A Domain Decomposition Method for the Diffusion of an Age-structured Population in a Multilayer Environment

Caterina Cusulin[1] and Luca Gerardo-Giorda[2]

[1] Faculty of Mathematics, University of Vienna, Nordbergstraße 15, 1090 Vienna, Austria. caterina.cusulin@univie.ac.at
[2] Dipartimento di Matematica, Università di Trento - Italy. gerardo@science.unitn.it

1 Introduction

The spatial spread of an age-structured population in an isolated environment is commonly governed by a partial differential equation with zero-flux boundary condition for the spatial domain. The variables involved are time, age and space, which will be denoted in the following by t, a and x, respectively. We denote the spatial domain by $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, and we assume the age of the population to be bounded, *i.e.* there exists $a_\dagger > 0$ such that $a \in [0, a_\dagger]$. Denoting the population density at time t per unit volume and age by $p(t, a, x)$, the total population at time t is given by

$$P(t) = \int_\Omega \int_0^{a_\dagger} p(t, a, x)\, da\, dx.$$

Let then $T > 0$, the population density $p(t, a, x)$ satisfies the following model problem.

Find $p(t, a, x) \in C(0, T; L^2(0, a_\dagger; H^1(\Omega)))$ such that

$$\begin{array}{rll} p_t + p_a + \mu(a)\, p - \text{div}\ (k(a,x)\nabla p) = g & \text{in } (0,T) \times (0,a_\dagger) \times \Omega \\ p(0,a,x) = p_0(a,x) & \text{in } (0,a_\dagger) \times \Omega \\ p(t,0,x) = \displaystyle\int_0^{a_\dagger} \beta(a) p(t,a,x)\ da & \text{in } (0,T) \times \Omega \\ \mathbf{n} \cdot (k(a,x)\nabla p) = 0 & \text{on } (0,T) \times (0,a_\dagger) \times \partial\Omega, \end{array} \tag{1}$$

where $\mathbf{n}$ denotes the outward normal to $\partial\Omega$, $\beta(a)$ is the age-specific fertility, and $\mu(a)$ is the age-specific mortality, such that

$$\int_0^{a_\dagger} \mu(a)\, da = +\infty. \tag{2}$$

We refer to [7] and references therein for issues concerning existence and uniqueness for the solution of problem (1).

1.1 The Reduced Model

In order to avoid the difficulties entailed by the presence of an unbounded coefficient in (1), it is usual to introduce the *survival probability*

$$\Pi(a) = \exp\left(-\int_0^a \mu(s)\, ds\right),$$

and a new variable

$$u(t,a,x) = \frac{p(t,a,x)}{\Pi(a)}.$$

Owing to (2), the survival probability at age $a_\dagger$ vanishes, ensuring that no individual exceeds the maximal age.
With these positions, (1) is equivalent to the following reduced model problem.

Find $u(t,a,x) \in C(0,T;L^2(0,a_\dagger;H^1(\Omega)))$ such that

$$\begin{array}{ll} u_t + u_a - \text{div }(k(a,x)\nabla u) = f & \text{in } (0,T)\times(0,a_\dagger)\times\Omega \\ u(0,a,x) = u_0(a,x) & \text{in } (0,a_\dagger)\times\Omega \\ u(t,0,x) = \displaystyle\int_0^{a_\dagger} m(a)u(t,a,x)\, da & \text{in } (0,T)\times\Omega \\ \mathbf{n}\cdot(k(a,x)\,\nabla u) = 0 & \text{on } (0,T)\times(0,a_\dagger)\times\partial\Omega, \end{array} \tag{3}$$

where $u_0(a,x) = p_0(a,x)/\Pi(a)$, $f = g/\Pi(a)$, and where $m(a) = \Pi(a)\,\beta(a)$ is called maternity function.

