{FEM/BEM}}}$$

and P_D denotes the $D_{\alpha,i}$-weighted projection.
E. g. for a face $F = \Gamma_i \cap \Gamma_j$:

$$(P_D u)_i(x) = \pm \frac{\alpha_j}{\alpha_i + \alpha_j} [u_i(x) - u_j(x)]$$

Proof performed by a partition of unity on Γ_i:

$$1 = \sum_{F \in \mathcal{F}_{\Gamma_i}} \theta_F + \sum_{E \in \mathcal{E}_{\Gamma_i}} \theta_E + \sum_{V \in \mathcal{V}_{\Gamma_i}} \theta_V$$

Generalization to unbounded domains

FETI/BETI theory might work quite analogous when the exterior domain is treated as a subdomain.

But following problems arise:

- S_{ext} has a trivial kernel
- different scalings: $H_i = \operatorname{diam} \Omega_i$ and $H_\Gamma = \operatorname{diam} \Omega$
- number of subdomains Ω_i touching the outer boundary Γ may be arbitrarily large
- processor balancing

Unbounded Domains – Numerical Results

$H_F = \max_{F \in \mathcal{F}_{\mathrm{DD}}} \operatorname{diam} F$ maximal diameter of a face
$H_\Omega = \operatorname{diam} \Omega$

2D, unbounded FETI/BETI (including the exterior domain)

$H_F/h = 4$ fixed

# subdom.	H_Ω/H_F	PCG it.
9	3	8
36	6	9
144	12	9
576	24	11
2304	48	12

Condition Estimate for Unbounded Domains

$H_F = \max_{F \in \mathcal{F}_{\mathrm{DD}}} \operatorname{diam} F$ maximal diameter of a face
$H_\Omega = \operatorname{diam} \Omega$

Conjecture

For the unbounded FETI/BETI setting the following condition number estimate holds:

$$\kappa(P M^{-1} P^\top P^\top F P) \leq C \left(1 + \log\left(\frac{H_F}{h}\right)\right)^2 \left(1 + \log\left(\frac{H_\Omega}{H_F}\right)\right)^\alpha,$$

C *independent of* h, H_F, H_Ω *and coefficient jumps.*

Proof in progess ...

Outline

Nonlinear Problems

Nonlinear Potential Problem:

$$-\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] = f \quad \text{in } \Omega_i$$

+ boundary / radiation conditions

+ transmission conditions

where $\nu_i \in \mathcal{C}^1(\mathbb{R}_0^+ \to \mathbb{R}^+)$
with $s \mapsto \nu_i(s)s$ strongly monotonic
increasing and Lipschitz continuous

Nonlinear Problems

Nonlinear Potential Problem:

$$-\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] = f \quad \text{in } \Omega_i$$

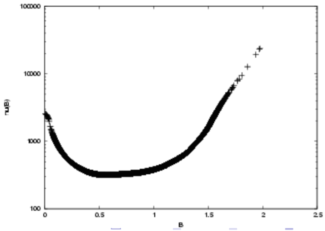
+ boundary / radiation conditions

+ transmission conditions

where $\nu_i \in \mathcal{C}^1(\mathbb{R}_0^+ \to \mathbb{R}^+)$
with $s \mapsto \nu_i(s)s$ strongly monotonic
increasing and Lipschitz continuous

e. g. ν_i determined from approximated
B-H-curve (Electromagnetics)
Jüttler/Pechstein, 2005

reluctivity ν_i



Newton's Method

$\longrightarrow$ Linearized Problem

$$- \nabla \cdot \left[\zeta_i(\nabla u^{(k)}) \nabla w^{(k)} \right] = r^{(k)} \quad \text{in } \Omega_i$$

+ boundary / radiation conditions

+ transmission conditions

Newton's Method

$\longrightarrow$ Linearized Problem

$$-\nabla\cdot\left[\zeta_i(\nabla u^{(k)})\nabla w^{(k)}\right] = r^{(k)} \quad \text{in } \Omega_i$$

+ boundary / radiation conditions

+ transmission conditions

Problem: Spectrum of $\zeta_i(\nabla u^{(k)})$

Rate of Variation: $\dfrac{\max_{x\in\Omega_i}\lambda_{\max}(\zeta_i(\nabla u^{(k)}(x)))}{\min_{x\in\Omega_i}\lambda_{\min}(\zeta_i(\nabla u^{(k)}(x)))}$

Rate of Anisotropy: $\max_{x\in\Omega_i}\dfrac{\lambda_{\max}(\zeta_i(\nabla u^{(k)}(x)))}{\lambda_{\min}(\zeta_i(\nabla u^{(k)}(x)))}$

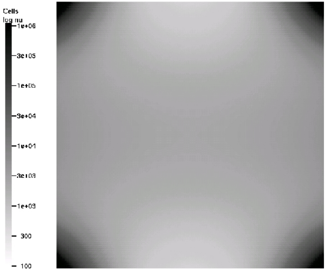
Newton's Method

$\longrightarrow$ Linearized Problem

$$-\nabla \cdot \left[\zeta_i(\nabla u^{(k)}) \nabla w^{(k)}\right] = r^{(k)} \quad \text{in } \Omega_i$$

