Fast and Reliable Pricing of American Options with Local Volatility

Ralf Forster[1*], Ralf Kornhuber[1], Karin Mautner[2], and Oliver Sander[1]

[1] FU Berlin, Institut für Mathematik, Arnimallee 6, 14195 Berlin, Germany
[2] HU Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany

Summary. We present globally convergent multigrid methods for the nonsymmetric obstacle problems as arising from the discretization of Black–Scholes models of American options with local volatilities and discrete data. No tuning or regularization parameters occur. Our approach relies on symmetrization by transformation and data recovery by superconvergence.

1 Introduction

Since Black and Scholes published their seminal paper [3] in 1973, the pricing of options by means of deterministic partial differential equations or inequalities has become standard practice in computational finance. An option gives the right (but not the obligation) to buy (call option) or sell (put option) a share for a certain value (exercise price K) at a certain time T (exercise date). On the exercise day T, the value of an option is given by its pay–off function $\varphi(S) = \max(K - S, 0) =: (K - S)_+$ for put options and $\varphi(S) = (S - K)_+$ for call options. In contrast to European options which can only be exercised at the expiration date T, American options can be exercised at any time until expiration. As a consequence, the pay–off function $\varphi(S)$ constitutes an a priori lower bound for the value V of American options which leads to an obstacle problem for V. While Black and Scholes started off with a constant risk–less interest rate and volatility, existence, uniqueness, and discretization is now well understood even for stochastic volatility [1]. On the other hand, an explosive growth of different kinds of equity derivatives on global markets has led to a great variety of well–tuned local volatility models, where the volatility is assumed to be a deterministic (and sometimes even smooth) function of time and space [5, 6]. As such kind of models are used for thousands and thousands of simulations each day, highly efficient and reliable solvers are an ongoing issue in banking practice. Particular difficulties arise from the spatial obstacle problems resulting from implicit time

* This work was supported in part by DFG as Project E7 of Matheon. We gratefully acknowledge the stimulating collaboration with Prof. Dr. P. Deuflhard (ZIB/FU Berlin) and with our project partners Dr. M. Overhaus and Dr. A. Ferraris (DBQuant, Deutsche Bank London).

discretization. The multigrid solver by Brandt and Cryer [4, 15, 16] lacks reliability in terms of a convergence proof and might fail in actual computations. Globally convergent multigrid methods with mesh–independent convergence rates [2, 12] are available for symmetric problems. Such algorithms were applied in [11] after symmetrization of the underlying bilinear form by suitable transformation. However, only constant coefficients were considered there.

In this paper, we present globally convergent multigrid methods for local volatility models with real–life data. To this end, we extend the above 'symmetrization by transformation' approach to variable coefficients. No continuous functions but only discrete market observations are available in banking practice. Therefore, we develop a novel recovery technique based on superconvergence in order to provide sufficiently accurate approximations of the coefficient functions and their derivatives. Finally, we present some numerical computations for an American put option with discrete dividends on a single share.

2 Continuous Problem and Semi–discretization in Time

The Black–Scholes model for the value $V(S,t)$ of an American put option at asset price $S \in \Omega_\infty = [0,\infty)$ and time $t \in [0,T)$ can be written as the following degenerate parabolic complementary problem [1, 5]

$$\begin{aligned} -\tfrac{\partial V}{\partial t} - \tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 V}{\partial S^2} - \mu S \tfrac{\partial V}{\partial S} + rV \geq 0 \,, \qquad V - \varphi \geq 0 \,, \\ \Big(- \tfrac{\partial V}{\partial t} - \tfrac{\sigma^2}{2} S^2 \tfrac{\partial^2 V}{\partial S^2} - \mu S \tfrac{\partial V}{\partial S} + rV \Big) \Big(V - \varphi \Big) = 0 \,, \end{aligned} \tag{1}$$

in backward time t with stopping condition $V(\cdot,T) = \varphi$ and the pay–off function $\varphi(S) = (K-S)_+$ with exercise price K. The risk–less interest rate $r(t)$, the strictly positive volatility surface $\sigma(S,t)$, and $\mu(t) = r(t) - d(t)$ with continuous dividend yield $d(t)$ are given functions.

