Schwarz Waveform Relaxation Algorithms

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Summary. Optimized Schwarz Waveform Relaxation algorithms have been developed over the last few years for the computation in parallel of evolution problems. In this paper, we present their main features.

1 Introduction

Model complexity in today's computation emphasizes the need for coupling models in different geographical zones. For evolution problems, it is desirable to design a coupling process where, the lesser subdomain boundary information is exchanged the better. This goal is different from the usual domain decomposition purpose, where either the scheme is explicit, and the exchange of information takes place every time-step, or the scheme is implicit (leading to a steady problem, usually elliptic), and domain decomposition techniques can be used, see [16] and [18].

The Schwarz Waveform Relaxation algorithms take their name from the waveform relaxation algorithms, developed in circuit simulation, see [3], and Schwarz method for solving in parallel, partial differential equations of elliptic type, see [17, 12]. The purpose is to solve the space-time partial differential equation in each subdomain in parallel, and to transmit domain boundary information to the neighbors at the end of the time interval. The basic idea comes from the world of absorbing boundary conditions: for a model problem, approximations of the Dirichlet-Neumann map are developed, which can be written in the Fourier variables. These approximations lead to transmission conditions which involve time and tangential derivatives. The coefficients in these transmission conditions are in turn computed so as to optimize the convergence factor in the algorithm. This process can be written as a complex best approximation problem of a homographic type, and solved either explicitly or asymptotically. This gives a convergent algorithm that we call Optimized Schwarz Waveform Relaxation algorithm, which outperforms the classical one, i.e. where transmission is made only by exchange of Dirichlet

data. It can be performed with or without overlap, and convergence is in any case much faster. It can be used with any high performance numerical method in the subdomains, extended to variables coefficients by a freezing process, [14], and to systems of equations, [13]. Finally, it can be used to couple different discretizations in different subdomains, and acts as a preconditioner for the full interface problem in space time [4].

The purpose of this paper is to present the main features of the optimized Schwarz waveform relaxation algorithms, to give a few proofs which are unpublished and which illustrate our mathematical techniques. For an extensive historical presentation, see [1], and for examples of applications see [9].

2 Description of the Schwarz Waveform Relaxation Algorithm

Suppose we want to solve an evolution equation $\mathcal{L}u = f$ in the domain Ω on the time domain $(0,T)$, with initial data u_0, and boundary conditions which will not be considered in this paper. Ω is split into subdomains Ω_j, $1 \leq j \leq J$, overlapping or non-overlapping. For each index j, $\mathcal{V}(j)$ is the set of indices of the neighbors of Ω_j, and we write for $k \geq 1$

$$\left\{ \begin{array}{rl} \mathcal{L}u_j^k = f & \text{in } \Omega_j \times (0,T), \\ u_j^k(\cdot,0) = u_0 & \text{in } \Omega_j, \\ \mathcal{B}_{jl}u_j^k = \mathcal{B}_{jl}u_l^{k-1} & \text{on } \partial\Omega_j \cap \bar{\Omega}_l \times (0,T), l \in \mathcal{V}(j), \end{array} \right. \tag{1}$$

with an initial guess $\mathcal{B}_{jl}u_l^0$ on the interfaces. This algorithm can be viewed as a Jacobi type iterative method, or as a preconditioner for an interface problem. We are only interested here in discussing the transmission conditions.

3 Classical Schwarz Waveform Relaxation Algorithm

As presented in the original paper [17], and analyzed in [12], the method concerns the Laplace equation and Dirichlet transmission conditions, *i.e.* $\mathcal{B}_j \equiv I$ and subdomains overlap. The algorithm is convergent, and the larger the overlap, the faster the method. In the evolution case, this algorithm is also convergent, but the mode of convergence depends on the type of equation. The starting point of our research was the example of the advection-diffusion equation, presented in DD11, [8]. The convergence curve exhibits a linear behavior for large time intervals, and is superlinear for small time intervals. The behavior is similar in higher dimension, [14].

For the wave equation, due to the finite speed of propagation, the convergence takes place in a finite number of steps $n_0 > \frac{cT}{L}$, where c is the wave speed, and L the size of the overlap. We showed in [7] an example, with a

finite differences discretization, which shows that the error decays very slowly until iteration n_0, and reaches 10^{-12} at iteration n_0.

The case of the Schrödinger equation is also very interesting, and has been addressed in [11]. See also the contribution by J. Szeftel in the minisymposium "Domain Decomposition Methods Motivated by the Physics of the Underlying Problem" in this issue. The Dirichlet transmission creates a highly oscillatory solution, and the convergence begins late, and this phenomenon gets worse as the final time increases.

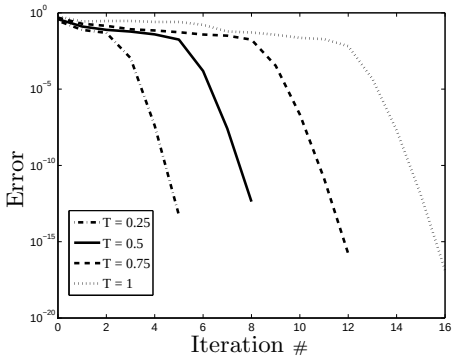

Fig. 1. Schrödinger equation: error with Dirichlet transmission conditions as a function of the iteration number for several final times.

We now describe a strategy to obtain more efficient transmission between subdomains.

4 Optimal Transmission Conditions

Let $\mathcal{L}$ denote a partial differential operator in time and space, which is elliptic in the space variables. For a domain Ω in $\mathbb{R}^n$, the Dirichlet-Neumann map $\mathcal{T}_\Omega$ maps a function h defined on $\partial\Omega \times (0,T)$ to the normal derivative $\frac{\partial u}{\partial n}$ where n is the unit exterior normal to $\partial\Omega$, and u the solution of $\mathcal{L}u = 0$ in $\Omega \times (0,T)$, with vanishing initial data. The importance of this map lies in the important result concerning a domain decomposition in layers: if it is used as transmission operator on the boundaries of the subdomains, then the convergence is achieved in J iterations, see in [15] a formal proof for an elliptic problem.