+ boundary / radiation conditions

+ transmission conditions



reluctivity $\nu_i(|\nabla u^{(k)}|)$

Problem: Spectrum of $\zeta_i(\nabla u^{(k)})$

Rate of Variation: $\dfrac{\max_{x \in \Omega_i} \lambda_{\max}(\zeta_i(\nabla u^{(k)}(x)))}{\min_{x \in \Omega_i} \lambda_{\min}(\zeta_i(\nabla u^{(k)}(x)))}$ can be large (10^3)

Rate of Anisotropy: $\max_{x \in \Omega_i} \dfrac{\lambda_{\max}(\zeta_i(\nabla u^{(k)}(x)))}{\lambda_{\min}(\zeta_i(\nabla u^{(k)}(x)))}$ here small

Preconditioning

First Preconditioner M_1^{-1}

Find bounds $\underline{\alpha}_i$, $\overline{\alpha}_i$:

$$\underline{\alpha}_i|\xi|^2 \leq (\zeta_i(\nabla u^{(k)}(x))\,\xi,\,\xi) \leq \overline{\alpha}_i|\xi|^2 \qquad \forall x \in \Omega_i \quad \forall i \in \mathcal{I}$$

M_1^{-1} is based on Steklov Poincaré operators with the constant scalar coefficients $\overline{\alpha}_i$

Theorem

M_1^{-1} fulfills

$$\kappa(P\,M_1^{-1}P^\top P^\top F\,P) \preceq \left(\max_{i\in\mathcal{I}} \frac{\overline{\alpha}_i}{\underline{\alpha}_i}\right)\left(1+\log\left(\frac{H}{h}\right)\right)^2$$

independent of coefficient jumps.

Preconditioning

First Preconditioner M_1^{-1}

Find bounds $\underline{\alpha}_i$, $\overline{\alpha}_i$:

$$\underline{\alpha}_i |\xi|^2 \leq (\zeta_i(\nabla u^{(k)}(x))\,\xi,\,\xi) \leq \overline{\alpha}_i |\xi|^2 \qquad \forall x \in \Omega_i \quad \forall i \in \mathcal{I}$$

M_1^{-1} is based on Steklov Poincaré operators with the constant scalar coefficients $\overline{\alpha}_i$

Theorem

M_1^{-1} fulfills

$$\kappa(P\,M_1^{-1} P^\top P^\top F\,P) \preceq \left(\max_{i\in\mathcal{I}} \frac{\overline{\alpha}_i}{\underline{\alpha}_i}\right)\left(1 + \log\left(\frac{H}{h}\right)\right)^2$$

independent of coefficient jumps.

Preconditioning

Second Preconditioner M_2^{-1}

Introduce mean coefficient field

$$\hat{\alpha}_i(x) = \max_{\xi \in \mathbb{R}^d \setminus \{0\}} \frac{(\zeta_i(\nabla u^{(k)}(x))\,\xi,\,\xi)}{|\xi|^2}$$

M_2^{-1} is based on Steklov-Poincaré operators with scalar but varying coefficients $\hat{\alpha}_i(x)$. Scaling matrices $D_{\hat{\alpha},i}$ involve $\hat{\alpha}_i(x)$.

Conjecture

M_2^{-1} leads to a better condition number than M_1^{-1} with a moderate dependence on the spectral variation.

Preconditioning

Second Preconditioner M_2^{-1}

Introduce mean coefficient field

$$\hat{\alpha}_i(x) = \max_{\xi \in \mathbb{R}^d \setminus \{0\}} \frac{(\zeta_i(\nabla u^{(k)}(x))\,\xi,\, \xi)}{|\xi|^2}$$

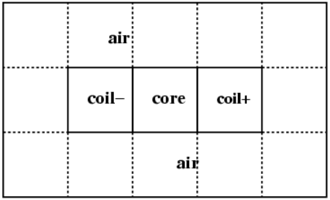
M_2^{-1} is based on Steklov-Poincaré operators with scalar but varying coefficients $\hat{\alpha}_i(x)$. Scaling matrices $D_{\hat{\alpha},i}$ involve $\hat{\alpha}_i(x)$.

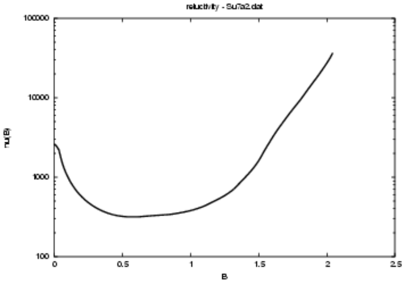
Conjecture

M_2^{-1} leads to a better condition number than M_1^{-1} with a moderate dependence on the spectral variation.