Numerical computations require bounded approximations of the unbounded interval Ω_∞. Additional problems result from the degeneracy at $S = 0$. Hence, Ω_∞ is replaced by the bounded interval $\Omega = [S_{\min}, S_{\max}] \subset \Omega_\infty$, $0 < S_{\min} < S_{\max} < \infty$. Appropriate boundary conditions will now be discussed for the example of a put option. Recall that a put option is the right to sell an asset for a fixed price K. If the price of the asset S tends to infinity, the option becomes worthless, because the holder would not like to lose an increasing amount of money by exercising it. Note that $\varphi(S_{\max}) = 0$ for sufficiently large $S_{\max}$. On the other hand, if the asset price tends to zero, then the holder would like to exercise the option almost surely to obtain almost maximal pay–off $\approx K \approx \varphi(S_{\min})$. Hence, we consider the *truncation* of (1) with $S \in \Omega$ and boundary conditions

$$V(S_{\min}) = \varphi(S_{\min}) \,, \quad V(S_{\max}) = \varphi(S_{\max}) \,. \tag{2}$$

Note that the boundary conditions are consistent with the stopping condition $V(T,\cdot) = \varphi$. As $S_{\min} \to 0$, $S_{\max} \to \infty$, the solutions of the resulting truncated problem converge to the solution of the original problem [1].

As usual, we replace backward time t by forward time $\tau = T - t$ to obtain an initial boundary value problem. We now apply a semidiscretization in time by the implicit Euler scheme using the given grid $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ with time steps $h_j := \tau_j - \tau_{j-1}$. We introduce the abbreviations $V_j = V(\cdot, \tau_j)$, $\sigma_j = \sigma(\cdot, \tau_j)$, $\mu_j := \mu(\tau_j)$, and $r_j := r(\tau_j)$. Starting with the initial condition $V_0 = \varphi$, the approximation V_j on time level $j = 1, \ldots, N$ is obtained from the complementary problem

$$\begin{aligned} &-\tfrac{\sigma_j^2}{2} S^2 V_j'' - \mu_j S V_j' + (h_j^{-1} + r_j) V_j - h_j^{-1} V_{j-1} \geq 0 , \quad V_j - \varphi \geq 0 , \\ &\Big(-\tfrac{\sigma_j^2}{2} S^2 V_j'' - \mu_j S V_j' + (h_j^{-1} + r_j) V_j - h_j^{-1} V_{j-1} \Big) \Big(V_j - \varphi \Big) = 0 , \end{aligned} \tag{3}$$

on Ω with boundary conditions taken from (2). For convergence results see [1].

3 Symmetrization and Spatial Discretization

We now derive a reformulation of the spatial problem (3) involving a non–degenerate differential operator in divergence form. To this end, we introduce the transformed volatilities and the transformed variables

$$\alpha(x) = \sigma_j(S(x)) , \quad u(x) = e^{-\beta(x)} V_j(S(x)) , \qquad S(x) = e^x , \quad x \in \overline{X} , \tag{4}$$

on the interval $X = (x_{\min}, x_{\max})$ with $x_{\min} = \log(S_{\min})$, $x_{\max} = \log(S_{\max})$, utilizing the function

$$\beta(x) = \tfrac{1}{2} x + \log\big(\alpha(x)\big) - \log\big(\alpha(0)\big) - \mu_j \int_0^x \frac{ds}{\alpha^2(s)} . \tag{5}$$

Observe that α, β usually vary in each time step.

