Boundary Value Problems in Ramified Domains with Fractal Boundaries

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Introduction

This work deals with some Poisson problems in a self-similar ramified domain of $\mathbb{R}^2$ with a fractal boundary (see Figure 1). We consider generalized Neumann condition on the fractal boundary. The first goal is to give a rigorous functional setting. The second goal is to propose a strategy for computing the solutions in simple subdomains obtained by stopping the construction after a finite number of steps. When the Neumann data belongs to the Haar basis associated to a dyadic decomposition of the fractal boundary, we show that the solution can be found by solving a sequence of boundary value problems in an elementary cell, with nonhomogeneous and nonlocal boundary conditions. For a general Neumann data g, the idea is to expand g on the Haar basis and use the linearity of the problem for deriving an expansion of the solution.
This work is an extension of [1], where the Hausdorff dimension of the fractal boundary was 1. Related results for the Helmholtz equation are contained in [2]. The proofs of the theoretical results below are given in [3].

1 The Geometry

Let a be a positive parameter. Consider the points of $\mathbb{R}^2$: $P_1 = (-1, 0)$, $P_2 = (1, 0)$, $P_3 = (-1, 1)$, $P_4 = (1, 1)$, $P_5 = (-1+a\sqrt{2}, 1+a\sqrt{2})$ and $P_6 = (1-a\sqrt{2}, 1+a\sqrt{2})$. Let Y^0 and F_i, $i = 1, 2$ be respectively the hexagonal subset of $\mathbb{R}^2$ and the similitudes defined by the following:

$$\begin{aligned}
&Y^0 = \text{Interior}\Big(\text{ Conv}(P_1, P_2, P_3, P_4, P_5, P_6)\Big),\\
&F_i(x) = \Big((-1)^i\Big(1 - \tfrac{a}{\sqrt{2}}\Big) + \tfrac{a}{\sqrt{2}}\Big(x_1 + (-1)^i x_2\Big), 1 + \tfrac{a}{\sqrt{2}} + \tfrac{a}{\sqrt{2}}\Big(x_2 + (-1)^{i+1}x_1\Big)\Big)\,.
\end{aligned}$$

The similitude F_i has the dilation ratio a and the rotation angle $(-1)^{i+1}\pi/4$. To prevent $F_1(Y^0)$ and $F_2(Y^0)$ from overlapping, one must choose $a \le \sqrt{2}/2$.

For $n \geq 1$, we call $\mathcal{A}_n$ the set containing all the 2^n mappings from $\{1, \ldots, n\}$ to $\{1, 2\}$. We define

$$\mathcal{M}_\sigma = F_{\sigma(1)} \circ \cdots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n, \tag{1}$$

and the ramified open domain, see Figure 1,

$$\Omega = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right). \tag{2}$$

Stronger constraints must be imposed on a to prevent the sets $\mathcal{M}_\sigma(\overline{Y^0})$, $\sigma \in \mathcal{A}_n$, $n > 0$, from overlapping. It can be shown that the condition is $2\sqrt{2}a^5 + 2a^4 + 2a^2 + \sqrt{2}a - 2 \leq 0$, *i.e.* , $a \leq a^* \sim 0.593465\ldots$

We call Γ^∞ the self similar set associated to the similitudes F_1 and F_2, *i.e.* the unique compact subset of $\mathbb{R}^2$ such that $\Gamma^\infty = F_1(\Gamma^\infty) \cup F_2(\Gamma^\infty)$. The Hausdorff dimension of Γ^∞ can be computed since Γ^∞ satisfies the Moran condition (open set condition) (see [6, 5]): $\dim_H(\Gamma^\infty) = -\log 2/\log a$. For instance, if $a = a^*$, then $\dim_H(\Gamma^\infty) \sim 1.3284371$.

We split the boundary of Ω into Γ^∞, $\Gamma^0 = [-1, 1] \times \{0\}$ and $\Sigma = \partial\Omega \backslash (\Gamma^0 \cup \Gamma^\infty)$. We define the polygonal open domain Y^N obtained by stopping the above construction at step $N + 1$,

$$Y^N = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right). \tag{3}$$

We also define the sets $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$ and $\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$.

