Class of Preconditioners for Discontinuous Galerkin Approximations of Elliptic Problems

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Summary. We present a class of Schwarz preconditioners for discontinuous Galerkin approximations of elliptic problems. We provide a unified framework for the construction and analysis of two-level methods which share the features of the classical Schwarz techniques for conforming finite element discretizations. Numerical experiments confirming the theoretical results are also included.

1 Introduction

Domain decomposition (DD) methods provide powerful preconditioners for the iterative solution of the large algebraic linear systems of equations that arise in finite element approximations of partial differential equations. Many DD algorithms can conveniently be described and analyzed as Schwarz methods, and, if on the one hand a general theoretical framework has been previously developed for classical conforming discretizations (see, e.g., [7]), on the other hand, only few results can be found for discontinuous Galerkin (DG) approximations (see, e.g., [6, 4, 2, 1]). Based on discontinuous finite element spaces, DG methods have become increasing popular thanks to their great flexibility for providing discretizations on matching and non-matching grids and their high degree of locality. In this paper we present and analyze, in the unified framework based on the *flux formulation* proposed in [3], a class of Schwarz preconditioners for DG approximations of second order elliptic problems. Schwarz methods for a wider class of DG discretizations are studied in [2, 1]. The issue of preconditioning non-symmetric DG approximations is also discussed. Numerical experiments to asses the performance of the proposed preconditioners and validate our convergence results are presented.

2 Discontinuous Galerkin Methods for Elliptic Problems

We consider the following model problem

$$-\Delta u = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a convex polygon or polyhedron and f a given function in $L^2(\Omega)$. Let $\mathcal{T}_h$ be a shape-regular *quasi-uniform* partition of Ω into disjoint open elements T (with diameter h_T), where each T is the affine image of a fixed master element $\widehat{T}$, i.e., $T = F_T(\widehat{T})$, and where $\widehat{T}$ is either the open unit d-simplex or the d-hypercube in $\mathbb{R}^d$, $d = 2, 3$. We define the mesh size h by $h = \max_{T\in\mathcal{T}_h} h_T$. We denote by $\mathscr{E}^I$ and $\mathscr{E}^B$ the sets of all interior and boundary faces of $\mathcal{T}_h$, respectively, and set $\mathscr{E} = \mathscr{E}^I \cup \mathscr{E}^B$. For a given approximation order $\ell_h \geq 1$, we define the discontinuous finite element spaces $V_h = \{v \in L^2(\Omega) : v|_T \circ F_T \in \mathcal{M}^{\ell_h}(\widehat{T})\ \forall T \in \mathcal{T}_h\}$ and $\boldsymbol{\Sigma}_h = [V_h]^d$, where $\mathcal{M}^{\ell_h}(\widehat{T})$ is either the space of polynomials of degree at most ℓ_h on $\widehat{T}$, if $\widehat{T}$ is the reference d-simplex, or the space of polynomials of degree at most ℓ_h in each variable on $\widehat{T}$, if $\widehat{T}$ is the reference d-hypercube.

For any internal face $e \in \mathscr{E}^I$ shared by two adjacent elements $T^\pm$ with outward normal unit vectors $\mathbf{n}^\pm$, we define the jump and *weighted* average operators, with $\delta \in [0, 1]$, by:

$$\begin{aligned} [\![\boldsymbol{\tau}]\!] &= \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-, \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-, \qquad e \in \mathscr{E}^I, \\ \{\!\{\boldsymbol{\tau}\}\!\}_\delta &= \delta\boldsymbol{\tau}^+ + (1-\delta)\boldsymbol{\tau}^-, \qquad \{\!\{\mathbf{v}\}\!\}_\delta = \delta\mathbf{v}^+ + (1-\delta)\mathbf{v}^-, \quad e \in \mathscr{E}^I, \end{aligned} \tag{2}$$

where $\boldsymbol{\tau}^\pm$ and $v^\pm$ denote the traces on $\partial T^\pm$ taken from the interior of $T^\pm$ of the (regular enough) functions $\boldsymbol{\tau}$ and v. On a boundary face $e \in \mathscr{E}^B$, we set

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau} \cdot \mathbf{n}, \quad [\![v]\!] = v\,\mathbf{n}, \quad \{\!\{\boldsymbol{\tau}\}\!\}_\delta = \boldsymbol{\tau}\,, \quad \{\!\{v\}\!\}_\delta = v, \quad e \in \mathscr{E}^B. \tag{3}$$

For $\delta = 1/2$ we write $\{\!\{\cdot\}\!\}$ in lieu of $\{\!\{\cdot\}\!\}_{1/2}$.

