Boundary Element Domain Decomposition Methods Challenges and Applications

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in collaboration with

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- Coupling of Finite and Boundary Element Methods
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- Symmetric Boundary Element Domain Decomposition Methods
  [Hsiao, Wendland '90; Carstensen, Kuhn, Langer '98; OS '96; ...]

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- Steklov–Poincaré Operator Domain Decomposition Methods
  [Agoshkov, Lebedev '85; Hsiao, Wendland '92; Hsiao, Schnack, Wendland '99, '00; OS '03; . . .]

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- Hybrid Domain Decomposition Methods (Mortar, Three Field, FETI)
  [Agouzal, Thomas '85; Bernardi, Maday, Patera '85; Wohlmuth '01; Brezzi, Marini '04; Farhat, Roux '91; Klawonn, Widlund '01; Toselli, Widlund '05; . . .]

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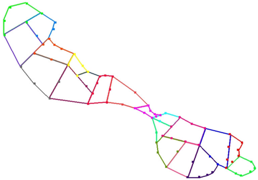
- Sparse Boundary Element Tearing and Interconnecting Methods
  [Langer, OS '03; Langer, Of, OS, Zulehner '05; Of '05]

Model Problem

$$-\mathrm{div}[\alpha(x)\nabla u(x)] \;=\; f(x) \quad \text{for } x \in \Omega, \quad u(x) \;=\; g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

Nonoverlapping Domain Decomposition



$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \quad \Gamma_i = \partial\Omega_i$$

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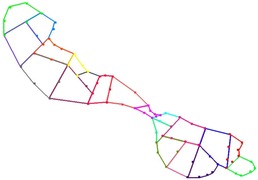

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Local Boundary Value Problems

$$-\alpha_i \Delta u_i(x) = f_i(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma$$

Transmission Boundary Conditions

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij} = \Gamma_i \cap \Gamma_j$$

Local Dirichlet Boundary Value Problem

$$-\alpha_i \Delta u_i(x) = f_i(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g_i(x) \quad \text{for } x \in \Gamma_i = \partial\Omega_i$$

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Representation Formula for $x \in \Omega_i$

$$u_i(x) = \int_{\Gamma_i} U^*(x,y) t_i(y) ds_y - \int_{\Gamma_i} g_i(y) \frac{\partial}{\partial n_y} U^*(x,y) ds_y + \frac{1}{\alpha_i} \int_{\Omega_i} U^*(x,y) f_i(y) dy$$

Fundamental Solution

$$U^*(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}, \qquad t_i(y) = \frac{\partial}{\partial n_y} u_i(y), \; y \in \Gamma_i$$

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Boundary Integral Equation for $x \in \Gamma_i$

$$\frac{1}{4\pi} \int_{\Gamma_i} \frac{t_i(y)}{|x-y|} ds_y = \frac{1}{2} g_i(x) + \frac{1}{4\pi} \int_{\Gamma_i} g_i(y) \frac{\partial}{\partial n_i(y)} \frac{1}{|x-y|} ds_y - \frac{1}{\alpha_i} \frac{1}{4\pi} \int_{\Omega_i} \frac{f(y)}{|x-y|} dy$$

Boundary Integral Equation for $x \in \Gamma_i$

$$(V_i t_i)(x) \;=\; \frac{1}{2} g_i(x) + (K_i g_i)(x) - \frac{1}{\alpha_i}(N_{0,i} f_i)(x)$$

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Single Layer Potential

$$V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i), \; \langle V_i w_i, w_i \rangle_{\Gamma_i} \geq c_1^{V_i} \| w_i \|^2_{H^{-1/2}(\Gamma_i)}, \; n = 2 : \operatorname{diam} \Omega_i < 1$$

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Dirichlet to Neumann Map

$$t_i = V_i^{-1}(\frac{1}{2} I + K_i) g_i - \frac{1}{\alpha_i} V_i^{-1} N_{0,i} f_i$$

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Steklov–Poincaré Operator

$$S_i = V_i^{-1}(\frac{1}{2} I + K_i)$$

Representation Formula for $x \in \Omega_i$

$$u_i(x) = \int_{\Gamma_i} U^*(x,y)t_i(y)ds_y - \int_{\Gamma_i} g_i(y)\frac{\partial}{\partial n_y}U^*(x,y)ds_y + \frac{1}{\alpha_i}\int_{\Omega_i} U^*(x,y)f_i(y)dy$$