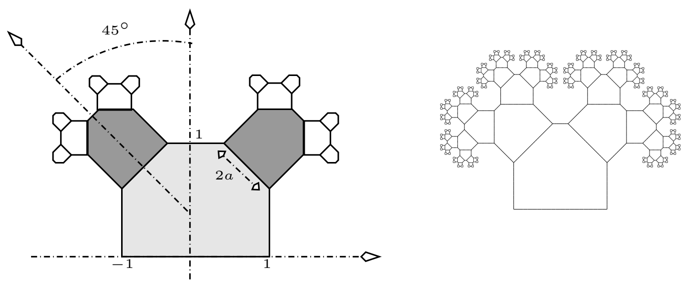

Fig. 1. The ramified domain Ω (only a few generations are displayed).

2 Functional Setting

Let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with first order partial derivatives in $L^2(\Omega)$. We also define

$$\mathcal{V}(\Omega) = \left\{v \in H^1(\Omega); v|_{\Gamma^0} = 0\right\} \quad \text{and} \quad \mathcal{V}(^n) = \left\{v \in H^1(Y^n); v|_{\Gamma^0} = 0\right\}.$$

Theorem 1. *There exists a constant $C > 0$, such that*

$$\forall u \in H^1(\Omega), \qquad \|u\|^2_{L^2(\Omega)} \le C \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)} \right). \tag{4}$$

The embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact.

For defining traces on Γ^∞, we need the classical result, see [4]:

Theorem 2. *There exists a unique Borel regular probability measure μ on Γ^∞ such that for any Borel set $A \subset \Gamma^\infty$,*

$$\mu(A) = 1/2\mu \left(F_1^{-1}(A) \right) + 1/2\mu \left(F_2^{-1}(A) \right). \tag{5}$$

The measure μ is called the *self-similar measure defined in the self similar triplet* $(\Gamma^\infty, F_1, F_2)$. Let L^2_μ be the Hilbert space of the functions on Γ^∞ that are μ-measurable and square integrable w.r.t. μ, with the norm $\|u\|_{L^2_\mu} = \sqrt{\int_{\Gamma^\infty} u^2 d\mu}$. A Hilbertian basis of L^2_μ can be constructed with e.g. Haar wavelets.
Consider the sequence of linear operators $\ell^n : H^1(\Omega) \to L^2_\mu$,

$$\ell^n(u) = \sum_{\sigma \in \mathcal{A}_n} \left(1/|\Gamma^\sigma| \int_{\Gamma^\sigma} u \, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}, \tag{6}$$

where $|\Gamma^\sigma|$ is the Lebesgue measure of Γ^σ.

Lemma 1. *The sequence $(\ell^n)_n$ converges in $\mathcal{L}(H^1(\Omega), L^2_\mu)$, to an operator that we call ℓ^∞. The operator ℓ^∞ can be seen as a renormalized trace operator.*

3 A Class of Poisson Problems

Take $g \in L^2_\mu$ and $u \in H^{\frac{1}{2}}(\Gamma^0)$. We look for $U(u,g) \in H^1(\Omega)$ s.t.

$$(U(u,g))|_{\Gamma^0} = u, \text{ and } \int_\Omega \nabla(U(u,g)) \cdot \nabla v = \int_{\Gamma^\infty} g \ell^\infty(v) \, d\mu, \; \forall v \in \mathcal{V}(\Omega). \tag{7}$$

If it exists, then $(U(u,g))$ satisfies $\Delta(U(u,g)) = 0$ in Ω, and $\partial_n(U(u,g)) = 0$ on Σ. We shall discuss the boundary condition on Γ^∞ after the following:

Theorem 3. *For $g \in L^2_\mu$ and $u \in H^{\frac{1}{2}}(\Gamma^0)$, (7) has a unique solution.*
Furthermore, if $g = \ell^\infty(\tilde{g})$, $\tilde{g} \in \mathcal{C}^1(\overline{\Omega})$, if $w_q \in H^1(Y^q)$ is the solution of:

$$\begin{aligned} &\Delta w_q = 0 \quad \text{in } Y^q, \qquad w_q|_{\Gamma^0} = u, \qquad \partial_n w_q = 0 \quad \text{on } \partial Y_q \backslash (\Gamma^0 \cup \Gamma^{q+1}), \\ &\partial_n w_q = (1/|\Gamma^{q+1}|) \tilde{g}|_{\Gamma^{q+1}} \quad \text{on } \Gamma^{q+1}, \end{aligned}$$

then $\lim_{q \to \infty} \|(U(u,g))|_{Y^q} - w_q\|_{H^1(Y^q)} = 0$.