The DG discretization based on the flux formulation proposed in [3] is defined by introducing an auxiliary variable $\boldsymbol{\sigma} = \nabla u$ and rewriting problem (1) as a first order system of equations. Further elimination of $\boldsymbol{\sigma}$, gives the *primal formulation* of DG methods:

$$\text{find } u_h \in V_h \text{ such that } A_h(u_h, v_h) = \int_\Omega f v_h \,\mathrm{d}x \qquad \forall\, v_h \in V_h\,. \tag{4}$$

Adopting the convention $\int_{\mathscr{E}} v_h \,\mathrm{d}s = \sum_{e\in\mathscr{E}} \int_e v_h \,\mathrm{d}s$, $A_h(\cdot,\cdot)$ is given by

$$\begin{aligned} A_h(u_h, v_h) = &\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}x + \int_{\mathscr{E}} [\![\widehat{u} - u_h]\!] \cdot \{\!\{\nabla_h v_h\}\!\} \,\mathrm{d}s \\ &+ \int_{\mathscr{E}^I} \{\!\{\widehat{u} - u_h\}\!\}\, [\![\nabla_h v_h]\!] \,\mathrm{d}s - \int_{\mathscr{E}} \{\!\{\widehat{\boldsymbol{\sigma}}\}\!\} \cdot [\![v_h]\!] \,\mathrm{d}s - \int_{\mathscr{E}^I} [\![\widehat{\boldsymbol{\sigma}}]\!]\, \{\!\{v_h\}\!\} \,\mathrm{d}s\,, \end{aligned} \tag{5}$$

where $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ are the scalar and vector numerical fluxes and ∇_h denotes the elementwise application of the operator ∇. By defining the numerical fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ as suitable linear combinations of averages and jumps of u_h and $\boldsymbol{\sigma}_h$,

we obtain different DG methods (see Table 1 for the choices considered in this work). The stability is achieved by penalizing the jumps of u_h over each face $e \in \mathscr{E}$. Therefore, $A_h(\cdot,\cdot)$ contains the stabilization term $\mathcal{S}^h(\cdot,\cdot)$ defined by

$$\mathcal{S}^h(u,v) = \sum_{e\in\mathscr{E}} \int_e \alpha\, h_e^{-1}\, [\![u]\!]\cdot[\![v]\!]\, \mathrm{d}s \quad \forall\, u,v \in V_h\,,$$

where h_e is the diameter of the face $e \in \mathscr{E}$. Here $\alpha \geq 1$ is a parameter (independent of the mesh size) that, for all but the LDG and NIPG methods, has to be chosen large enough to ensure the coercivity of the bilinear form. From now on, we drop the subindex h from the finite element functions. In matrix notation, problem (4) is written as the linear system $\mathbf{Au} = \mathbf{f}$.

Table 1. Numerical fluxes on interior faces.

Method	$\widehat{u}(u_h)$	$\widehat{\boldsymbol{\sigma}}(\boldsymbol{\sigma}_h, u_h)$	Symmetry
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \alpha\, h_e^{-1}\, [\![u_h]\!]$	Yes
SIPG(δ)	$\{\!\{u_h\}\!\}_{(1-\delta)}$	$\{\!\{\nabla_h u_h\}\!\}_\delta - \alpha\, h_e^{-1}\, [\![u_h]\!]$	Yes
NIPG	$\{\!\{u_h\}\!\} + [\![u_h]\!]\cdot \mathbf{n}_T$	$\{\!\{\nabla_h u_h\}\!\} - \alpha\, h_e^{-1}\, [\![u_h]\!]$	No
IIPG	$\{\!\{u_h\}\!\} + 1/2\, [\![u_h]\!]\cdot \mathbf{n}_T$	$\{\!\{\nabla_h u_h\}\!\} - \alpha\, h_e^{-1}\, [\![u_h]\!]$	No
LDG	$\{\!\{u_h\}\!\} - \boldsymbol{\beta}\cdot [\![u_h]\!]$	$\{\!\{\boldsymbol{\sigma}_h\}\!\} + \boldsymbol{\beta}\cdot [\![\boldsymbol{\sigma}_h]\!] - \alpha\, h_e^{-1}\, [\![u_h]\!]$	Yes

Remark 1. The results we present here apply to more general elliptic equations with possibly smooth variable coefficients, and remain valid for more general partitions (non necessarily matching).

