Extending Theory for Domain Decomposition Algorithms to Irregular Subdomains

Clark R. Dohrmann[1], Axel Klawonn[2], and Olof B. Widlund[3]

[1] Sandia National Laboratories, Albuquerque, USA. crdohrm@sandia.gov
[2] Universität Duisburg-Essen, Essen, Germany. axel.klawonn@uni-due.de
[3] Courant Institute, New York University, New York, USA. widlund@cims.nyu.edu

1 Introduction

In the theory of iterative substructuring domain decomposition methods, we typically assume that each subdomain is quite regular, e.g., the union of a small set of coarse triangles or tetrahedra; see, e.g., [13, Assumption 4.3]. However, this is often unrealistic especially if the subdomains result from using a mesh partitioner. The subdomain boundaries might then not even be uniformly Lipschitz continuous. We note that existing theory establishes bounds on the convergence rate of the algorithms which are insensitive to even large jumps in the material properties across subdomain boundaries as reflected in the coefficients of the problem. The theory for overlapping Schwarz methods is less restrictive as far as the subdomain shapes are concerned, see e.g. [13, Chapter 3], but little has been known on the effect of large changes in the coefficients; see however [11] and recent work [6] and [12].

The purpose of this paper is to begin the development of a theory under much weaker assumptions on the partitioning. We will focus on a recently developed overlapping Schwarz method, see [4], which combines a coarse space adopted from an iterative substructuring method, [13, Algorithm 5.16], with local preconditioner components selected as in classical overlapping Schwarz methods, i.e., based on solving problems on overlapping subdomains. This choice of the coarse component will allow us to prove results which are independent of coefficient jumps. We note that there is an earlier study of multigrid methods [5] in which the coarsest component is similarly borrowed from iterative substructuring algorithms.

We will use nonoverlapping subdomains, and denote them by $\Omega_i, i = 1, \ldots, N$, as well as overlapping subdomains $\Omega'_j, j = 1, \ldots, N'$. The interface between the Ω_i will be denoted by Γ.

So far, complete results have only been obtained for problems in the plane. Although our results also hold for compressible plane elasticity, we will confine ourselves to scalar elliptic problems of the following form:

$$-\nabla \cdot (\rho(x)\nabla u(x)) \;=\; f(x), \quad x \in \Omega \subset \mathbb{R}^2, \tag{1}$$

with a Dirichlet boundary condition on a measurable subset $\partial\Omega_D$ of $\partial\Omega$, the boundary of Ω, and a Neumann condition on $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. The coefficient $\rho(x)$ is strictly positive and assumed to be a constant ρ_i for $x \in \Omega_i$. We use piecewise linear, continuous finite elements and triangulations with shape regular elements and assume that each subdomain is the union of a set of quasi uniform elements. The weak formulation of the elliptic problem is written in terms of a bilinear form,

$$a(u,v) \ := \ \sum_{i=1}^{N} a_i(u,v) \ := \ \sum_{i=1}^{N} \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v dx.$$

Our study requires the generalization of some technical tools used in the proof of a bound of the convergence rate of this type of algorithm; see [3, 8]. Some of the standard tools are no longer available and we have to modify the basic line of reasoning in the proof of our main result. Three auxiliary results, namely a Poincaré inequality, a Sobolev-type inequality for finite element functions, and a bound for certain edge terms, will be required in our proof; see Lemmas 2, 3, and 4. We will work with John domains, see Section 2, and will be able to express our bounds on the convergence of our algorithm in terms of a few geometric parameters.

2 John Domains and a Poincaré Inequality

We first give a definition of a John domain; see [7] and the references therein.

Definition 1 (John domain). *A domain $\Omega \subset \mathbb{R}^n$ – an open, bounded, and connected set – is a John domain if there exists a constant $C_J \geq 1$ and a distinguished central point $x_0 \in \Omega$ such that each $x \in \Omega$ can be joined to it by a curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $dist(\gamma(t), \partial\Omega) \geq C_J^{-1}|x - \gamma(t)|$ for all $t \in [0,1]$.*

This condition can be viewed as a twisted cone condition. We note that certain snowflake curves with fractal boundaries are John domains and that the length of