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Computation of the Normal Derivative

$$\begin{aligned} t_i(x) &= \frac{1}{2}t_i(x) + \int_{\Gamma_i} \frac{\partial}{\partial n_x}U^*(x,y)t_i(y)ds_y - \frac{\partial}{\partial n_x}\int_{\Gamma_i} g_i(y)\frac{\partial}{\partial n_y}U^*(x,y)ds_y \\ &\quad + \frac{1}{\alpha_i}\frac{\partial}{\partial n_x}\int_{\Omega_i} U^*(x,y)f_i(y)dy \end{aligned}$$

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Hypersingular Boundary Integral Equation for $x \in \Gamma_i$

$$t_i(x) = \frac{1}{2} t_i(x) + (K_i' t_i)(x) + (D_i g_i)(x) + \frac{1}{\alpha_i} (N_{1,i} f_i)(x)$$

Dirichlet to Neumann Map

$$t_i = D_i g_i + (\frac{1}{2} I + K_i') t_i + \frac{1}{\alpha_i} N_{1,i} f_i$$

Dirichlet to Neumann Map

$$\begin{aligned} t_i &= D_i g_i + (\frac{1}{2} I + K_i') t_i + \frac{1}{\alpha_i} N_{1,i} f_i \\ &= D_i g_i + (\frac{1}{2} I + K_i') [V_i^{-1} (\frac{1}{2} I + K_i) g_i - \frac{1}{\alpha_i} V_i^{-1} N_{0,i} f_i] + \frac{1}{\alpha_i} N_{1,i} f_i \end{aligned}$$

Dirichlet to Neumann Map

$$
\begin{aligned}
t_i &= D_i g_i + (\frac{1}{2} I + K_i') t_i + \frac{1}{\alpha_i} N_{1,i} f_i \\
&= D_i g_i + (\frac{1}{2} I + K_i') [V_i^{-1} (\frac{1}{2} I + K_i) g_i - \frac{1}{\alpha_i} V_i^{-1} N_{0,i} f_i] + \frac{1}{\alpha_i} N_{1,i} f_i \\
&= S_i g_i - \frac{1}{\alpha_i} N_i f_i
\end{aligned}
$$

Steklov–Poincaré Operator

$$
S_i = V_i^{-1} (\frac{1}{2} I + K_i) = D_i + (\frac{1}{2} I + K_i') V_i^{-1} (\frac{1}{2} I + K_i) = \ldots
$$

Dirichlet to Neumann Map

$$
\begin{aligned}
t_i &= D_i g_i + (\frac{1}{2}I + K_i')t_i + \frac{1}{\alpha_i}N_{1,i}f_i \\
&= D_i g_i + (\frac{1}{2}I + K_i')[V_i^{-1}(\frac{1}{2}I + K_i)g_i - \frac{1}{\alpha_i}V_i^{-1}N_{0,i}f_i] + \frac{1}{\alpha_i}N_{1,i}f_i \\
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- Mapping Properties of $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$
- Definition of S_i via Domain Variational Formulation (FEM)

Boundary Integral Equations (Calderon Projector) for $f = 0$

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix}$$

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Corollary [Plemelj 1911]

$$K_i V_i = V_i K_i', \quad V_i D_i = \frac{1}{4} I - K_i^2$$

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Theorem [OS, Wendland 2001]

$$\|(\frac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \leq c_K(\Gamma_i)\,\|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i)$$

with

$$c_K(\Gamma_i) = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^{V_i} c_1^{D_i}} < 1$$

shape sensitive

Theorem

$$\|S_i v_i\|_{V_i} \leq c_K(\Gamma_i)\, \|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i)$$

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$$\langle S_i v_i, v_i\rangle_{\Gamma_i} \geq [1 - c_K(\Gamma_i)]\,\|v_i\|^2_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i), v_i \perp 1$$

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Define

$$\langle \widetilde{S}_i u_i, v_i\rangle_{\Gamma_i} = \langle S_i u_i, v_i\rangle_{\Gamma_i} + \beta_i \langle u_i, w_{\mathrm{eq,i}}\rangle_{\Gamma_i} \langle v_i, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}, \qquad V_i w_{\mathrm{eq,i}} = 1$$

Theorem

$$\|S_i v_i\|_{V_i} \le c_K(\Gamma_i)\, \|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i)$$