3 Non-Overlapping Schwarz Methods

We consider three level of *nested* partitions of the domain Ω satisfying the previous assumptions: a subdomain partition $\mathcal{T}_{N_s}$ made of N_s non-overlapping subdomains Ω_i, a coarse partition $\mathcal{T}_H$ (with mesh size H) and a fine partition $\mathcal{T}_h$ (with mesh size h). For each subdomain $\Omega_i \in \mathcal{T}_{N_s}$ we denote by $\mathscr{E}_i$ the set of all faces of $\mathscr{E}$ belonging to $\overline{\Omega}_i$, and set $\mathscr{E}_i^I = \{e \in \mathscr{E}_i \,:\, e \subset \Omega_i\}$, $\mathscr{E}_i^B = \{e \in \mathscr{E}_i \,:\, e \subset \partial\Omega_i \cap \partial\Omega\}$. The set of all (internal) faces belonging to the skeleton of the subdomain partition will be denoted by Γ, i.e., $\Gamma = \bigcup_{i=1}^{N_s} \Gamma_i$ with $\Gamma_i = \{e \in \mathscr{E}_i^I \,:\, e \subset \partial\Omega_i\}$. For $i = 1,\ldots,N_s$, we define the local spaces by $V_h^i = \{u \in L^2(\Omega_i) \,:\, v|_T \circ F_T \in \mathcal{M}^{\ell_h}(\widehat{T})\ \forall T \in \mathcal{T}_h, T \subset \Omega_i\}$ and $\boldsymbol{\Sigma}_h^i = [V_h^i]^d$, and the *prolongation* operators $R_i^T : V_h^i \longrightarrow V_h$ as the classical inclusion operators from V_h^i to V_h. For vector-valued functions R_i^T is defined componentwise. We observe that $V_h = R_1^T V_h^1 \oplus \ldots \oplus R_{N_s}^T V_h^{N_s}$ and $\boldsymbol{\Sigma}_h = R_1^T \boldsymbol{\Sigma}_h^1 \oplus \cdots \oplus R_{N_s}^T \boldsymbol{\Sigma}_h^{N_S}$. The restriction operators R_i, are defined as the transpose of R_i^T with respect to the L^2–inner product.

Local solvers: we consider the DG approximation of the problem:

$$-\Delta u_i = f|_{\Omega_i} \quad \text{in } \partial\Omega_i, \quad u_i = 0 \quad \text{on } \Omega_i\,.$$

In view of (5), the local bilinear forms $A_i : V_h^i \times V_h^i \longrightarrow \mathbb{R}$ are defined by

$$A_i(u_i, v_i) = \int_{\Omega_i} \nabla_h u_i \cdot \nabla_h v_i \,\mathrm{d}x + \int_{\mathscr{E}_i} [\![\widehat{u}_i - u_i]\!] \cdot \{\!\{\nabla_h v_i\}\!\} \,\mathrm{d}s + \int_{\mathscr{E}_i^I} \{\!\{\widehat{u}_i - u_i\}\!\} [\![\nabla_h v_i]\!] \,\mathrm{d}s - \int_{\mathscr{E}_i} \{\!\{\widehat{\boldsymbol{\sigma}}_i\}\!\} \cdot [\![v_i]\!] \,\mathrm{d}s - \int_{\mathscr{E}_i^I} [\![\widehat{\boldsymbol{\sigma}}_i]\!] \{\!\{v_i\}\!\} \,\mathrm{d}s,$$