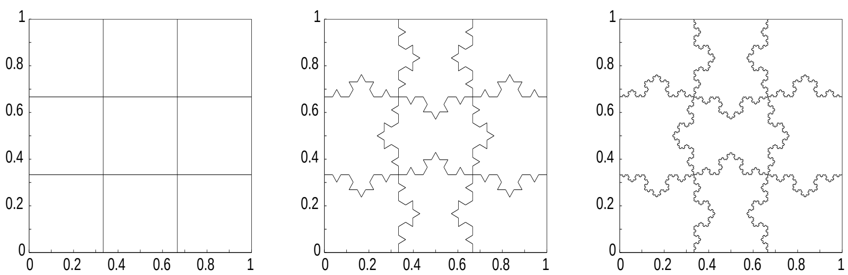

Fig. 1. The subdomains are obtained by first partitioning the unit square into smaller squares. We then replace the middle third of each edge by the other two edges of an equilateral triangle, increasing the length by a factor 4/3. The middle third of each of the resulting shorter edges is then replaced in the same way and this process is repeated until we reach the length scale of the finite element mesh.

the boundary of a John domain can be arbitrarily much larger than its diameter; see Figure 1.

In any analysis of any domain decomposition method with more than one level, we need a Poincaré inequality. This inequality is closely related to an isoperimetric inequality; see [10].

Lemma 1 (Isoperimetric inequality). *Let $\Omega \subset \mathbb{R}^n$ be a domain and let f be sufficiently smooth. Then,*

$$\inf_{c\in\mathbb{R}} \left(\int_\Omega |f - c|^{n/(n-1)} \, dx \right)^{(n-1)/n} \leq \gamma(\Omega, n) \int_\Omega |\nabla f| \, dx,$$

if and only if,

$$[\min(|A|, |B|)]^{1-1/n} \leq \gamma(\Omega, n)|\partial A \cap \partial B|. \tag{2}$$

Here, $A \subset \Omega$ is arbitrary, and $B = \Omega \setminus A$; $\gamma(\Omega, n)$ is the best possible constant and $|A|$ is the measure of the set A, etc.

We note that the domain does not need to be star-shaped or Lipschitz. For two dimensions, we immediately obtain a standard Poincaré inequality by using the Cauchy-Schwarz inequality.

Lemma 2 (Poincaré's inequality). *Let $\Omega \subset \mathbb{R}^2$ be a domain. Then,*

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L_2(\Omega)} \leq (\gamma(\Omega, 2))^2 |\Omega| \|\nabla u\|^2_{L_2(\Omega)}, \quad \forall u \in H^1(\Omega).$$

For $n = 3$ such a bound is obtained by using Hölder's inequality several times. In Lemma 2, we then should replace $|\Omega|$ by $|\Omega|^{2/3}$. The best choice of c is $\bar{u}_\Omega$, the average of u over the domain.

Throughout, we will use a weighted $H^1(\Omega_i)-$norm defined by

$$\|u\|^2_{H^1(\Omega_i)} := |u|^2_{H^1(\Omega_i)} + 1/H_i^2 \|u\|^2_{L_2(\Omega_i)} := \int_{\Omega_i} \nabla u \cdot \nabla u dx + 1/H_i^2 \int_{\Omega_i} |u|^2 dx.$$

Here, H_i is the diameter of Ω_i. The weight originates from a dilation of a domain with diameter 1. We will use Lemma 2 to remove L_2-terms from full H^1-norms.

3 The Algorithm, Technical Tools, and the Main Result

The domain $\Omega \subset \mathbb{R}^2$ is decomposed into nonoverlapping subdomains Ω_i, each of which is the union of finite elements, and with the finite element nodes on the boundaries of neighboring subdomains matching across the interface Γ, which is the union of the parts of the subdomain boundaries which are common to at least two subdomains. The interface Γ is composed of edges and vertices. An edge $\mathcal{E}^{ij}$ is an open subset of Γ, which contains the nodes which are shared by the boundaries of a particular pair of subdomains Ω_i and Ω_j. The subdomain vertices $\mathcal{V}^k$ are end points of edges and are typically shared by more than two; see [9, Definition 3.1] for more details on how these sets can be defined for quite general situations. We denote the standard finite element space of continuous, piecewise linear functions on Ω_i by $V^h(\Omega_i)$ and assume that these functions vanish on $\partial\Omega_i \cap \partial\Omega_D$.