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Theorem

$$c_1^{\widetilde{S}_i} \langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i} \le \langle \widetilde{S}_i v_i, v_i\rangle_{\Gamma_i} \le c_2^{\widetilde{S}_i} \langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i}$$

with

$$c_1^{\widetilde{S}_i} = \min\{1 - c_K(\Gamma_i), \beta_i \langle 1, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}\}, \qquad c_2^{\widetilde{S}_i} = \max\{c_K(\Gamma_i), \beta_i \langle 1, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}\}$$

Theorem

$$\|S_i v_i\|_{V_i} \;\le\; c_K(\Gamma_i)\,\|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i)$$

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$$c_1^{\widetilde{S}_i}\,\langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i} \;\le\; \langle \widetilde{S}_i v_i, v_i\rangle_{\Gamma_i} \;\le\; c_2^{\widetilde{S}_i}\,\langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i}$$

with

$$c_1^{\widetilde{S}_i} = \min\{1 - c_K(\Gamma_i), \beta_i \langle 1, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}\}, \qquad c_2^{\widetilde{S}_i} = \max\{c_K(\Gamma_i), \beta_i \langle 1, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}\}$$

Optimal Scaling

$$\beta_i = \frac{1}{2\langle 1, w_{\mathrm{eq,i}}\rangle_{\Gamma_i}}$$

Dirichlet to Neumann Map (Partial Differential Equation)

$$\alpha_i t_i(x) = \alpha_i (S_i u_i)(x) - (N_i f_i)(x) \quad \text{for } x \in \Gamma_i$$

Dirichlet Boundary Condition

$$u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma$$

Transmission Conditions

$$u_i(x) = u_j(x), \quad \alpha_i t_i(x) + \alpha_j t_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}$$

Dirichlet to Neumann Map (Partial Differential Equation)

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Transmission Conditions

$$u_i(x) = u_j(x), \quad \alpha_i t_i(x) + \alpha_j t_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}$$

Dirichlet Domain Decomposition Approach

Find $u \in H^{1/2}(\Gamma_S)$ such that $u(x) = g(x)$ for $x \in \Gamma$ and

$$\alpha_i (S_i u_{|\Gamma_i})(x) + \alpha_j (S_j u_{|\Gamma_j})(x) = (N_i f_i)(x) + (N_j f_j)(x) \quad \text{for } x \in \Gamma_{ij}$$

Variational Problem

Find $u \in H^{1/2}(\Gamma_S)$ such that $u(x) = g(x)$ for $x \in \Gamma$ and

$$\sum_{i=1}^{p} \alpha_i \int_{\Gamma_i} (S_i u_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} (N_i f_i)(x) v_{|\Gamma_i}(x) ds_x$$

for all $v \in H^{1/2}(\Gamma_S)$, $v(x) = 0$ for $x \in \Gamma$

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for all $v \in H^{1/2}(\Gamma_S)$, $v(x) = 0$ for $x \in \Gamma$

Galerkin Variational Problem

Find $u_{0,h} \in S_h^1(\Gamma_S) \subset H_0^{1/2}(\Gamma_S)$ such that

$$\sum_{i=1}^{p} \alpha_i \int_{\Gamma_i} (S_i u_{0,h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} [(N_i f_i)(x) - \alpha_i (S_i u_g)(x)] v_{|\Gamma_i}(x) ds_x$$

for all $v_h \in S_h^1(\Gamma_S)$, where u_g is some bounded extension of g.

Linear System

$$\sum_{i=1}^{p} \alpha_i A_i^\top S_{i,h} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top \underline{f}_i, \quad S_{i,h}[\ell, k] = \langle S_i \varphi_k^i, \varphi_\ell^i \rangle_{\Gamma_i}$$

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Non–Symmetric BEM Approximation

$$S_i u_i = V_i^{-1}(\frac{1}{2} I + K_i) u_i = w_i : \langle V_i w_i, \tau_i \rangle_{\Gamma_i} = \langle (\frac{1}{2} I + K_i) u_i, \tau_i \rangle_{\Gamma_i}$$

Linear System

$$\sum_{i=1}^{p} \alpha_i A_i^\top S_{i,h} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top \underline{f}_i, \quad S_{i,h}[\ell, k] = \langle S_i \varphi_k^i, \varphi_\ell^i \rangle_{\Gamma_i}$$