1.2 Space-time Discretization

Classical approaches to the numerical solution of (3) integrate along the characteristics in age and time (see for instance [4, 5, 6]). However, the presence of different time scales in the dynamics suggests the use of different steps in the discretization of time and age (see [1, 2]).
Let us consider a discretization of the interval $(0,T)$ into N subintervals of length $\Delta t = T/N$ (for simplicity we consider a uniform discretization, adaptivity in time being beyond the scope of this paper). For equation (3) we advance in time by means of a backward Euler scheme, where the initial condition in age is computed at the previous time step. At each time step, we solve the parabolic problem in age and space:

Find $u^n \in L^2(0,a_\dagger;H^1(\Omega))$ such that

$$\begin{cases} \dfrac{d}{da}\,\langle u^{n+1},v\rangle + A(a;u^{n+1},v) = (f,v) + \dfrac{1}{\Delta t}(u^n,v) & \forall v\in H^1(\Omega) \\ u^{n+1}(0,x) = \displaystyle\int_0^{a_\dagger} m(a)u^n(a,x)\, da \end{cases} \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^1(\Omega)$ and $H^{-1}(\Omega)$, and where $A(a;\cdot,\cdot)$ is the bilinear form given by

$$A(a;w,v) = \int_\Omega k(a,x)\nabla w\cdot\nabla v + \frac{1}{\Delta t}\int_\Omega wv.$$

We discretize equation (4) in space by means of finite elements (see e.g. [8] for an introduction to finite element methods). Let $\Omega = \bigcup_{j=1}^{N} K_j$, where each $K_j = T_{K_j}(E)$ is an element of the triangulation, E is the reference simplex, and T_{K_j} is an invertible affine map. The associated finite element space is then

$$V_h = \left\{ \varphi_h \in C^0(\Omega) \,|\, \varphi_{h|K_j} \circ T_{K_j} \in \mathbb{P}_1(E) \right\},$$

where $\mathbb{P}_1(E)$ is the space of polynomials of degree at most one in each variable on E. A semi-discrete problem in space is then obtained by applying a Galerkin procedure and choosing a finite element basis for V_h. Since the finite element basis functions depend only on space, we can rewrite problem (4) as

$$\begin{cases} M \dfrac{d\mathbf{u}_h^{n+1}}{da} + \mathcal{A}(a)\mathbf{u}_h^{n+1} = \mathbf{f} + \dfrac{1}{\Delta t} M \mathbf{u}^n \\ \mathbf{u}_h^{n+1}(0,x) = \displaystyle\int_0^{a_\dagger} m(a)\mathbf{u}_h^n(a,x)\, da \end{cases} \tag{5}$$

where M is the mass matrix ($M_{ij} = \int_\Omega \varphi_j \varphi_i \, dx$) and $\mathcal{A}(a)$ is the stiffness matrix associated to the bilinear form $A(a;\cdot,\cdot)$, ($(\mathcal{A}(a))_{ij} = A(a;\varphi_j,\varphi_i)$).

2 Diffusion in a Multilayer Environment and Domain Decomposition

We consider a population spreading in a stratified environment composed of m layers, with zero flux boundary conditions. We refer the interested reader to [10] for issues concerning the motivations of such model. We suppose that the age-specific fertility and the age-specific mortality depend only on the layer, while the diffusion coefficients depend both on the age and on the layer. On the interface between the j-th and the $(j+1)$-th layer we have to impose the continuity of the trace and the normal flux, thus the equation in the j-th layer reads