We will view our algorithm as an additive Schwarz method, as in [13, Chapter 2], being defined in terms of a set of subspaces. To simplify the discussion, we will use exact solvers for both the coarse problem and the local ones. All that is then required for the analysis of our algorithm is an estimate of a parameter in a stable decomposition of any elements in the finite element space; see [13, Assumption 2.2 and Lemma 2.5]. Thus, we need to estimate C_0^2 in

$$\sum_{j=0}^{N'} a(u_j, u_j) \leq C_0^2 a(u, u), \quad \forall u \in V^h,$$

for some $\{u_j\}$, such that

$$u = \sum_{j=0}^{N'} R_j^T u_j, \quad u_j \in V_j.$$

Here $R_j^T : V_j \longrightarrow V^h$ is an interpolation operator from the space of the j-th subproblem, defined on Ω_j', into the space V^h. By using [13, Lemmas 2.5 and 2.10], we find that the condition number $\kappa(P_{ad})$ of the additive Schwarz operator can be bounded by $(N^C + 1)C_0^2$ where N^C is the minimal number of colors required to color the subdomains Ω_j' such that no pair of intersecting subdomains have the same color.

Associated with each space V_j is a projection P_j defined by

$$a(\tilde{P}_j u, v) = a(u, v), \ \forall v \in V_j, \text{ and } P_j = R_j^T \tilde{P}_j.$$

The additive Schwarz operator, the preconditioned operator used in our iteration, is

$$P_{ad} = \sum_{j=0}^{N'} P_j.$$

The coarse space V_0, which is described differently in [4], is spanned by functions defined by their values on the interface and extended as discrete harmonic functions into the interior of the subdomains Ω_i. The discrete harmonic extensions minimize the energy; see [13, Section 4.4]. There is one basis function, $\theta_{\mathcal{V}^k}(x)$, for each subdomain vertex; it is the discrete harmonic extension of the standard nodal basis function. There is also a basis function, $\theta_{\mathcal{E}^{ij}}(x)$, for each edge $\mathcal{E}^{ij}$, which equals 1 at all nodes on the edge and vanishes at all other interface nodes. The vertex and edge functions provide a partition of unity.

The local spaces $V_j, j = 1, \ldots N'$, are defined as

$$V_j \ = \ V^h(\Omega_j') \cap H_0^1(\Omega_j').$$

This is the standard choice as in [13, Chapter 3]. We assume that each Ω_j' has a diameter comparable to those of the subdomains Ω_i which intersect Ω_j'; we also assume that neighboring subdomains Ω_i and Ω_j have comparable diameters. The overlap between the subdomains is characterized by parameters δ_j, as in [13, Assumption 3.1]; δ_j is the minimum width of the subset Ω_{j,δ_j} of Ω_j' which is also covered by neighboring overlapping subdomains. We will assume that the width of Ω_{j,δ_j} is on the order of δ_j everywhere; our arguments can easily be extended to a more general case.

We can now formulate our main result, which is also valid for compressible elasticity with piecewise constant Lamé parameters, provided that the coarse space is enriched as in [4].

Theorem 1 (Condition number estimate). *Let $\Omega \subset \mathbb{R}^2$ be an arbitrary John domain with a shape regular triangulation. The condition number then satisfies*

$$\kappa(P_{ad}) \leq C\,(1 + H/\delta)(1 + \log(H/h))^2,$$

where $C > 0$ is a constant which only depends on the John and Poincaré parameters, the number of colors required for the overlapping subdomains, and the shape regularity of the finite elements.

Here, H/h is shorthand for $\max_i(H_i/h_i)$, as in many domain decomposition papers; h_i is the diameter of the smallest element of Ω_i. Similarly, H/δ is the largest ratio of H_i and the smallest of the δ_j of the subregions Ω_j' that intersect Ω_i.

The logarithmic factors of our main result can be improved to a first power if a sufficiently large subset of each subdomain edge is Lipschitz. If the coefficients do not have large jumps across the interface, the coarse space is suitably enriched, and the subregions satisfy [13, Assumption 4.3], we can eliminate the logarithmic factors altogether.

To prove this theorem, we need two auxiliary results, in addition to Poincaré's inequality. The first is a discrete Sobolev inequality:

Lemma 3 (Discrete Sobolev inequality).

$$\|u\|^2_{L_\infty(\Omega_i)} \leq C(1 + \log(H/h))\|u\|^2_{H^1(\Omega_i)}, \quad \forall u \in V^h(\Omega_i). \tag{3}$$

The constant $C > 0$ depends only on the John parameter and the shape regularity of the finite elements.

The inequality (3) is well-known in the theory of iterative substructuring methods. Proofs for domains satisfying an interior cone condition are given in [1] and in [2, Sect. 4.9].

The second important tool provides estimates of the edge functions.