Theorem 3 says in particular that (7) has an intrinsic meaning for a large class of data g. From the definition of w_q, we may say that $U(u,g)$ satisfies a Neumann condition on Γ^∞ with datum g.

4 A Strategy for Computing $U(u,g)|_{Y^n}$

4.1 The Case when $g = 0$.

We use the notation $\mathcal{H}(u) = U(u,0)$. Call T the Dirichlet-Neumann operator from $H^{\frac{1}{2}}(\Gamma^0)$ to $(H^{\frac{1}{2}}(\Gamma^0))'$, $Tu = \partial_n \mathcal{H}(u)|_{\Gamma^0}$. We remark that $T \in \mathbb{O}$, the cone containing the self-adjoint, positive semi-definite, bounded linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to $(H^{\frac{1}{2}}(\Gamma^0))'$ which vanish on the constants.
If T is available, the self-similarity implies that $\mathcal{H}(u)|_{Y^0} = w$, where w is s.t.

$$\Delta w = 0 \quad \text{in } Y^0, \qquad \frac{\partial w}{\partial n}|_{\partial Y^0 \setminus (\Gamma^0 \cup \Gamma^1)} = 0, \tag{8}$$

$$w|_{\Gamma^0} = u, \tag{9}$$

$$\frac{\partial w}{\partial n} + \frac{1}{a}\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = 0 \quad \text{on } F_i(\Gamma^0),\ i = 1, 2. \tag{10}$$

We stress the fact that (8)-(10) is well posed, from the observation on T above. Since (10) allows computing $\mathcal{H}(u)|_{Y^0}$, it is called a transparent boundary condition. The construction may be generalized to $\mathcal{H}(u)|_{Y^{n-1}}$, $n \geq 1$:

Proposition 1. *For $u \in H^{\frac{1}{2}}(\Gamma^0)$, $\mathcal{H}(u)|_{Y^{n-1}}$ can be found by successively solving $1 + 2 + \cdots + 2^{n-1}$ boundary value problems in Y^0:*
• *Loop: for $p = 0$ to $n-1$,*
  • • *Loop : for $\sigma \in \mathcal{A}_p$, (at this point, if $p \geq 1$, $(\mathcal{H}(u))|_{\Gamma^\sigma}$ is known)*
    • •• *Find $w \in H^1(Y^0)$ satisfying the boundary value problem (8), (10), and either (9) if $p = 0$, or $w|_{\Gamma^0} = \mathcal{H}(u)|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma$ if $p > 0$.*
    • •• *Set $\mathcal{H}(u)|_{Y^0} = w$ if $p = 0$. If $p > 0$, set $\mathcal{H}(u)|_{\mathcal{M}_\sigma(Y^0)} = w \circ (\mathcal{M}_\sigma)^{-1}$.*

We are left with computing T: in Theorem 4 below, we show that T can be obtained as the limit of a sequence of operators constructed by a simple induction. This is the consequence of the following result:

Proposition 2. *There exists a constant $\rho < 1$ such that for any $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Omega^\sigma} |\nabla \mathcal{H}(u)|^2 \leq \rho^p \int_{\Omega} |\nabla \mathcal{H}(u)|^2, \quad \forall p > 0. \tag{11}$$

In order to compute T, we introduce the mapping $\mathbb{M} : \mathbb{O} \mapsto \mathbb{O}$: for any $Z \in \mathbb{O}$,

$$\forall u \in H^{\frac{1}{2}}(\Gamma^0), \qquad \mathbb{M}(Z)u = \partial_n w|_{\Gamma^0}, \tag{12}$$

where w satisfies (8), (9) and $\frac{\partial w}{\partial n} + \frac{1}{a}\left(Z(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = 0$ on $F_i(\Gamma^0)$.

Theorem 4. *The operator T is the unique fixed point of $\mathbb{M}$. Moreover, if ρ, $0 < \rho < 1$, is the constant appearing in (11), then, for all $Z \in \mathbb{O}$, $\exists C > 0$ s.t.*

$$\|\mathbb{M}^p(Z) - T\| \leq C \rho^{\frac{p}{4}}, \quad \forall p \geq 0. \tag{13}$$

In what follows, we propose a method for computing $(U(0,g))|_{Y^{n-1}}$, (n is some fixed positive integer). We first distinguish the case when g belongs to the Haar basis associated to the dyadic decomposition of Γ^∞.

