Schwarz waveform relaxation algorithms : theory and applications

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DD17. July 2006

Outline

Coupling process

Issues

♢ For a given problem, split the domain : domain decomposition.

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- ♢ For a given problem, different numerical methods in different zones : FEM/FD, SM/FEM, AMR.
- ♢ Couple two different models in different zones.
- ♢ Furthermore the codes can be on distant sites.

DDM for evolution problems

Usual methods

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◇ Explicit + interpolation − > exchange of information every time step
− > time consuming, possibly unstable for hyperbolic problems.

◇ Implicit − > uniform time step.

DDM for evolution problems

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- Different time and space steps in different subdomains,
- Different models in different subdomains,
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- Easy to use, fast and cheap.

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- Work on the PDE level, globally in time,
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Optimized Schwarz Waveform Relaxation

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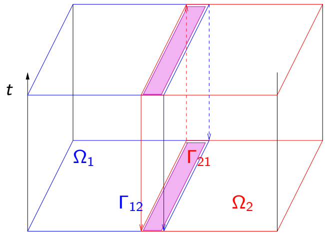
The Schwarz algorithm

$$\mathcal{L}u := \partial_t u + a\partial_x u + (\mathbf{b}\cdot\nabla)u - \nu\Delta u + cu \text{ in } \Omega\times(0,T)$$

$$\nu > 0.$$



$$\begin{cases} \mathcal{L}u_1^{k+1} = f & \text{in } \Omega_1\times(0,T) \\ u_1^{k+1}(\cdot,0) = u_0 & \text{in } \Omega_1 \\ \mathcal{B}_1 u_1^{k+1} = \mathcal{B}_1 u_2^k & \text{on } \Gamma_{12}\times(0,T) \end{cases}$$

$$\begin{cases} \mathcal{L}u_2^{k+1} = f & \text{in } \Omega_2\times(0,T) \\ u_2^{k+1}(\cdot,0) = u_0 & \text{in } \Omega_2 \\ \mathcal{B}_2 u_2^{k+1} = \mathcal{B}_2 u_1^k & \text{on } \Gamma_{21}\times(0,T) \end{cases}$$

How to choose the transmission operators ?

Transmission conditions

$$\mathcal{B}_1 u_1^{k+1} = \mathcal{B}_1 u_2^k \text{ on } \Gamma_{12} \times (0, T), \qquad \mathcal{B}_2 u_2^{k+1} = \mathcal{B}_2 u_1^k \text{ on } \Gamma_{21} \times (0, T)$$

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Classical Schwarz

$\mathcal{B}_j \equiv I$ AND overlap.

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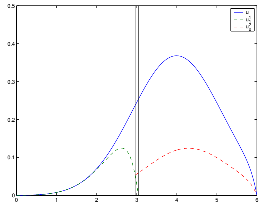
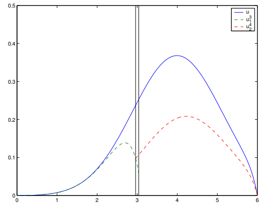
1D Numerical experiment

$$a = 1, \nu = 0.2, \Omega = (0, 6), T = 2.5, L = 0.08.$$

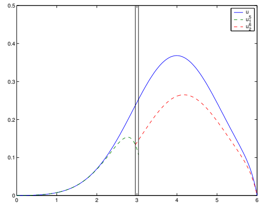
$u_1^1(\cdot, T), u_2^2(\cdot, T)$



$u_1^3(\cdot, T), u_2^4(\cdot, T)$



$u_1^5(\cdot, T), u_2^6(\cdot, T)$



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$\mathcal{B}_j \equiv I$ AND overlap.

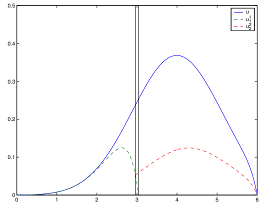
Optimized Schwarz Waveform relaxation

$\mathcal{B}_j \equiv$ absorbing boundary operator+optimization WITH OR WITHOUT overlap

How to choose the transmission operators ?