Theorem 1. *Assume $\sigma_j \in C^2(\overline{\Omega})$ and $\sigma_j(S) \geq c > 0$ for all $S \in \Omega$. Then the linear complementary problem*

$$-(au')' + bu - f \geq 0 , \quad u - \psi \geq 0 , \quad \big(-(au')' + bu - f \big)\big(u - \psi \big) = 0 \tag{6}$$

with coefficients

$$a = \tfrac{\alpha^2}{2} , \qquad b = h_j^{-1} + r_j + \tfrac{1}{8\alpha^2}\big(\alpha^2 - 2\mu_j\big)^2 - \tfrac{\alpha''\alpha^2 + 2\mu_j\alpha'}{2\alpha} , \tag{7}$$

right hand side $f = h_j^{-1} e^{-\beta} V_{j-1}(S(\cdot))$, obstacle $\psi = e^{-\beta}\varphi(S(\cdot))$, and boundary conditions $u(x_{\min}) = \psi(x_{\min})$, $u(x_{\max}) = \psi(x_{\max})$ is equivalent to (3) *in the sense that u defined in* (4) *solves* (6)*, if and only if V_j solves* (3).

The proof follows from basic calculus. Observe that b might become negative for strongly varying $\alpha(x) = \sigma_j(S(x))$ due to the last term in the definition of b, which could even lead to a stability constraint on the time step h_j. We never encountered such difficulties for realistic data.

For a given spatial grid $x_{\min} = x_0 < x_1 \cdots < x_M = x_{\max}$ the finite element discretization of (6) can be written as the discrete convex minimization problem

$$U = \operatorname*{argmin}_{v \in \mathcal{K}} \int_X \tfrac{1}{2}\big(a(v')^2 + bv^2\big) - fv \, dx \tag{8}$$

with $\mathcal{K}$ denoting the discrete, closed, convex set

$$\mathcal{K} = \{v \in C(X) \mid v|_{[x_{i-1},x_i]} \text{ is linear }, \ v(x_i) \geq \psi(x_i) \ \forall i = 1, \ldots, M, \\ v(x_0) = \psi(x_0), \ v(x_M) = \psi(x_M)\} \ .$$

The fast and reliable solution of (8) can be performed, e.g., by globally convergent multigrid methods [2, 12].

4 Data Recovery

In reality, $r(t)$, $\mu(t)$, and $\sigma(S,t)$ are not available as given functions but have to be calibrated from discrete market observations. To this end, it is common practice in computational finance to introduce sufficient smoothness, e.g., by cubic spline approximation of the local volatility [7, 14] which would suggest even C^4–regularity of $\sigma(S,t)$. We refer to [1] for further information. From now on we assume that the data are given in vectors or matrices of point values of sufficiently smooth functions. The grid points usually have nothing to do with the computational grid.

Intermediate function values can be approximated to second order by piecewise linear interpolation. As our transformation technique also requires $\frac{\partial \sigma}{\partial S}$ and $\frac{\partial^2 \sigma}{\partial S^2}$, we now derive an algorithm for the approximation of higher derivatives by successive linear interpolation in suitable superconvergence points. Note that superconvergence has a long history in finite elements (cf., e.g., [13] and the literature cited therein). For two–dimensional functions such as $\sigma(S,t)$, this recovery technique can be applied separately in both variables.

From now on, let $w_k = w(s_k)$ denote given function values at given grid points $s_0 < s_1 < \cdots < s_K$ with mesh size $h = \max_{k=1,\ldots,K}(s_k - s_{k-1})$. Starting with $s_k^{(0)} = s_k$, we introduce a hierarchy of pivotal points

$$s_k^{(n)} = \frac{s_k + \cdots + s_{k-n}}{n+1} \ , \qquad k = n, \ldots, K \ , \quad n \leq K \ . \tag{9}$$

Note that $s_n^{(n)} < s_{n+1}^{(n)} < \cdots < s_K^{(n)}$ with $s_k^{(n)} \in (s_{k-1}^{(n-1)}, s_k^{(n-1)})$ and

$$0 \leq \max_{k=n+1,\ldots,K} (s_k^{(n)} - s_{k-1}^{(n)}) \leq h \ . \tag{10}$$

In the case of equidistant grids the pivotal points either coincide with grid points (n even) or with midpoints (n uneven). Let

$$L_{k-1}^{(n)}(s) = \frac{s_k^{(n)} - s}{s_k^{(n)} - s_{k-1}^{(n)}} \ , \qquad L_k^{(n)}(s) = \frac{s - s_{k-1}^{(n)}}{s_k^{(n)} - s_{k-1}^{(n)}}$$