Non–Symmetric BEM Approximation

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$$w_{i,h} \in S_h^0(\Gamma_i) : \langle V_i w_{i,h}, \tau_{i,h} \rangle_{\Gamma_i} = \langle (\frac{1}{2} I + K_i) u_i, \tau_{i,h} \rangle_{\Gamma_i}$$

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Stability Condition

$$c_S \, \|v_{i,h}\|_{H^{1/2}(\Gamma_i)} \le \sup_{0 \ne \tau_{i,h} \in S_h^0(\Gamma_i)} \frac{\langle v_{i,h}, \tau_{i,h} \rangle_{\Gamma_i}}{\|\tau_{i,h}\|_{H^{-1/2}(\Gamma_i)}} \quad \text{for all } v_{i,h} \in S_h^1(\Gamma_i)$$

Symmetric BEM Approximation

$$S_i u_i = D_i u_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)u_i = D_i u_i + (\frac{1}{2}I + K_i')w_i$$

where

$$\langle V_i w_i, \tau_i \rangle_{\Gamma_i} = \langle (\frac{1}{2}I + K_i)u_i, \tau_i \rangle_{\Gamma_i}$$

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Stability

$$\langle \widehat{S}_i v_i, v_i \rangle_{\Gamma_i} \geq \langle D_i v_i, v_i \rangle_{\Gamma_i} \geq c_1^{D_i} |v_i|^2_{H^{1/2}(\Gamma_i)}$$

Symmetric BEM Approximation

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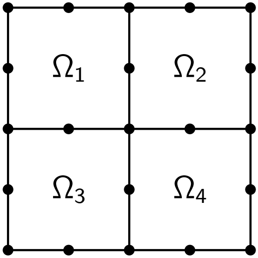
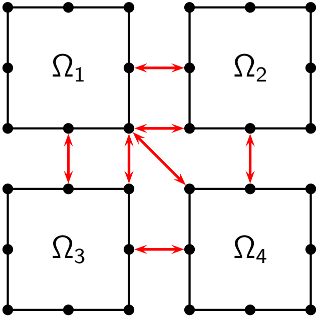
FEM Approximation

$$\widehat{S}_{i,h} = K_{C_i C_i} - K_{C_i I_i}^\top K_{I_i I_i}^{-1} K_{C_i I_i}$$

Tearing and Interconnecting



Tearing and Interconnecting



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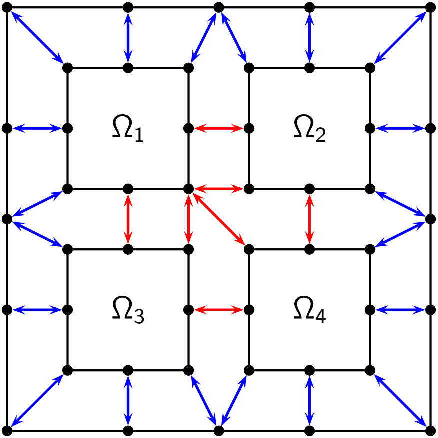

All–Floating BETI [Of '05]

Total FETI [Dostal et. al. '05]

Linear System

$$\begin{pmatrix} \alpha_1 \widehat{S}_{1,h} & & & -B_1^\top \\ & \ddots & & \vdots \\ & & \alpha_p \widehat{S}_{p,h} & -B_p^\top \\ B_1 & \ldots & B_p & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{g} \end{pmatrix}$$

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Local System

$$\alpha_i \widehat{S}_{i,h} \underline{u}_i = \underline{f}_i + B_i^\top \underline{\lambda}, \qquad \widetilde{S}_{i,h} \underline{1} = \underline{0}$$

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Solvability Condition

$$(\underline{f}_i + B_i^\top \underline{\lambda}, \underline{1}) = 0$$

Linear System

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Extended Linear System

$$\alpha_i \widetilde{S}_{i,h} \underline{u}_i = \alpha_i [\widehat{S}_{i,h} + \beta_i \underline{a}_i \underline{a}_i^\top] \underline{u}_i = \underline{f}_i + B_i^\top \underline{\lambda}, \qquad a_{i,k} = \langle \varphi_k^i, w_{\text{eq,i}} \rangle_{\Gamma_i}$$

Linear System

$$\begin{pmatrix} \alpha_1 \widehat{S}_{1,h} & & & -B_1^\top \\ & \ddots & & \vdots \\ & & \alpha_p \widehat{S}_{p,h} & -B_p^\top \\ B_1 & \dots & B_p & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{g} \end{pmatrix}$$