Comparison

$u_1^1(\cdot, T), u_2^2(\cdot, T)$



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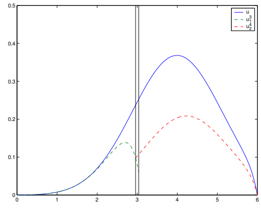

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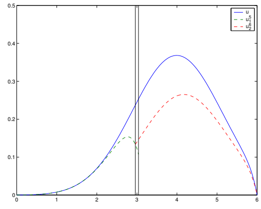

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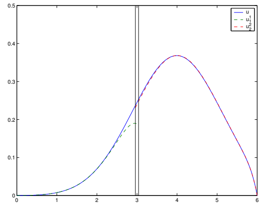

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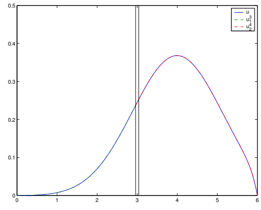

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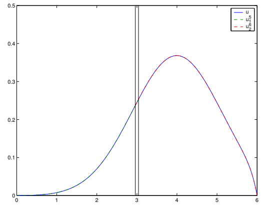

The optimal SWR algorithm

$$\Omega_1 = (-\infty, L) \times \mathbb{R}^n, \quad \Omega_2 = (0, \infty) \times \mathbb{R}^n.$$

$$\mathcal{B}_j \equiv \partial_x + S_j(\partial_t, \partial_y)$$

$a > 0$, Fourier transform $t \leftrightarrow \omega, y \leftrightarrow \kappa$

$$S_1(i\omega, i\kappa) = \frac{\delta^{1/2} - a}{2\nu}, S_2(i\omega, i\kappa) = \frac{\delta^{1/2} + a}{2\nu}.$$

$$\delta(\omega, k) = a^2 + 4\nu((i(\omega + \mathbf{b} \cdot \mathbf{k}) + \nu|k|^2 + c)$$

Convergence in 2 iterations (I if I subdomains).

Two options :

- Use the optimal transmission condition (easier in 1D)

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- Use the optimal transmission condition (easier in 1D)
- Approximate the optimal $->$ optimized transmission conditions

Design of approximate SWR algorithms

boundary operators

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$$\tilde{S}_1(i\omega, i\kappa) = \frac{P - a}{2\nu}, P(\omega, k) = \mathbf{p} + \mathbf{q}(i(\omega + \mathbf{b} \cdot \mathbf{k}) + \nu |k|^2), (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^2.$$

$$\mathcal{B}_1 u := \partial_x u - \frac{a - \mathbf{p}}{2\nu} u + \mathbf{q}(\partial_t + \mathbf{b} \cdot \nabla u - \nu \Delta_y u)$$

Well-posedness and convergence

Transmission conditions

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Convergence factor

$$\rho(\omega, k, P, L) = \left(\frac{P-\delta^{1/2}}{P+\delta^{1/2}}\right)^2 e^{-2\delta^{1/2}L/\nu}$$

$$\widehat{e_j^{k+2}}(\omega, 0, k) = \rho(\omega, k, P, L)\widehat{e_j^k}(\omega, 0, k)$$

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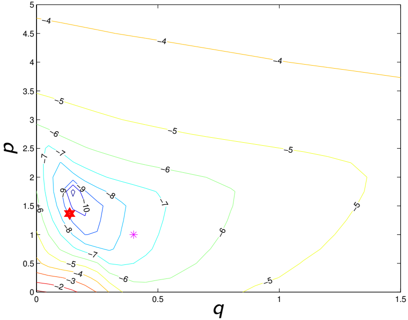
THEOREM

For $p, q > 0$, $p > \frac{a^2}{4\nu} q$, the algorithm is well-posed in suited Sobolev spaces and converges with and without overlap.