$$\begin{array}{rll} \partial_t u_j + \partial_a u_j - \mathrm{div}\, (k_j(a,x)\nabla u_j) = f_j & \text{in } (0,T)\times(0,a_\dagger)\times\Omega_j \\ u_j(0,a,x) = u_{0,j}(a,x) & \text{in } (0,a_\dagger)\times\Omega_j \\ u_j(t,0,x) = \displaystyle\int_0^{a_\dagger} m_j(a)u_j(t,a,x)\, da & \text{in } (0,T)\times\Omega_j \\ \mathbf{n}_j\cdot(k_j(a,x)\nabla u_j) = 0 & \text{on } (0,T)\times(0,a_\dagger)\times(\partial\Omega\cap\partial\Omega_j) \\ u_j(t,a,x) = u_{j-1}(t,a,x) & \text{on } (0,T)\times(0,a_\dagger)\times(\overline{\Omega}_j\cap\overline{\Omega}_{j-1}) \\ u_j(t,a,x) = u_{j+1}(t,a,x) & \text{on } (0,T)\times(0,a_\dagger)\times(\overline{\Omega}_j\cap\overline{\Omega}_{j+1}) \\ \mathbf{n}_j\cdot(k_j\nabla u_j) = \mathbf{n}_j\cdot(k_{j-1}\nabla u_{j-1}) & \text{on } (0,T)\times(0,a_\dagger)\times(\overline{\Omega}_j\cap\overline{\Omega}_{j-1}) \\ \mathbf{n}_j\cdot(k_j\nabla u_j) = \mathbf{n}_j\cdot(k_{j+1}\nabla u_{j+1}) & \text{on } (0,T)\times(0,a_\dagger)\times(\overline{\Omega}_j\cap\overline{\Omega}_{j+1}) \end{array} \tag{6}$$

A domain decomposition procedure to solve equation (6) is thus straightforward. After time discretization we have to solve a parabolic problem: the age-space domain is naturally decomposed in strips $(0,a_\dagger)\times\Omega_j$ and the global solution is obtained by a standard waveform relaxation procedure (see e.g. [9]) we outline in the following section.

2.1 A Waveform Relaxation Procedure

For sake of simplicity in presentation, we give here the two-domain formulation of the domain decomposition algorithm, and, in order to improve readability, we drop any index referring to time discretization. We set $\Omega = \Omega_1 \cup \Omega_2$, we denote the interface between the two subdomains by $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, the space of traces on Γ of functions in $H^1(\Omega)$ by $\Lambda = H^{1/2}(\Gamma)$, and we set $V_i = H^1(\Omega_i)$ $(i = 1, 2)$. At each time step, the coupled problem reads as follows:

Find $u_1 \in L^2(0, a_\dagger; H^1(\Omega_1))$ and $u_2 \in L^2(0, a_\dagger; H^1(\Omega_2))$ such that

$$\begin{cases} \dfrac{d}{da}\langle u_1, v_1\rangle + A_1(a; u_1, v_1) = (f_1, v_1) & \forall v_1 \in V_1 \\ \dfrac{d}{da}\langle u_2, v_2\rangle + A_2(a; u_2, v_2) = (f_2, v_2) & \forall v_2 \in V_2 \\ u_1 = u_2 & \text{on } (0, a_\dagger) \times \Gamma \\ \dfrac{d}{da}\langle u_2, R_2\mu\rangle + A_2(a; u_2, R_2\mu) = & \forall \mu \in \Lambda \\ \qquad = (f, R_2\mu) + (f, R_1\mu) - \dfrac{d}{da}\langle u_1, R_1\mu\rangle - A_1(a; u_1, R_1\mu), & \end{cases} \tag{7}$$

where $A_i(a; \cdot, \cdot)$ denotes the restriction of the bilinear form $A(a; \cdot, \cdot)$ to Ω_i, whereas $R_i\mu$ denotes any possible extension of μ to Ω_i $(i = 1, 2)$.
We apply a balancing Neumann-Neumann waveform relaxation procedure to enforce the interface continuities of equation (7).

Step 1. At each time step, given an initial value $\lambda^0 \in L^2((0, a_\dagger) \times \Gamma)$, solve:

$$\begin{cases} \dfrac{d}{da}\langle u_1^{k+1}, v_1\rangle + A_1(a; u_1^{k+1}, v_1) = (f_1, v_1) & \forall v_1 \in V_1 \\ u_1^{k+1} = \lambda^k & \text{on } (0, a_\dagger) \times \Gamma \\ u_1^{k+1}(0, x) = u_1^0(x) & \end{cases}$$

and

$$\begin{cases} \dfrac{d}{da}\langle u_2^{k+1}, v_2\rangle + A_2(a; u_2^{k+1}, v_2) = (f_2, v_2) & \forall v_2 \in V_2 \\ u_2^{k+1} = \lambda^k & \text{on } (0, a_\dagger) \times \Gamma \\ u_2^{k+1}(0, x) = u_2^0(x). & \end{cases}$$