where $\widehat{u}_i$ and $\widehat{\boldsymbol{\sigma}}_i$ are the *local* numerical fluxes. On $e \in \mathscr{E}_i^I$, $\widehat{u}_i$ and $\widehat{\boldsymbol{\sigma}}_i$ are defined as the numerical fluxes $\widehat{u}$, $\widehat{\boldsymbol{\sigma}}$ of the global DG method on interior faces, and on $e \in \mathscr{E}_i^B \cup \Gamma_i$ as $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ on boundary faces. Note that, each $e \in \Gamma_i$ is a boundary face for the local partition but an interior face for the global partition. From the definition of $A_i(\cdot,\cdot)$, and taking into account the different definition (2)-(3) of the average operator on interior and boundary faces (implying that $\{\!\{R_i^T v_i\}\!\}_\delta = \delta v_i$ *but* $\{\!\{v_i\}\!\}_\delta = v_i$ on $e \in \Gamma_i$), it follows that we are using *approximate* local solvers, that is, $A_h(R_i^T u_i, R_i^T u_i) \leq \omega A_i(u_i, u_i)$, with $\omega \neq 1$.

Coarse solver: for all $u_0, v_0 \in V_h^0 = \{v_H \in L^2(\Omega) : v_H|_T \in \mathcal{M}^{\ell_H}(T)\ \forall\, T \in \mathcal{T}_H\}$, with $0 \leq \ell_H \leq \ell_h$, the coarse solver $A_0 : V_h^0 \times V_h^0 \longrightarrow \mathbb{R}$ is defined by $A_0(u_0, v_0) = A_h(R_0^T u_0, R_0^T v_0)$, where R_0^T is the classical injection operator.

Remark 2. We notice that, in all the previously proposed Schwarz methods (see, e.g., [6, 4]) *exact* local solvers were employed.

3.1 Schwarz Methods: Variational and Algebraic Formulations

For $i = 0, \ldots, N_s$, and for all $v_i \in V_h^i$, we define the projection operators $\widetilde{P}_i : V_h \longrightarrow V_h^i$ by $A_i(\widetilde{P}_i u, v_i) = A_h(u, R_i^T v_i)$ and set $P_i = R_i^T \widetilde{P}_i : V_h \longrightarrow V_h$. The additive and multiplicative Schwarz operators we consider are defined by

$$P_{ad} = \sum_{i=0}^{N_s} P_i, \quad P_{mu} = I - (I - P_{N_s})(I - P_{N_s-1}) \cdots (I - P_0),$$

respectively, where $I : V_h \longrightarrow V_h$ is the identity operator. We also define the error propagation operator $E_{N_s} = (I - P_{N_s})(I - P_{N_s-1}) \cdots (I - P_0)$ and observe that $P_{mu} = I - E_{N_s}$. The Schwarz methods can be written as the product of a suitable preconditioners, namely $\mathbf{B}_{ad}$ or $\mathbf{B}_{mu}$, and $\mathbf{A}$. In fact, for $i = 0, \ldots, N_s$, it is straightforward to note that the matrix representation of the projection operators P_i, is given by $\mathbf{P}_i = \mathbf{R}_i^T \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{A}$. Then,

$$\mathbf{P}_{ad} = \sum_{i=0}^{N_s} \mathbf{P}_i = \mathbf{B}_{ad}\mathbf{A}, \qquad \mathbf{P}_{mu} = \mathbf{I} - (\mathbf{I} - \mathbf{P}_{N_s}) \cdots (\mathbf{I} - \mathbf{P}_0) = \mathbf{B}_{mu}\mathbf{A}.$$

The additive Schwarz operator P_{ad} is symmetric for all symmetric DG approximations, while, the multiplicative operator P_{mu} is non symmetric (see [1] for a symmetrized version). Therefore, suitable iterative methods have to be used for solving the resulting linear systems: for the former case we use the *conjugate gradient* method while for the latter case we use the *generalized minimal residual* (GMRES) linear solver.

4 Convergence Results

In this section we present the convergence results for the proposed two-level Schwarz methods. We refer to [2, 1] for their proofs and further discussions on the convergence analysis. In what follows N_c denotes the maximum number of adjacent subdomains a given subdomain can have, and C is a positive constant independent of the mesh size.

Theorem 1. *Let $A_h(\cdot,\cdot)$ be the bilinear form of a symmetric DG method given in Table 1. Then, the condition number of P_{ad}, $\kappa(P_{ad})$, satisfies*

$$\kappa(P_{\mathrm{ad}}) \leq C\,\alpha\,\frac{H}{h}(1+\omega(1+N_c))\,. \tag{6}$$

Remark 3. Note that Theorem 1 shows that $\kappa(P_{\mathrm{ad}})$ depends linearly on the penalty parameter α.