One dimension : influence of the parameters

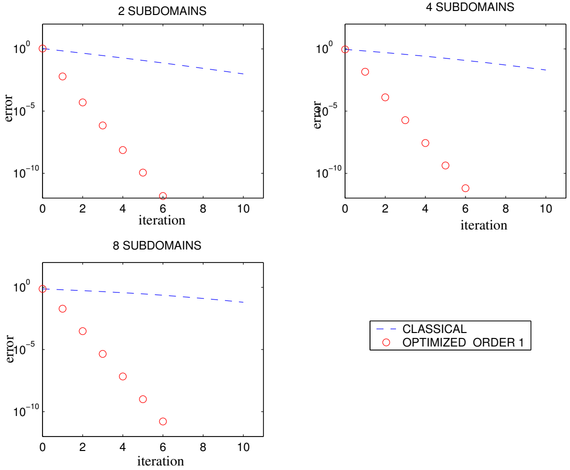


Error obtained running the algorithm with first order transmission conditions for 5 steps and various choices of p and q.

p^*, q^* : theoretical values ,

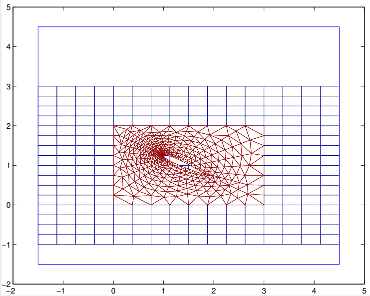
p^*, q^* : Taylor approximations.

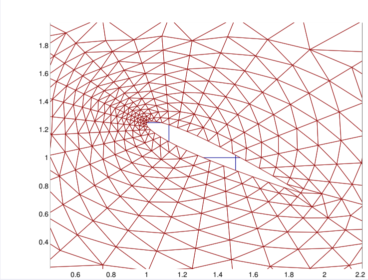
One dimension : comparison



Two dimensions : coupling different numerical methods

The heat bubble hitting an airfoil





Evolution of a heat bubble around an airfoil.

Coupling through `Corba`, *"Common Object Request Broker Architecture"*.

Two dimensions : coupling different numerical methods

Programming

- F.E in Ω_1, $F.D$ in Ω_2,
- Write the interface problem,
- solve by Krylov,

Results for a time window=10 timesteps

the steady algorithm is :

```
do time iterations 1 :N
 do Krylov iterations
 with preconditioning
  residual vectors =
   size of interface
```

15 iterations $\times 10$.

the unsteady algorithm is :

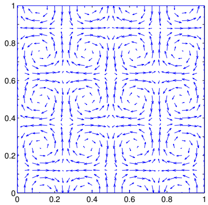
```
 do Krylov iterations
do time iterations 1 :N
 residual vectors =
size of interface x N
```

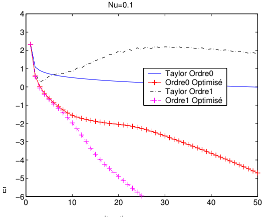
100 iterations.

P.d'Anfray, J. Ryan,L.H. M2AN 2002

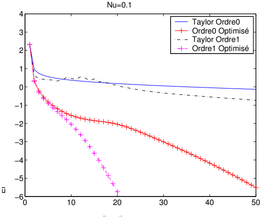
Robustness : rotating velocities

$$a(x,y) = 0.32\pi \sin(4\pi x)\sin(4\pi y),$$
$$b(x,y) = 0.32\pi \cos(4\pi y)\cos(4\pi x).$$

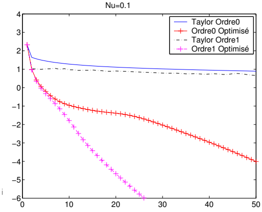




interface 0.3



interface 0.4



interface 0.5

Optimization of the convergence factor

$$\delta(z) = a^2 + 4\nu c + 4\nu z, z = i(\omega + \mathbf{b} \cdot \mathbf{k}) + \nu |k|^2$$

$$\rho(z, P, L) = \left(\frac{P(z) - \delta^{1/2}(z)}{P(z) + \delta^{1/2}(z)} \right)^2 e^{-2\delta^{1/2} L}$$

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- Best approximation

$$\inf_{P \in \mathbb{P}_n} \sup_{z \in K} |\rho(z, P, L)|, \quad K = (\frac{\pi}{T}, \frac{\pi}{\Delta t}), k_j \in (\frac{\pi}{X_j}, \frac{\pi}{\Delta x_j})$$

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THEOREM

For any n, for $L = 0$ or sufficiently small, the problem has a unique solution characterized by an equioscillation property.