Step 2. Solve

$$\begin{cases} \dfrac{d}{da}\langle \psi_1^{k+1}, v_1\rangle + A_1(a; \psi_1^{k+1}, v_1) = (f_1, v_1) & \forall v_1 \in V_1 \\ \dfrac{d}{da}\langle \psi_1^{k+1}, R_1\mu\rangle + A_1(a; \psi_1^{k+1}, R_1\mu) = & \forall \mu \in \Lambda \\ \qquad = \dfrac{d}{da}\langle u_1^{k+1}, R_1\mu\rangle + \dfrac{d}{da}\langle u_2^{k+1}, R_2\mu\rangle & \\ \qquad\quad + A_1(a; u_1^{k+1}, R_1\mu) + A_2(a; u_2^{k+1}, R_2\mu) - (f, R_2\mu) - (f, R_1\mu) & \\ \psi_1^{k+1}(0, x) = 0 & \end{cases}$$

and

$$\begin{cases} \frac{d}{da}\langle \psi_2^{k+1}, v_2\rangle + A_2(a;\psi_2^{k+1}, v_2) = (f_2, v_2) & \forall v_2 \in V_2 \\ \frac{d}{da}\langle \psi_2^{k+1}, R_2\mu\rangle + A_2(a;\psi_2^{k+1}, R_2\mu) = & \forall \mu \in \Lambda \\ \qquad = \frac{d}{da}\langle u_1^{k+1}, R_1\mu\rangle + \frac{d}{da}\langle u_2^{k+1}, R_2\mu\rangle \\ \qquad\quad + A_1(a;u_1^{k+1}, R_1\mu) + A_2(a;u_2^{k+1}, R_2\mu) - (f, R_2\mu) - (f, R_1\mu) \\ \psi_2^{k+1}(0,x) = 0. \end{cases}$$

Step 3. Set

$$\lambda^{k+1} = \lambda^k - \vartheta\left(\frac{k_1}{k_1+k_2}\psi_1^{k+1} - \frac{k_2}{k_1+k_2}\psi_2^{k+1}\right)_{|(0,a_\dagger)\times\Gamma}$$

and iterate until convergence.

For a more detailed description of the algorithm we refer to [3].

3 Numerical Results

In this section we consider a population spreading in a one dimensional environment constituted of two layers. We solve problem (7) on the domain $\Omega = [0,1]$, and we assume $a_\dagger = 100$ as maximal age. In the numerical tests we choose $\Delta a = 2$, as well as $\Delta t = 1$. We let $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 = (0,\alpha)$, $\Omega_2 = (\alpha, 1)$, and we discretize problem (7) in space via $\mathbb{P}_1$ finite elements. We solve each subproblem by computing the integral in (5) via a Simpson quadrature rule, and by advancing implicitly in age (5) via a backward Euler scheme. For a more detailed description of the numerical approximation of (3) in a single domain we refer to [2]. We consider

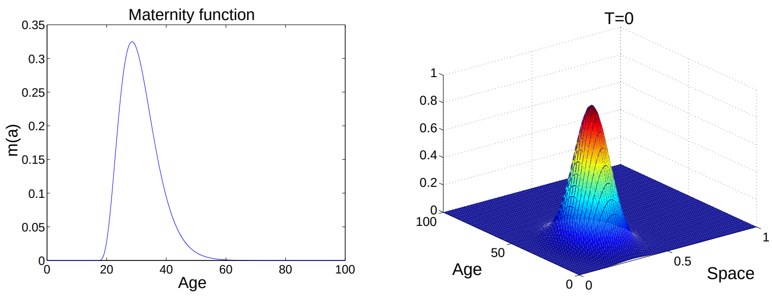

Fig. 1. Maternity function (left) and age-space initial profile (right) for the test cases

diffusion coefficients that are uniform in age and heterogeneous in space, with the ratio $\delta k = k_1/k_2$ up to 10^4. The maternity function and initial profile are given in