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Solution

$$\underline{u}_i = \frac{1}{\alpha_i} \widetilde{S}_{i,h}^{-1} [\underline{f}_i + B_i^\top \underline{\lambda}] + \gamma_i \underline{1}, \quad (\underline{f}_i + B_i^\top \underline{\lambda}, \underline{1}) = 0$$

Dual Problem

$$\sum_{i=1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}_i^{-1} B_i^\top \underline{\lambda} + \sum_{i=1}^{p} \gamma_i B_i \underline{1} = - \sum_{i=1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}_{i,h}^{-1} \underline{f}_i, \quad (\underline{f}_i + B_i^\top \underline{\lambda}, \underline{1}) = 0$$

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Projection

$$P^\top = I - G(G^\top G)^{-1} G^\top, \quad P^\top G \underline{\gamma} = \underline{0}$$

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Preconditioner

$$C_F \sim F = \sum_{i=1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}_i^{-1} B_i^\top$$

Dual Problem

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$$C_F \sim F = \sum_{i=1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}_i^{-1} B_i^\top$$

Scaled Hypersingular BETI Preconditioner [Langer, OS '03]

$$C_F^{-1} = (BC_\alpha^{-1} B^\top)^{-1} BC_\alpha^{-1} D_h C_\alpha^{-1} B^\top (BC_\alpha^{-1} B^\top)^{-1}$$

Coupled Linear System

$$
\begin{pmatrix}
\alpha_1 V_{1,h} & & & -\alpha_1 \widetilde{K}_{1,h} & & & \\
& \ddots & & & \ddots & & \\
& & \alpha_p V_{p,h} & & & -\alpha_p \widetilde{K}_{p,h} & \\
\alpha_1 \widetilde{K}_{1,h}^\top & & & \alpha_1 \widetilde{D}_{1,h} & & & -B_1^\top \\
& \ddots & & & \ddots & & \vdots \\
& & \alpha_p \widetilde{K}_{p,h}^\top & & & \alpha_p \widetilde{D}_{p,h} & -B_p^\top \\
& & & B_1 & \dots & B_p &
\end{pmatrix}
\begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix}
=
\begin{pmatrix} \underline{f}_{0,1} \\ \vdots \\ \underline{f}_{0,p} \\ \underline{f}_{1,1} \\ \vdots \\ \underline{f}_{1,p} \\ \underline{g} \end{pmatrix}
$$

Coupled Linear System

$$\begin{pmatrix} \alpha_1 V_{1,h} & & & -\alpha_1 \widetilde{K}_{1,h} & & & \\ & \ddots & & & \ddots & & \\ & & \alpha_p V_{p,h} & & & -\alpha_p \widetilde{K}_{p,h} & \\ \alpha_1 \widetilde{K}_{1,h}^\top & & & \alpha_1 \widetilde{D}_{1,h} & & & -B_1^\top \\ & \ddots & & & \ddots & & \vdots \\ & & \alpha_p \widetilde{K}_{p,h}^\top & & & \alpha_p \widetilde{D}_{p,h} & -B_p^\top \\ & & & B_1 & \dots & B_p & \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_{0,1} \\ \vdots \\ \underline{f}_{0,p} \\ \underline{f}_{1,1} \\ \vdots \\ \underline{f}_{1,p} \\ \underline{g} \end{pmatrix}$$

Twofold Saddle Point Problem

$\rightarrow$ Transformation into positive definite symmetric system

[Bramble, Pasciak '88; Zulehner '02; Langer, Of, OS, Zulehner '05]

$\rightarrow$ Scaling of Preconditioners is needed!

Preconditioner for Discrete Steklov–Poincaré Operator: $C_{S_i} \sim \widetilde{S}_{i,h}$

- Single Layer Potential [OS, Wendland '95, '98]

$$C_{S_i}^{-1} = \bar{M}_{i,h}^{-1} \bar{V}_{i,h} \bar{M}_{i,h}^{-1}$$

- Geometric/Algebraic Multigrid for discrete Hypersingular Integral Operator $D_{i,h} \sim \widetilde{S}_{i,h}$

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Preconditioner for Discrete Single Layer Potential

- Geometric Multigrid/Multilevel [Maischak, Stephan, Tran,...]
- Artificial Multilevel Preconditioning [OS '03]
- Algebraic Multigrid for Sparse BEM [Langer, Pusch '05; Of '05]

Galerkin Discretisation of Boundary Integral Operators

- non–local kernels $\longrightarrow$ dense stiffness matrices
- singular fundamental solution $\longrightarrow$ singular surface integrals
- integration by parts $\longrightarrow$ weakly singular surface integrals only

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Fast Boundary Element Methods

- Fast Multipole Algorithm [Greengard, Rokhlin '87; . . .]
- Panel Clustering [Hackbusch, Nowak '89; . . .]
- Adaptive Cross Approximation [Bebendorf, Rjasanow '03; . . .]
- Hierarchical Matrices [Hackbusch '99; . . .]
- Wavelets [Dahmen, Prößdorf, Schneider '93; . . .]