Asymptotic results

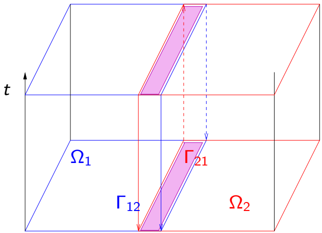
Example : overlapping case, $L \approx C\Delta x$

- Dirichlet transmission conditions : $|\rho| \approx 1 - \alpha\Delta x$,
- Taylor approximation : $|\rho| \approx 1 - \beta\sqrt{\Delta x}$,
- Optimization : $p \approx C_p \Delta x^{-\frac{1}{5}}$, $q \approx C_q \Delta x^{\frac{3}{5}}$, $|\rho| \approx 1 - O(\Delta x^{\frac{1}{5}})$.

Outline

Schwarz Waveform relaxation algorithm

$$\mathcal{L}u := u_{tt} - c^2 \Delta u, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^m$$



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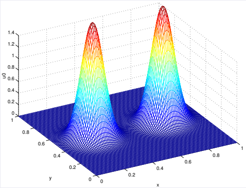
A numerical experiment

Data

$c = 1, \ T = 1,$
$\Omega = (0,1) \times (0,1).$
Two subdomains, overlap
$L = 0.08.$



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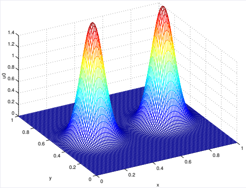
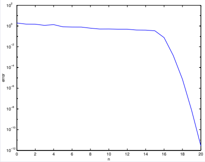


Convergence history : Dirichlet transmission conditions with overlap



Convergence after $n > \frac{cT}{L} = 12$ iterations

Other transmission conditions

General transmission operators

$$\mathcal{B}_1 = \prod_{j=1}^{J}(\partial_x + \alpha_j \partial_t), \mathcal{B}_2 = \prod_{j=1}^{J}(\partial_x - \alpha_j \partial_t).$$

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Plane waves analysis

$$e_1^k = a_1^k(\omega, k)e^{\sigma(x-L)}, e_2^k = a_2^k(\omega, k)e^{\sigma x}.$$

$$\sigma = \begin{cases} \frac{|\omega|}{c}\sqrt{\left(\frac{ck}{\omega}\right)^2 - 1}, & \text{evanescent waves,} \\ \frac{i\omega}{c}\sqrt{1 - \left(\frac{ck}{\omega}\right)^2}, & \text{propagating waves.} \end{cases}$$

$$|\rho| = \begin{cases} e^{-L\frac{|\omega|}{c}\sqrt{\left(\frac{ck}{\omega}\right)^2 - 1}}, & \text{evanescent waves,} \\ \prod_{j=1}^{J}\left|\frac{\alpha_j - \sqrt{1 - \left(\frac{ck}{\omega}\right)^2}}{\alpha_j + \sqrt{1 - \left(\frac{ck}{\omega}\right)^2}}\right| & \text{propagating waves.} \end{cases}$$

Plane wave analysis : continue

Convergence factor, propagating case

θ angle of incidence on the interface, $\sin\theta = \frac{ck}{\omega}$.

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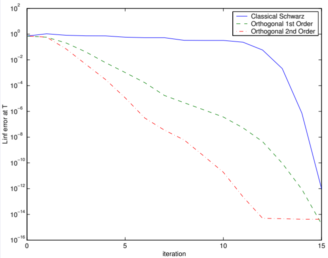
Strategy 1 : orthogonal absorption $\alpha = 1$



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Given *eps*, find n and $\alpha(n)$ such that

1. The overlap takes care of the wide angles $\theta \geq \theta_{max}(n) = \arccos(\frac{nL}{cT})$,

Plane wave analysis : continue

Convergence factor, propagating case

θ angle of incidence on the interface, $\sin\theta = \frac{ck}{\omega}$.

$$|\rho| = \prod_{j=1}^{J} \left| \frac{\alpha_j - \sqrt{1 - \left(\frac{ck}{\omega}\right)^2}}{\alpha_j + \sqrt{1 - \left(\frac{ck}{\omega}\right)^2}} \right| = \prod_{j=1}^{J} \left| \frac{\alpha_j - \cos\theta}{\alpha_j + \cos\theta} \right|.$$

Strategy 2 : optimization

Given *eps*, find n and $\alpha(n)$ such that

1. The overlap takes care of the wide angles $\theta \geq \theta_{max}(n) = \arccos(\frac{nL}{cT})$,
2. the convergence rate ρ is optimized by $\rho(\theta_{max}(n))^n < eps$.