Nonlinear Problems – Numerical Results

Model problem





Nonlinear Problems – Numerical Results

$\|f\|_{L_\infty} = 2.3 \cdot 10^{-6}$ $\varepsilon_{\text{lin}} = 10^{-8}$ $\varepsilon_{\text{Newton}} = 10^{-6}$

d.o.f.	Lagr.	H/h	anis.	var.	Newt.	PCG iterations M_1^{-1}	M_2^{-1}	ref
897	178	4	6.1	13.1	2	14.5	10.0	11
3713	354	8	6.1	13.7	2	17.0	12.0	12
15105	706	16	6.1	13.8	2	19.0	13.0	13
60929	1410	32	6.1	13.9	2	19.5	15.0	14
224737	2818	64	6.1	13.9	2	20.0	15.0	14

15 subdomains, FETI

Nonlinear Problems – Numerical Results

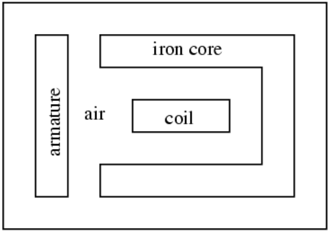
$\|f\|_{L_\infty} = 8 \cdot 10^{-6}$ $\varepsilon_{\text{lin}} = 10^{-8}$ $\varepsilon_{\text{Newton}} = 10^{-6}$

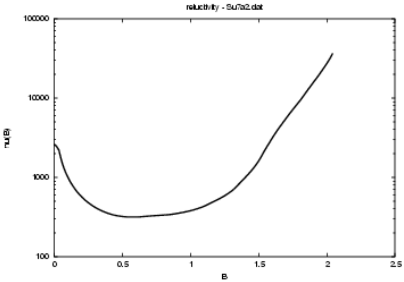
d.o.f.	Lagr.	H/h	anis.	var.	Newt.	PCG iterations M_1^{-1}	M_2^{-1}	ref
897	178	4	4.8	4.8	2	14.5	10.0	11
3713	354	8	4.9	8.5	2	16.5	12.0	12
15105	706	16	5.0	26.6	3	22.7	12.7	13
60929	1410	32	11.2	331.6	3	26.0	14.7	14
224737	2818	64	12.3	2657.1	3	88.3	16.7	14

15 subdomains, FETI

Nonlinear Problems – Numerical Results

Simplified model of a magnetic valve





Nonlinear Problems – Numerical Results

preconditioner M_2^{-1} FETI

$\varepsilon_{\text{lin}} = 10^{-8}$ $\varepsilon_{\text{Newton}} = 10^{-6}$

d.o.f.	Lagr.	H/h	anis.	var.	Newt.	PCG it.	ref
806	408	3	12.0	187.4	6	14.0	12
2539	777	6	12.0	469.9	4	17.8	14
14357	1515	12	12.0	852.4	4	21.3	17
57833	2991	24	12.0	1495.4	3	25.3	19
232145	5943	48	12.4	2670.9	4	27.8	21

70 subdomains

Nonlinear Problems – Numerical Results

preconditioner M_2^{-1} FETI/BETI

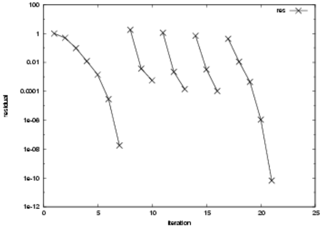
$\varepsilon_{\text{lin}} = 10^{-8}$ $\varepsilon_{\text{Newton}} = 10^{-6}$

d.o.f.	Lagr.	H/h	anis.	var.	Newt.	PCG it.	ref
806	408	3	12.0	187.4	6	13.7	11
2539	777	6	12.0	469.9	4	18.0	13
14357	1515	12	12.0	852.4	5	21.0	16
57833	2991	24	12.0	1495.4	4	24.0	18
232145	5943	48	12.4	2670.9	5	27.2	20

70 subdomains

Nonlinear Problems – Numerical Results

distribution of $\nu(|\mathbf{B}|)$ for the solution of the nonlinear problem



Newton residual (nested iteration)

Conclusion

Generalization of coupled FETI/BETI framework to

1. unbounded domains
2. varying coefficients $\longrightarrow$ nonlinear problems

Two conjectures driven from numerical results:

1. leads to weak dependence on H_Ω / H_F
2. leads to weak dependence of the rate of variation

in the condition estimate.

Acknowledgements: Many thanks to **Olaf Steinbach** and **Günther Of** for endless discussions.
Financial support of the FWF within the SFB F013.

Conclusion

Generalization of coupled FETI/BETI framework to

1. unbounded domains
2. varying coefficients $\longrightarrow$ nonlinear problems

Two conjectures driven from numerical results:

1. leads to weak dependence on H_Ω / H_F
2. leads to weak dependence of the rate of variation

in the condition estimate.

Acknowledgements: Many thanks to **Olaf Steinbach** and **Günther Of** for endless discussions.
Financial support of the FWF within the SFB F013.

Conclusion

Generalization of coupled FETI/BETI framework to

1. unbounded domains
2. varying coefficients $\longrightarrow$ nonlinear problems

Two conjectures driven from numerical results:

1. leads to weak dependence on H_Ω / H_F
2. leads to weak dependence of the rate of variation

in the condition estimate.

Acknowledgements: Many thanks to **Olaf Steinbach** and **Günther Of** for endless discussions.
Financial support of the FWF within the SFB F013.