4.2 The Case when g Belongs to the Haar Basis

The case when g is a Haar wavelet is particularly favorable because transparent boundary conditions may be used, thanks to self-similarity.
Let us call $e_F = U(0, 1_{\Gamma^\infty})$.
We introduce the linear operator B, bounded from $(H^{\frac{1}{2}}(\Gamma^0))'$ to $L^2(\Gamma^0)$, by: $Bz = -\frac{\partial w}{\partial x_2}|_{\Gamma^0}$, where $w \in \mathcal{V}(Y^0)$ is the unique weak solution to

$$\Delta w = 0 \quad \text{in } Y^0, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial Y_0 \backslash (\Gamma^0 \cup \Gamma^1), \tag{14}$$

$$\frac{\partial w}{\partial x_2}|_{F_i(\Gamma^0)} + \frac{1}{a}\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -z \circ F_i^{-1}, \qquad i = 1, 2. \tag{15}$$

The self-similarity in the geometry and the scale-invariance of the equations are the fundamental ingredients for proving the following theorem:

Theorem 5. *The normal derivative y_F of e_F on Γ^0 belongs to $L^2(\Gamma^0)$ and is the unique solution to: $y_F = By_F$ and $\int_{\Gamma^0} y_F = -1$.*
For all $n \geq 1$, the restriction of e_F to Y^{n-1} can be found by successively solving $1 + 2 + \cdots + 2^{n-1}$ boundary value problems in Y^0, as follows:
•Loop: for $p = 0$ to $n - 1$,
• • Loop : for $\sigma \in \mathcal{A}_p$, (at this point, if $p > 0$, $e_F|_{\Gamma^\sigma}$ is known)
• •• Solve the boundary value problem in Y^0: find $w \in H^1(\Omega)$ satisfying (14), with $w|_{\Gamma^0} = 0$ if $p = 0$, $w|_{\Gamma^0} = e_F|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma$ if $p > 0$, and

$$\frac{\partial w}{\partial n} + \frac{1}{a}\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -\frac{1}{2^{p+1}a} y_F \circ F_i^{-1}, \quad \text{on } F_i(\Gamma^0),\ i = 1, 2.$$

• •• Set $e_F|_{Y^0} = w$ if $p = 0$, else set $e_F|_{\mathcal{M}_\sigma(Y^0)} = w \circ (\mathcal{M}_\sigma)^{-1}$.

When g is a Haar wavelet on Γ^∞, the knowledge of T, e_F and y_F permits $U(0, g)$ to be computed: call $g^0 = 1_{F_1(\Gamma^\infty)} - 1_{F_2(\Gamma^\infty)}$ the Haar mother wavelet, and define $e^0 = U(0, g^0)$. One may compute $e^0|_{Y^n}$ by using the following:

Proposition 3. *We have $e^0|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (14) and*

$$\frac{\partial w}{\partial n} + \frac{1}{a}\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = \frac{(-1)^i}{2a} y_F \circ F_i^{-1} \text{ on } F_i(\Gamma^0),\ i = 1, 2, \tag{16}$$

Furthermore, for $i = 1, 2$,

$$e^0|_{F_i(\Omega_0)} = (-1)^{i+1}/2\ e_F \circ F_i^{-1} + \left(\mathcal{H}(e^0|_{F_i(\Gamma_0)} \circ F_i)\right) \circ F_i^{-1}. \tag{17}$$

For a positive integer p , take $\sigma \in \mathcal{A}_p$. Call g^σ the Haar wavelet on Γ^∞, defined by $g^\sigma|_{\mathcal{M}_\sigma(\Gamma^\infty)} = g^0 \circ \mathcal{M}_\sigma^{-1}$, and $g^\sigma|_{\Gamma^\infty \backslash \mathcal{M}_\sigma(\Gamma^\infty)} = 0$, and call $e^\sigma = U(0, g^\sigma)$, and y^σ (resp. y^0) the normal derivative of e^σ (resp. e^0) on Γ^0. The following result shows that (e^σ, y^σ) can be computed by induction:

Proposition 4. *The family (e^σ, y^σ) is defined by induction: assume that $\mathcal{M}_\sigma = F_i \circ \mathcal{M}_\eta$ for some $i \in \{1, 2\}$, $\eta \in \mathcal{A}_{p-1}$, $p > 1$. Then $e^\sigma|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (14) and*

$$\frac{\partial w}{\partial n} + \frac{1}{a}\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -\frac{1}{2a} y^\eta \circ F_i^{-1} \quad \text{on } F_i(\Gamma^0),\ i = 1, 2. \tag{18}$$

Then, with $j = 1 - i$, $e^\sigma|_{\Omega\setminus Y^0}$ is given by

$$
\begin{aligned}
e^\sigma|_{F_i(\Omega)} &= \frac{1}{2} e^\eta \circ F_i^{-1} + \big(\mathcal{H}(e^\sigma|_{F_i(\Gamma_0)} \circ F_i)\big) \circ F_i^{-1}, \\
e^\sigma|_{F_j(\Omega)} &= \big(\mathcal{H}(e^\sigma|_{F_j(\Gamma_0)} \circ F_j)\big) \circ F_j^{-1}.
\end{aligned}
\tag{19}
$$

If $\mathcal{M}_\sigma = F_i$, $i = 1, 2$, then y^η (resp. e^η) must be replaced by y^0 (resp. e^0) in (18), (resp.(19)).

What follows indicates that for $n \geq 0$ fixed, $\|\nabla e^\sigma\|_{L^2(Y^n)}$, $\sigma \in \mathcal{A}_p$, decays exponentially as $p \to \infty$:

Theorem 6. $\exists C > 0$ *and* ρ, $0 < \rho < 1$ *s.t.*

$$
\|\nabla e^\sigma\|_{L^2(Y^n)} \leq C 2^{-n} \rho^{p-n}, \quad \forall \sigma \in \mathcal{A}_p, 0 \leq n < p - 1. \tag{20}
$$

4.3 The General Case

Consider now the case when g is a general function in L^2_μ. It is no longer possible to use the self-similarity in the geometry for deriving transparent boundary conditions for $U(0, g)$. The idea is different: one expands g on the Haar basis, and use the linearity of (7) with respect to g for obtaining an expansion of $U(0, g)$ in terms of e_F, e^0, and e^σ, $\sigma \in \mathcal{A}_p$, $p > 1$. Indeed, one can expand $g \in L^2_\mu$ as follows:

$$
g = \alpha_F 1_{\Gamma^\infty} + \alpha_0 g^0 + \sum_{p=1}^{\infty} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma g^\sigma. \tag{21}
$$

The following result, which is a consequence of Theorem 6, says that $(U(0, g))|_{Y^n}$ can be expanded in terms of $e_F|_{Y^n}$, $e^0|_{Y^n}$, and $e^\sigma|_{Y^n}$, $\sigma \in \mathcal{A}_p$, $p \geq 1$. Moreover, a few terms in the expansion are enough to approximate $(U(0, g))|_{Y^n}$ with a good accuracy:

Proposition 5. *Assume (21) and call r^P the error $r^P = U(0, g) - \alpha_F e_F - \alpha_0 e^0 - \sum_{p=1}^{P} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e^\sigma$. $\exists C$ (independent of g) s.t.*

$$
\|r^P\|_{H^1(Y^n)} \leq C \sqrt{2^{-P}} \rho^{P-n} \|g\|_{L^2_\mu}, \quad \forall n, P,\ 0 \leq n < P - 1. \tag{22}
$$

Generalizations. Here we discuss possible generalizations of the example above. The geometrical construction only depends on three basic elements: the elementary cell Y^0 and the similitudes F_1 and F_2 (dilation ratii a_1 and a_2, $0 < a_i < 1$, rotation angles α_1 and α_2). The following conditions must be satisfied: 1) the elementary cell Y^0 is a Lipschitz domain. 2) The domain Ω defined by (2) is a connected open set. 3) For $\sigma_1, \sigma_2 \in \cup_{n \in \mathbb{N}} \mathcal{A}_n$, $\sigma_1 \neq \sigma_2$, $\mathcal{M}_{\sigma_1}(Y^0) \cap \mathcal{M}_{\sigma_2}(Y^0) = \emptyset$. If these conditions are fulfilled, all the above results apply. The important point is to use the measure μ defined in Theorem 2. Of course, one can consider constructions with to more than two similitudes, i.e. F_i, $i = 1, \ldots, p$, with respective dilation ratio $a_i > 0$ and angles α_i.