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Complexity

$$\mathcal{O}(N_i \log^q N_i) \quad (\text{Fast Multipole: } q = 2)$$

Theorem

Requirement: algebraic multigrid preconditioner for $V_{i,h}$ is optimal.
Solving by Bramble Pasciak transformation and conjugate gradient method:

Twofold saddle point problem of the standard BETI method:

- number of iterations $\mathcal{O}((1 + \log(H/h))^2)$
- $\mathcal{O}((H/h)^2(1 + \log(H/h))^4)$ arithmetical operations

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Twofold saddle point problem of the all–floating BETI method:

- number of iterations $\mathcal{O}(1+\log(H/h))$
- $\mathcal{O}((H/h)^2(1+\log(H/h))^3)$ arithmetical operations

Theorem

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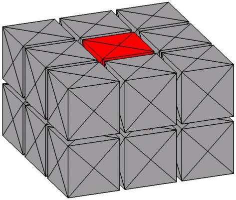
The use of fast boundary element methods does not perturb the convergence rates of the approximation.

Example: Linear elasticity (steel and concrete)



18 subdomains

	BETI		all–floating	
L	t_2	It.	t_2	It.
0	31	19(21(10))	39	22(17(10))
1	217	28(33(14))	170	24(27(14))
2	2129	35(44(14))	1437	27(33(14))
3	14149	42(51(14))	9005	32(36(14))
4	116404	47(54(14))	77111	38(38(15))

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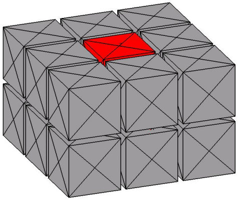
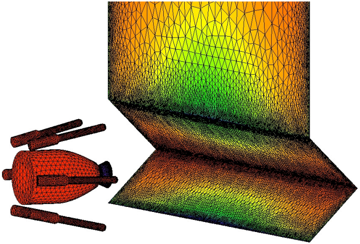
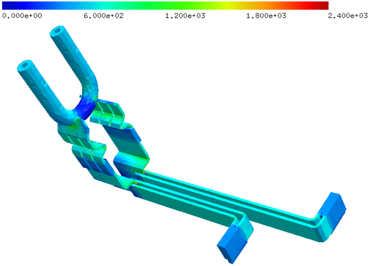


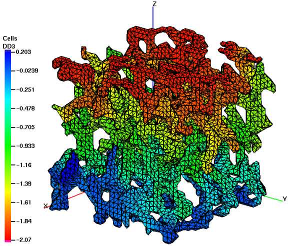
18 subdomains

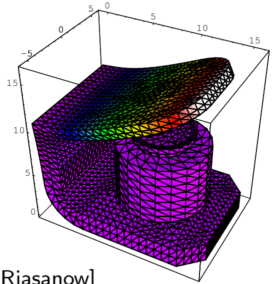
L	BETI t_2	BETI It.	all–floating t_2	all–floating It.
0	31	19(21(10))	39	22(17(10))
1	217	28(33(14))	170	24(27(14))
2	2129	35(44(14))	1437	27(33(14))
3	14149	42(51(14))	9005	32(36(14))
4	116404	47(54(14))	77111	38(38(15))

L	N_i	Dirichlet DD t_2	Dirichlet DD It.	BETI t_2	BETI It.	all–floating t_2	all–floating It.
0	24	7	53(10)	7	78	8	65
1	96	25	110(14)	19	100	19	82
2	384	181	130(14)	112	114	115	85
3	1536	986	148(14)	562	129	476	95
4	6144	6902	154(14)	4352	153	3119	105
5	24576	59264	166(16)	31645	172	23008	120









[Courtesy to Z. Andjelic, H. Andrä, J. Breuer, G. Of, S. Rjasanow]

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