Fig. 1. In Table 1 and 2 we report the iteration counts at different time levels for two different positions of the interface, $\alpha = 0.5$ and $\alpha = 0.7$, with a mesh size of $h = 1/100$ in both subdomains. The stopping criterion is given by $\|\lambda^{k+1} - \lambda^k\|_0/\|\lambda^k\|_0 < 10^{-6}$. The number of iterations increases with the amplitude of the jumps in the diffusive coefficients, but the algorithm appears to be robust with respect to the position of the interface and with respect to the evolution in time. In Table 3 we report the iteration counts at different time levels for different mesh sizes in Ω_1 and Ω_2. We choose $k_1 = .1$, $k_2 = .01$ and the interface is $\alpha = .6$. The algorithm appears insensitive to the difference of mesh sizes between the two subdomains. In Figure 2 we report the time evolution of the space profile of individuals of age 20 with $\delta k = 100$ and the evolution of the iteration counts (with $\alpha = 0.5$ and $\delta k = 1, 10, 100$) for a longer simulation, with stopping criterion set at 10^{-10}. The jump in the normal derivative due to the high heterogeneity of the spatial medium is clearly visible, and the robustness of the algorithm with respect to the evolution in time is evident. Finally, in Figure 3 we report the age-space profile of the solution at time $T = 5$, with $\delta k = 100$.
The numerical tests are performed with MATLAB® 6.1. A more detailed description of the test cases as well as further numerical results in two dimensions in space can be found in a forthcoming paper ([3]).

Table 1. Two subdomains, $\alpha = 0.5$, $h_1 = h_2 = 1/100$: iteration counts per time step: $\|\lambda^{k+1} - \lambda^k\|_0/\|\lambda^k\|_0 < 10^{-6}$.

δk	$T = 1$	$T = 3$	$T = 6$	$T = 9$	$T = 12$	$T = 15$	$T = 20$
1	13	11	11	10	10	10	10
10	17	15	14	14	14	14	13
10^2	23	20	20	20	19	19	19
10^3	26	23	23	23	22	22	22
10^4	32	27	27	26	26	25	25

Table 2. Two subdomains, $\alpha = 0.7$, $h_1 = h_2 = 1/100$: iteration counts per time level: $\|\lambda^{k+1} - \lambda^k\|_0/\|\lambda^k\|_0 < 10^{-6}$.

δk	$T = 1$	$T = 3$	$T = 6$	$T = 9$	$T = 12$	$T = 15$	$T = 20$
1	17	11	11	10	10	10	10
10	22	14	14	14	14	13	13
10^2	32	20	19	19	19	18	18
10^3	37	23	22	22	22	21	21
10^4	44	26	26	25	25	25	25

Table 3. Two subdomains, $\alpha = 0.6$, $\nu_1 = .1$, $\nu_2 = .01$: iteration counts per time level: $\|\lambda^{k+1} - \lambda^k\|_0 / \|\lambda^k\|_0 < 10^{-6}$.

h_1/h_2	$T = 1$	$T = 3$	$T = 6$	$T = 9$	$T = 12$	$T = 15$	$T = 20$
1	19	14	14	14	14	13	13
10	19	14	14	14	14	13	13
50	19	14	14	14	14	13	13