The multiplicative operator is non-symmetric and in Theorem 2, we show that the energy norm of the error propagation operator is strictly less than one.

Theorem 2. *Let $A_h(\cdot,\cdot)$ be the bilinear form of a symmetric DG method given in Table 1. Then, there exists $\widetilde{\alpha} > 0$ such that if $\alpha \geq \widetilde{\alpha}$*

$$\|E_{N_s}\|_A^2 = \sup_{\substack{u\in V_h\\ u\neq 0}} \frac{A_h(E_{N_s}u, E_{N_s}u)}{A_h(u,u)} \leq 1 - \frac{2-\omega}{C\,\alpha\,(1+2\,\omega^2(N_c+1)^2)}\frac{h}{H} < 1.$$

Theorem 2 also guarantees that the multiplicative Schwarz method can be accelerated with the GMRES linear solver (see [5]).

Remark 4. As in the classical Schwarz theory, our convergence result for P_{mu} relies upon the hypothesis that $\omega \in (0,2)$. Since we are using approximate local solvers, we need a technical assumption on the size of the penalty parameter to guarantee $\omega \in (0,2)$. Nevertheless, we wish to stress that the assumed size of $\widetilde{\alpha}$ is moderate (see [1] for details).

Remark on Schwarz methods for the non-symmetric NIPG and IIPG approximations. In Table 2, we numerically demonstrate that the minimum eigenvalue of the *symmetric part* of the additive and multiplicative operators, denoted by $\lambda_{\min}(P_{ad})$ and $\lambda_{\min}(P_{mu})$, respectively, might be negative. As a consequence, the [5] GMRES convergence theory cannot be applied to explain the observed optimal performance of the proposed preconditioners (see Sect. 5).

Table 2. NIPG method ($\alpha = 1$) : $\ell_h = \ell_H = 1$, $N_s = 16$, Cartesian grids. Minimum eigenvalue of the *symmetric part* of P_{ad} (left) and P_{mu} (right).

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	0.06	-0.16	-0.31	-0.40
$H_0/2$	0.64	0.01	-0.26	-0.40
$H_0/4$	-	0.63	-0.02	-0.27
$H_0/8$	-	-	0.62	-0.05

(a) $\lambda_{\min}(P_{ad})$

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	0.16	-0.09	-0.27	-0.38
$H_0/2$	1.00	0.01	-0.21	-0.38
$H_0/4$	-	1.00	0.09	-0.20
$H_0/8$	-	-	1.00	-0.03

(b) $\lambda_{\min}(P_{mu})$

5 Numerical Results

We take $\Omega = (0,1) \times (0,1)$ and we choose f so that the exact solution of problem (1) (with non-homogeneous boundary conditions) is given by $u(x,y) = \exp(xy)$. The subdomain partitions consist of N_s squares, $N_s = 4, 16$ (see Fig. 1 for $N_s = 4$). We consider both matching and non-matching Cartesian grids (see Fig. 1 where the initial coarse and fine non-matching grids are depicted. The corresponding matching grids are obtained by gluing together all the elements that have at least a hanging-node). We denote by H_0 and h_0 the corresponding initial coarse and fine mesh sizes, respectively, and we consider n successive global uniform refinements of these initial grids so that the resulting mesh sizes are $H_n = H_0/2^n$ and $h_n = h_0/2^n$, with $n = 0, 1, 2, 3$. The tolerance is set to 10^{-9}.

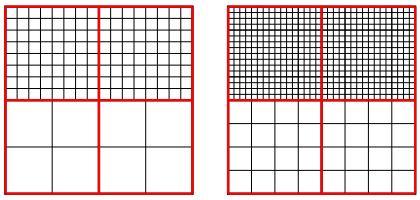

Fig. 1. Sample of a $N_s = 4$ subdomain partition of $\Omega = (0,1) \times (0,1)$ with the initial coarse (left) and fine (right) non-matching meshes.