denote the linear Lagrange polynomials on the interval $[s_{k-1}^{(n)}, s_k^{(n)}]$. We now introduce piecewise linear approximations p_n of $w^{(n)}$ by successive piecewise interpolation. More precisely, we set

$$p_0(s) = \sum_{j=k-1}^{k} w(s_j) L_j^{(0)}(s) \ , \qquad p_n(s) = \sum_{j=k-1}^{k} p'_{n-1}(s_j^{(n)}) L_j^{(n)}(s) \tag{11}$$

for $s \in [s_{k-1}, s_k]$, $k = 1, \ldots, K$, and $s \in [s_{k-1}^{(n)}, s_k^{(n)}]$, $k = n+1, \ldots, K$, respectively. The approximation p_n can be regarded as the piecewise linear interpolation of divided differences.

Lemma 1. *The derivative p'_{n-1} has the representation*

$$p'_{n-1}(s_k^{(n)}) = n!\, w[s_{k-n}, \dots, s_k]\ , \quad k = n, \dots, K\ , \tag{12}$$

where $w[s_{k-n}, \dots, s_k]$ denotes the divided differences of w with respect to $s_{k-n}, \dots, s_k$.

Proof. Recall that $s_k^{(n)} \in (s_{k-1}^{(n-1)}, s_k^{(n-1)})$. Using the definitions (9), (11), we immediately get

$$p'_{n-1}(s_k^{(n)}) = \frac{p_{n-1}(s_k^{(n-1)}) - p_{n-1}(s_{k-1}^{(n-1)})}{s_k^{(n-1)} - s_{k-1}^{(n-1)}} = \frac{n\left(p'_{n-2}(s_k^{(n-1)}) - p'_{n-2}(s_{k-1}^{(n-1)})\right)}{s_k - s_{k-n}}$$

so that the assertion follows by straightforward induction.

We are now ready to state the main result of this section.

Theorem 2. *Assume that $w \in C^{n+2}[s_0, s_K]$ and let p_n be defined by (11). Then*

$$\max_{s\in[s_n^{(n)}, s_K^{(n)}]} |w^{(n)}(s) - p_n(s)| \le (n + \tfrac{1}{2})\|w^{(n+2)}\|_\infty\, h^2$$

holds with $\|w^{(n+2)}\|_\infty = \max_{x\in[s_0,s_K]} |w^{(n+2)}(x)|$.

Proof. Let $s \in [s_{k-1}^{(n)}, s_k^{(n)}]$ and denote $\varepsilon_n(s) = w^{(n)}(s) - p'_{n-1}(s)$. Exploiting the linearity of interpolation and a well–known interpolation error estimate (cf., e.g., [8, Theorem 7.16]), we obtain

$$w^{(n)}(s) - p_n(s) = \tfrac{w^{(n+2)}(\zeta)}{2}(s - s_{k-1}^{(n)})(s - s_k^{(n)}) + L_{k-1}^{(n)}(s)\varepsilon_n(s_{k-1}^{(n)}) + L_k^{(n)}(s)\varepsilon_n(s_k^{(n)})$$

with some $\zeta \in (s_{k-1}^{(n)}, s_k^{(n)})$. In the light of (10), it is sufficient to show that $|\varepsilon_n(s_{k-1}^{(n)})| + |\varepsilon_n(s_k^{(n)})| \le n\|w^{(n+2)}\|_\infty h^2$. Utilizing (9) and Lemma 1, we get

$$\varepsilon_n(s_k^{(n)}) = w^{(n)}\left(\frac{1}{n+1}\sum_{i=k-n}^{k} s_i\right) - n!\, w[s_{k-n}, \dots, s_k] =: A - B\ .$$

The Hermite–Genocchi formula (cf., e.g., [8, Theorem 7.12]) yields

$$B = n! \int_{\Sigma^n} w^{(n)}\left(\sum_{i=k-n}^{k} x_i s_i\right)\, dx\ ,$$

where Σ^n denotes the n–dimensional unit simplex

$$\Sigma^n = \{x \in \mathbb{R}^{n+1} \,|\, \textstyle\sum_{i=0}^n x_i = 1 \text{ and } x_i \ge 0\}\ .$$