Lemma 4 (Edge functions). *The edge function $\theta_{\mathcal{E}^{ij}}$ can be bounded as follows:*

$$\|\theta_{\mathcal{E}^{ij}}\|^2_{H^1(\Omega_i)} \leq C(1 + \log(H_i/h_i)), \tag{4}$$

and

$$\|\theta_{\mathcal{E}^{ij}}\|^2_{L_2(\Omega_i)} \leq CH_i^2(1 + \log(H_i/h_i)). \tag{5}$$

Proofs of Lemmas 3 and 4 are given in [3] and [8], respectively. We note that inequality (4) can be established using ideas similar to those in [13, Section 4.6.3]. The proof of inequality (5) requires a new idea. We note that a uniform L_2-bound holds for more regular edges or if all angles of the triangulation are acute.

4 Proof of Theorem 1

As in many other proofs of results on domain decomposition algorithms, we can work on one subdomain at a time. With local bounds, there are no difficulties in handling variations of the coefficients across the interface.

We recall that the coarse space is spanned by the $\theta_{\mathcal{V}^k}$, the discrete harmonic extensions of the restrictions of the standard nodal basis functions to Γ, and the edge functions $\theta_{\mathcal{E}^{ij}}$. The vertex basis functions have bounded energy, while, according to (4), the edge functions have an energy that grows in proportion to $(1+\log(H/h))$. The coarse space component $u_0 \in V_0$ in the decomposition of an arbitrary finite element function $u(x)$ is chosen as

$$u_0(x) \;=\; \sum_k u(\mathcal{V}^k)\theta_{\mathcal{V}i^k}(x) + \sum_{ij} \bar{u}_{\mathcal{E}^{ij}}\theta_{\mathcal{E}^{ij}}(x).$$

Here, $\bar{u}_{\mathcal{E}^{ij}}$ is the average of u over the edge. This interpolation formula is the two-dimensional analog of [13, Formula (5.13)] and it reproduces constants. In the case of regular edges, we can estimate the edge averages by using the Cauchy–Schwarz inequality and an elementary trace theorem. In our much more general case, we instead get two logarithmic factors by estimating the edge averages by $\|u\|_{L_\infty}$ and using Lemmas 3 and 4. The norms of the vertex terms of u_0 are bounded by one logarithmic factor. Replacing $u(x)$ by $u(x) - \bar{u}_{\Omega_i}$ and using Lemma 2, to remove the L_2-terms of the H^1-norms, we find that

$$|u_0|^2_{H^1(\Omega_i)} \le C(1+\log(H/h))^2 |u|^2_{H^1(\Omega_i)},$$

and

$$a(u_0,u_0) \le C(1+\log(H/h))^2 a(u,u).$$

Similarly, we can prove

$$\|u-u_0\|^2_{L_2(\Omega_i)} \le C(1+\log(H/h))^2 H_i^2 |u|^2_{H^1(\Omega_i)}. \tag{6}$$

In the case of regular subdomain boundaries, or if all angles of the triangulation are acute, no logarithmic factors are necessary in (6).

We now turn to the estimate related to the local spaces. Again, we will carry out the work on one subdomain Ω_i at a time. Let $w := u - u_0$ and define a local term in the decomposition by $u_j = I^h(\theta_j w)$. We will borrow extensively from [13, Sections 3.2 and 3.6]. Thus, I^h interpolates into V^h and the θ_j, supported in Ω'_j, provide a partition of unity. These functions vary between 0 and 1 and their gradients are bounded by $|\nabla\theta_j| \le C/\delta_j$ and they vanish outside the areas of overlap.

We note only a fixed number of Ω'_j intersect Ω_i; we will only consider the contribution from one of them, Ω'_j. As in our earlier work, the only term that requires a careful estimate is $\nabla\theta_j w$. We cover the set Ω_{j,δ_j} with patches of diameter δ_j and note that on the order of H_i/δ_j of them will suffice. Just as in the proof of [13, Lemma 3.11], we have

$$\int_{\Omega_i} |\nabla\theta_j w|^2 \le C/\delta_j^2 \big(\delta_j^2 |w|^2_{H^1(\Omega_i)} + (H_i/\delta_j)\delta_j^2 \|w\|^2_{H^1(\Omega_i)}\big).$$

The proof is completed by using (6) and the bound on the energy of u_0.

Acknowledgement. The authors wish to thank Professor Fanghua Lin of the Courant Institute for introducing us to John domains and Poincaré's inequality for very general domains and Dr. Oliver Rheinbach for producing Figure 1.

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

This work was supported in part by the U.S. Department of Energy under contracts DE-FG02-06ER25718 and DE-FC02-01ER25482 and in part by National Science Foundation Grant DMS-0513251.

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