We first address the scalability of the proposed additive Schwarz method, i.e., the independence of the convergence rate of the number of subdomains. In Table 3 we compare the condition number estimates for the SIP method ($\alpha = 10$) with $\ell_h = \ell_H = 1$ obtained on non-matching Cartesian grids (see Fig. 1) with $N_s = 4, 16$. As stated in Theorem 1, our preconditioner seems to

be insensitive on the number of subdomains, and the expected convergence rates are clearly achieved.

Table 3. SIPG method ($\alpha = 10$): $\ell_h = \ell_H = 1$, non-matching Cartesian grids.

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	31.4	65.9	137.2	277.8
$H_0/2$	6.3	32.8	67.1	137.0
$H_0/4$	-	6.3	33.0	67.1
$H_0/8$	-	-	6.4	33.0
$\kappa(\mathbf{A})$	4.3e3	1.7e4	7.0e4	2.8e5

(a) $\kappa(P_{ad})$: $N_s = 4$

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	29.3	63.3	133.9	272.9
$H_0/2$	6.1	31.5	65.5	135.8
$H_0/4$	-	6.4	32.8	66.9
$H_0/8$	-	-	6.4	33.0
$\kappa(\mathbf{A})$	4.3e3	1.7e4	7.0e4	2.8e5

(b) $\kappa(P_{ad})$: $N_s = 16$

In Table 4 we compare the GMRES iteration counts obtained with our additive and multiplicative Schwarz methods. More precisely, the results reported in Table 4 have been obtained on Cartesian grids with the LDG method ($\alpha = 1$, $\beta = (0.5, 0.5)^T$), by using $\ell_h = 2$, $\ell_H = 1$ and $N_s = 16$. The crosses in the last row of Table 4 mean that the GMRES fails to converge due to its large memory requirements. In both cases we observe optimal convergence rates (we note however, that for the multiplicative preconditioner, the hypothesis on the size of α required in Theorem 2 is not satisfied).

Table 4. LDG method ($\alpha = 1$, $\beta = (0.5, 0.5)^T$): $\ell_h = 2$, $\ell_H = 1$, Cartesian grids.

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	49	68	95	128
$H_0/2$	33	46	64	88
$H_0/4$	-	33	47	65
$H_0/8$	-	-	34	48
#iter($\mathbf{A}$)	112	210	403	x

(a) $\mathbf{B}_{ad}\mathbf{A}\mathbf{u} = \mathbf{B}_{ad}\mathbf{f}$: $N_s = 16$

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	22	30	40	53
$H_0/2$	14	17	23	32
$H_0/4$	-	12	16	21
$H_0/8$	-	-	10	13
#iter($\mathbf{A}$)	112	210	403	x

(b) $\mathbf{B}_{mu}\mathbf{A}\mathbf{u} = \mathbf{B}_{mu}\mathbf{f}$: $N_s = 16$

Finally, we present some numerical computations carried out with the non-symmetric NIPG method ($\alpha = 1$). In Table 5 we compared the GMRES iteration counts obtained with $\ell_h = \ell_H = 1$ on Cartesian grids and by preconditioning with the proposed additive and multiplicative Schwarz preconditioners. Clearly, the GMRES applied to the preconditioned systems

converges in a finite number of steps and, as in the symmetric case, the iteration counts remain unchanged whenever we decrease both the fine and the coarse mesh keeping their ratio constant. In all the cases addressed, the multiplicative Schwarz method seems to be approximately twice faster than the additive preconditioner.

Table 5. NIPG ($\alpha = 1$): GMRES iteration counts, $\ell_h = \ell_H = 1$, Cartesian grids.

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	25	26	29	36
$H_0/2$	14	21	24	28
$H_0/4$	-	14	20	23
$H_0/8$	-	-	14	19
#iter($\mathbf{A}$)	33	61	117	227

(a) $\mathbf{B}_{ad}\mathbf{A}\mathbf{u} = \mathbf{B}_{ad}\mathbf{f}$: $N_s = 16$

$H \downarrow \ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	12	13	16	20
$H_0/2$	1	9	11	14
$H_0/4$	-	1	8	10
$H_0/8$	-	-	1	7
#iter($\mathbf{A}$)	33	61	117	227

(b) $\mathbf{B}_{mu}\mathbf{A}\mathbf{u} = \mathbf{B}_{mu}\mathbf{f}$: $N_s = 16$

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