As $|\Sigma^n| = 1/n!$, the value A is just the centroid formula for the quadrature of the integral B [9]. It is obtained by simply replacing the integrand by its barycentric value. Using a well–known error estimate [10], we obtain

$$|\varepsilon_n(s_k^{(n)})| \le \frac{\|w^{(n+2)}\|_\infty}{2(n+1)(n+2)} \sum_{i=k-n}^{k} |s_i - s_k^{(n)}|^2\ .$$

Now the assertion follows from the straightforward estimate $|s_i - s_k^{(n)}| \le nh$.

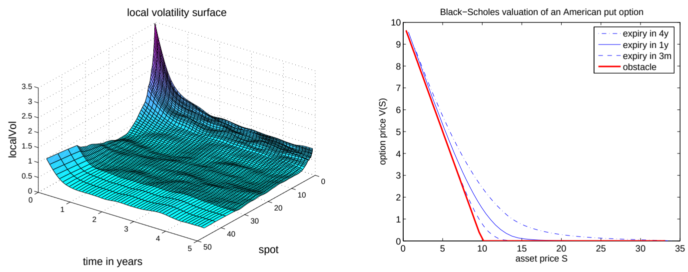

Fig. 1. Local volatility σ and computed values V at time $t = 0$.

In the remaining boundary regions $s \in [s_0, s_n^{(n)}]$ and $s \in [s_K^{(n)}, s_K]$, the function p_n can still be defined according to (11) once a hierarchy of additional pivotal points $s_k^{(n)}$ for $k = 0, \ldots, n-1$ and $k = K+1, \ldots, K+n$ has been selected. However, the approximation in such regions then reduces to first order, unless additional boundary conditions of u at s_0 and s_K are incorporated.

5 Numerical Results

For confidentiality reasons, we consider an American put option on an artificial single share with EURIBOR interest rates, strike price $K = 10$€, and an artificial but typical volatility surface σ as depicted in the left picture of Figure 1 (see also [5, 6]) for the different expiry dates $T = 3/12$, 1, and 4 years. Discrete dividends of $\delta_i = 0.3$ € are paid after $t_i = 4/12, 16/12, 28/12, 40/12$ years. In order to incorporate discrete dividend payments into our model (1), $V(S)$ is replaced by $\tilde{V}(\tilde{S})$, φ, σ are replaced by the shifted functions $\tilde{\varphi}(\tilde{S}) = \varphi(\tilde{S} + D)$, $\tilde{\sigma}(\tilde{S}, \cdot) = \sigma(\tilde{S} + D, \cdot)$ and we set $d = 0$. Here, $D(t)$ is the present value of all dividends yet to be paid until maturity [5, p. 7f.]. We set $\tilde{S}_{\min} = e^{-1}$ and $\tilde{S}_{\max} = e^{3.5}$. Finally, $V(S) = \tilde{V}(S - D)$ is the desired value of the option.

Local volatility data are given on a grid $S_0 = 0.36 < S_1 < \cdots < S_K = 100$. The transformed grid points $x_k = \log(S_k)$ are equidistant for $S_k < 4$, $S_k > 30$ while the original grid points S_k are equidistant for $4 < S_k < 30$ thus reflecting nicely the slope of the volatility surface for small S. To approximate α', α'' occurring in

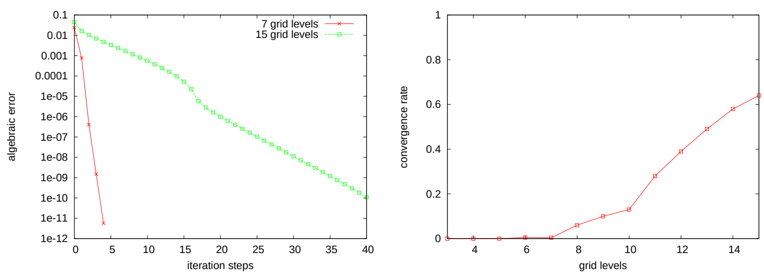

Fig. 2. Iteration history and averaged convergence rates

Theorem 1, we use the recovery procedure (11) with respect to an extension of the hierarchy $S_k^{(2)}$ as defined in (9), though second order accuracy is only guaranteed for $s \in [S_2^{(2)}, S_K^{(2)}]$ (cf. Theorem 2). For the actual data set, the coefficient b is positive and thus the transformed problem (6) is uniquely solvable, if the time steps satisfy $h_j < 0.35$ years. Note that much smaller time steps are required for accuracy reasons.