Comparison

Example : $eps = 10^{-2}$

First order : $n = 3.7459 \approx 3 - 4$,$\theta_{\max} \approx 73^o$.
Second order : $n = 1.9540 \approx 2$,$\theta_{\max} \approx 81^o$.

Iteration	0	1	2	3	4	5
Dirichlet	0.7059	1.0555	0.8146	0.7340	0.7321	0.5760
Orthogonal O1	0.7059	0.5793	0.2035	0.0413	0.0061	0.0010
Optimized O1	0.7059	0.4403	0.1132	0.0216	0.0062	0.0018
Orthogonal O2	0.7059	0.5853	0.0701	0.0045	0.0003	0.0000
Optimized O2	0.7059	0.5847	0.0415	0.0099	0.0030	0.0004

Theoretical results

Continuous level

- Well-posedness of the best approximation problems (explicit),
- Well-posedness of the subdomain problems (Kreiss theory),
- Convergence of the algorithm (Fourier analysis, “à la” Engquist-Majda).

Discrete level

- Discretization by finite volumes schemes,
- Well-posedness of the discrete algorithm,1D case.
- Convergence of the discrete algorithm (Fourier analysis + energy estimates) also nonconforming discretization in time. 1D case.
- Error estimates for non conforming grids in time.

Outline

Parabolic problems

- 1D theoretical analysis (M. Gander and L.H.)
- 2D with non constant velocity (V. Martin)
- Shallow water (V. Martin)
- Non conformal coupling (M.G., L.H., C. Japhet and M. Kern)

Hyperbolic problems

- 1D heterogeneous (M. Gander and L.H.) optimal SWR.
- 2D homogeneous overlapping SWR (M. Gander and L.H.)
- 1D Mesh refinement,
- Nonoverlapping SWR in 2D (M. Gander and L.H.)
- Nonlinear waves in 1D (L.H and J. Szeftel),

Mixed

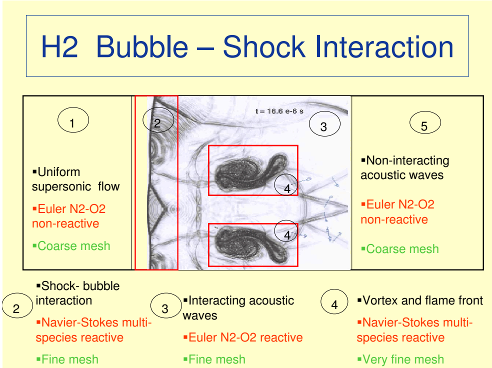
- coupling a large scale oceanic model and a coastal model,
- coupling Euler and Navier-Stokes in an AMR frame.
- coupling ocean and atmosphere models.

Collaborators

- Mostly : M. Gander (Université Genève).
- 1D wave equation : F. Nataf (CNRS P6).
- 2D advection-diffusion : P. D'Anfray et J. Ryan (ONERA). V. Martin (Amiens).
- Heterogeneous problems (application to oceanography) : C. Japhet (P13), M. Kern (INRIA), E. Blayo (Grenoble).
- Schrödinger equation and non linear models : J. Szeftel.
- Application to micromagnetism : S. Labbé (P11) et K. Santugini(Genève)

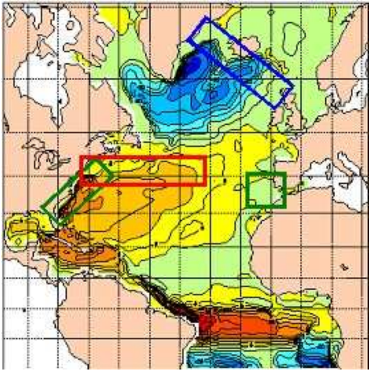
http ://www.math.univ-paris13.fr/ halpern See MS M04 today at 4pm.

Two applications



Combustion

Two applications



Ocean and ocean-atmosphere computations