5 Numerical Results

To transpose the strategies described above to finite element methods, one needs to use self-similar triangulations of Ω: we first consider a regular family of meshes $\mathcal{T}_h^0$ of Y^0, with the special property that for $i = 1, 2$, the set of nodes of $\mathcal{T}_h^0$ lying on $F_i(\Gamma^0)$ is the image by F_i of the set of nodes lying on Γ^0. Then one can construct self-similar meshes of Ω by $\mathcal{T}_h = \cup_{p=0}^{\infty} \cup_{\sigma \in \mathcal{A}_p} \mathcal{M}_\sigma(\mathcal{T}_h^0)$, with self-explanatory notations. With such meshes and conforming finite elements, one can transpose everything to the discrete level.

An Example. The aim is to compute $U(0, g)|_{Y^5}$, with $g(s) = (1_{s<0} - 1_{s>0}) \cos(3\pi s/2)$, where $s \in [-1, 1]$ is a parametrization of Γ^∞. We first compute the operator T by the method in § 4.1 and $e^\sigma|_{Y^5}$, for $\sigma \in \mathcal{A}_p$, $p \leq 5$ by the method in § 4.2. Then we expand g on the Haar basis and use the expansion in Proposition 5.

In the top of Figure 2, we plot two approximations of $U(0, g)|_{Y^5}$; we have used the expansion in Proposition 5., with $P = 5$ on the left, and $P = 2$ on the right. We see that taking $P = 2$ is enough for approximating $U(0, g)|_{Y^0}$, but not for $U(0, g)|_{Y^j}$, $j \geq 1$. In the bottom of Figure 2, we plot (in log scales) the errors $\| \sum_{p=i}^{5} \sum_{\sigma \in \mathcal{A}_p} \alpha^\sigma e_h^\sigma \|_{L^2(Y^j)}$, for $i = 2, 3, 4$ and $j = 0, 1, 2, 3, 4$, where α^σ are the coefficients of the wavelet expansion of g. The behavior is the one predicted by Proposition 5.

Again, we stress that there is no error from the domain truncation, and that we did not solve any boundary value problem in Y^5, but a sequence of boundary problems in Y^0. Nevertheless, the function smoothly matches at the interfaces between the subdomains.

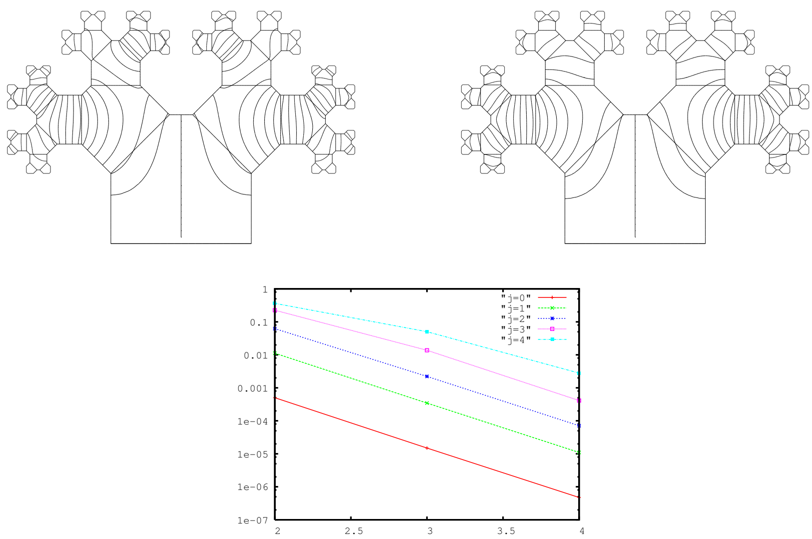

Fig. 2. Top: Contours of the approximations of $U(0, g)|_{Y^5}$ by taking $P = 5$ (left) and $P = 2$ (right). Bottom: $\| \sum_{p=i}^{5} \sum_{\sigma \in \mathcal{A}_p} \alpha^\sigma e_h^\sigma \|_{L^2(Y^j)}$ for $i = 2, 3, 4$ and $j = 0, 1, 2, 3, 4$.

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