The transformed interval $\overline{X} = [-1, 3.5]$ is discretized by an equidistant grid with mesh size $H = 1/128 = 2^{-7}$ and we use the uniform time step $h = T/100$ years, for simplicity. Such step sizes are typical as to desired accuracies. The solutions at time $t = 0$ for the different expiry dates are depicted in the right picture of Figure 1. Note that only the options with the long maturity of 1 or 4 years are influenced by dividend payments until expiry date. The spatial problems of the form (6) were solved by truncated monotone multigrid [12] with respect to $J = 7$ grid levels as obtained by uniform coarsening. The initial iterates on time level j were taken from the preceding time level for $j > 1$ and from the obstacle function ψ for $j = 1$. We found that two or three $V(1,1)$ sweeps were sufficient to reduce the algebraic error $\|u_j - u_j^\nu\|$ in the energy norm below 10^{-10}. The corresponding iteration history on the initial time level is shown in the left picture of Figure 2. The iteration history for $H = 1/32768 = 2^{-15}$ and the averaged convergence rates as depicted in the right picture illustrate the convergence behavior for decreasing mesh size.

References

[1] Y. Achdou and O. Pironneau. *Computational Methods for Option Pricing.* SIAM, Philadelphia, 2005.

[2] L. Badea. Convergence rate of a Schwarz multilevel method for the constrained minimization of nonquadratic functionals. *SIAM J. Numer. Anal.*, 44:449–477, 2006.

[3] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Polit. Econ.*, 81:637–659, 1973.

[4] A. Brandt and C.W. Cryer. Multigrid algorithms for the solution of linear complementary problems arising from free boundary problems. *SIAM J. Sci. Stat. Comput.*, 4:655–684, 1983.

[5] O. Brockhaus, M. Farkas, A. Ferraris, D. Long, and M. Overhaus. *Equity Derivatives and Market Risk Models.* Risk Books, 2000.

[6] O. Brockhaus, A. Ferraris, C. Gallus, D. Long, R. Martin, and M. Overhaus. *Modelling and Hedging Equity Derivatives.* Risk Books, 1999.

[7] T.F. Coleman, Y. Li, and A. Verma. Reconstructing the unknown local volatility function. *J. Comp. Finance*, 2:77–102, 1999.

[8] P. Deuflhard and A. Hohmann. *Numerical Analysis in Modern Scientific Computing: An Introduction.* Springer, 2003.

[9] I.J. Good and R.A. Gaskins. The centroid method of numerical integration. *Numer. Math.*, 16:343–359, 1971.

[10] A. Guessab and G. Schmeißer. Convexity results and sharp error estimates in approximate multivariate integration. *Math. Comp.*, 73:1365–1384, 2004.

[11] M. Holtz and A. Kunoth. B-spline based monotone multigrid methods. *SIAM J. Numer. Anal.*, to appear.

[12] R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. *Numer. Math.*, 69:167 – 184, 1994.

[13] M. Křížek and P. Neittaanmäki. On superconvergence techniques. *Acta Appl. Math.*, 9:175–198, 1987.

[14] R. Lagnado and S. Osher. A technique for calibrating derivative security pricing models: Numerical solution of an inverse problem. *J. Comp. Finance*, 1:13–25, 1997.

[15] C.W. Oosterlee. On multigrid for linear complementarity problems with application to American-style options. *ETNA*, 15:165–185, 2003.

[16] C. Reisinger and G. Wittum. On multigrid for anisotropic equations and variational inequalities "pricing multi–dimensional European and American options". *Comput. Vis. Sci.*, 7:189–197, 2004.