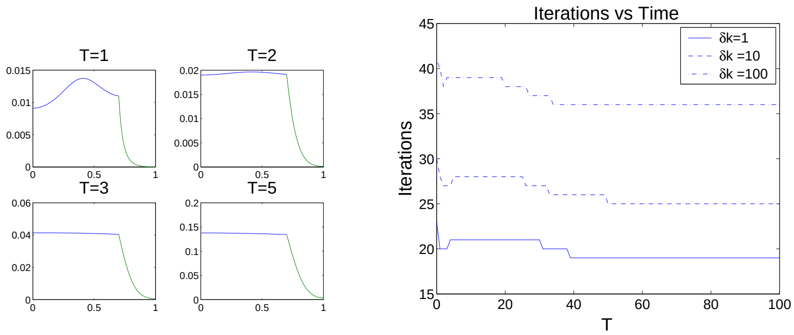

Fig. 2. Time evolution of the profile at age $a = 20$ (left, $\delta k = 100$) and Iterations count vs Time evolution (right, $\alpha = 0.5$, $\|\lambda^{k+1} - \lambda^k\|_0 / \|\lambda^k\|_0 < 10^{-10}$).

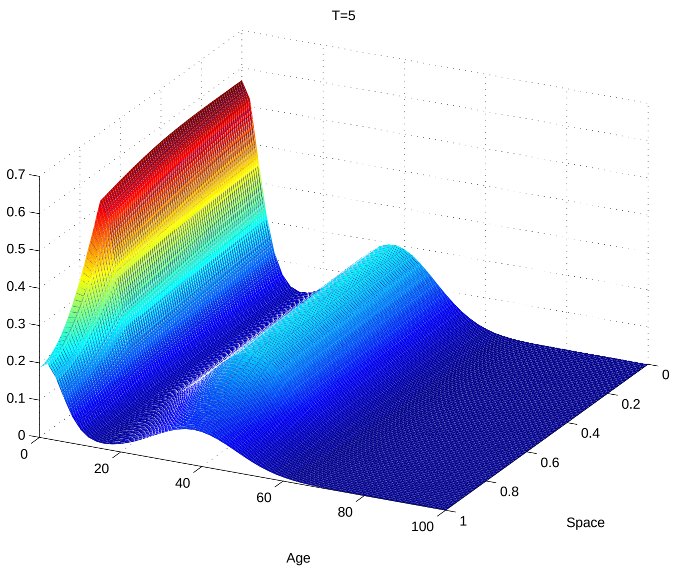

Fig. 3. Age-space profile at time $T = 5$, $\delta k = 100$.

4 Conclusions

We proposed here a balancing Neumann-Neumann procedure to approximate the solution of the diffusion of an age-structured population in a multilayer environment. The proposed algorithm appears to be very robust in terms of iteration counts with respect to the mesh size, the position of the interface, and the heterogeneities in the viscosity coefficients.

References

[1] B.P. Ayati and T. Dupont. Galerkin methods in age and space for a population model with nonlinear diffusion. *SIAM J. Numer. Anal.*, 40(3):1064–1076, 2002.

[2] C. Cusulin and L. Gerardo-Giorda. A FEM-Galerkin approximation for diffusion in age-structured population dynamics. Technical report, Dep. Math., University of Trento, 2006. In preparation.

[3] C. Cusulin and L. Gerardo-Giorda. Numerical approximation of the diffusion of an age-structured population in a multi-layer environment. Technical report, Dep. Math., University of Trento, 2006. In preparation.

[4] M.-Y. Kim. Galerkin methods for a model of population dynamics with nonlinear diffusion. *Numer. Methods Partial Differential Equations*, 12:59–73, 1996.

[5] M.-Y. Kim and E.-J. Park. Characteristic finite element methods for diffusion epidemic models with age-structured populations. *Comput. Math. Appl.*, 97:55–70, 1998.

[6] F.A. Milner. A numerical method for a model of population dynamics with spatial diffusion. *Comput. Math Appl.*, 19(31), 1990.

[7] A. Okubo and S.A. Levin. *Diffusion and Ecological Problems: Modern Perspectives.* Springer, New York, 2001.

[8] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations.* Springer-Verlag, Berlin, 1994.

[9] A. Quarteroni and A. Valli. *Domain Decompostion Methods for Partial Differential Equations.* Oxford University Press, 1999.

[10] N. Shigesada and K. Kawasaki. *Biological Invasions: Theory and Practice.* Oxford University Press, New York, 1997.