4.1 Example: the Advection-Diffusion Equation

We consider the operator

$$\mathcal{L} = \partial_t + (\mathbf{a} \cdot \boldsymbol{\nabla}) - \nu\Delta + cId,$$

with $\mathbf{a} = (a, \mathbf{b})$, $a \in \mathbb{R}^+, \mathbf{b} \in \mathbb{R}^{n-1}$. We search for the Dirichlet-Neumann map for a half-space $\mathbb{R}^{\pm} \times \mathbb{R}^{n-1}$, and denote by x the first coordinate, and $\mathbf{y}$ the $(n-1)-$dimensional coordinate. By Fourier transform $t \leftrightarrow \omega, \mathbf{y} \leftrightarrow \boldsymbol{\zeta}$, we see that the Fourier transform $\hat{w} = \mathcal{F}w$ of any solution of $\mathcal{L}w = 0$ is solution of the ordinary differential equation

$$-\nu \frac{\partial^2 \hat{w}}{\partial x^2} + a \frac{\partial \hat{w}}{\partial x} + \big(i(\omega + \mathbf{b} \cdot \boldsymbol{\zeta}) + \nu |\boldsymbol{\zeta}|^2 + c\big)\hat{w} = 0, \tag{2}$$

with characteristic roots

$$r_{\pm}(\boldsymbol{\zeta}, \omega) = \frac{a \pm \sqrt{\delta(\boldsymbol{\zeta}, \omega)}}{2\nu}, \quad \delta(\boldsymbol{\zeta}, \omega) = a^2 + 4\nu(i(\omega + \mathbf{b} \cdot \boldsymbol{\zeta}) + \nu|\boldsymbol{\zeta}|^2 + c). \tag{3}$$

The complex square root in this text is always with strictly positive real part. In order to work with at least square integrable functions in time and space, we seek for solutions which do not increase exponentially in time. Since the real parts of the roots, $\Re r_{\pm}$, are such that $\Re r_+ > 0$ and $\Re r_- < 0$, the Dirichlet-Neumann map $\mathcal{T}_+$ for $(L, +\infty) \times \mathbb{R}^{n-1}$ (resp. $\mathcal{T}_-$ for $(-\infty, L) \times \mathbb{R}^{n-1}$) is given by

$$\begin{aligned} \mathcal{F}\mathcal{T}_{\pm}(h)(\boldsymbol{\zeta}) &= r_{\mp}(\boldsymbol{\zeta}, \omega)\mathcal{F}h(\boldsymbol{\zeta}), \\ \mathcal{T}_{\pm} &= \frac{a \mp \sqrt{a^2 + 4\nu(\partial_t + \mathbf{b} \cdot \boldsymbol{\nabla} - \nu\Delta_{\mathbf{y}} + c)}}{2\nu}. \end{aligned} \tag{4}$$

Since the operator $\mathcal{L}$ has constant coefficients, the Dirichlet-Neumann maps do not depend on L. We consider J subdomains $\Omega_j = (a_j, b_j) \times \mathbb{R}^{n-1}$, with $a_1 = -\infty$, $b_J = +\infty$, and $a_j \leq b_{j-1} < b_j$ for $1 \leq j \leq J$, and transmission operators of the form

$$\mathcal{B}_{ij} \equiv \partial_x - S_{ij}(\partial_{\mathbf{y}}, \partial_t).$$

The Fourier transform of the error $e_j^k = u - u_j^k$ in each subdomain is given by

$$\mathcal{F}e_j^k = \alpha_j^k e^{r_+ x} + \beta_j^k e^{r_- x}, \qquad 1 \leq j \leq J, \tag{5}$$

with $\beta_1^k = 0$ and $\alpha_J^k = 0$.

Theorem 1. *With the transmission conditions $S_{jj-1} = \mathcal{T}_-, S_{jj+1} = \mathcal{T}_+$, the algorithm is optimal: the convergence is achieved in J iterations.*

Proof. Inserting (5) in the transmission conditions, we get at each step k a system of $2J$ equations with $2J$ unknowns (α_j^k, β_j^k). In the case of the theorem, the system reduces to $\alpha_j^k = \alpha_{j+1}^{k-1}$ and $\beta_j^k = \beta_{j-1}^{k-1}$. Since $\beta_1^1 = 0$ and $\alpha_J^1 = 0$, we deduce that $\alpha_j^J = 0$ and $\beta_j^J = 0$ for all j. Thus at iteration J, the solution of the algorithm is equal to u in each subdomain.

Note that the result still holds for any partial differential equation with constant coefficients, like Schrödinger equations (see [11]), or even systems, like the shallow-water system, see [13].

4.2 The Quasi Optimal Algorithm in One Dimension

The transparent operator is global in time and space. When used in the context of absorbing boundary conditions, enormous efforts have been made for the approximation of the Dirichlet-Neumann map by local operators, in order to obtain sparse matrices in the actual computations (see [10] in the context of parabolic equations). However in one dimension, we have to transmit informations on the interface over the whole time interval. Therefore we can afford to use pseudodifferential operators in time. For the advection-diffusion equation, we find in [6] a discrete Dirichlet-Neumann map for several numerical schemes, *e.g.* Euler and Crank-Nicolson. We have used this strategy for the Schrödinger equation, and we have shown that the convergence is achieved in very few iterations, even with non constant potential, see [11]. We intend to explore this direction in higher dimension.

5 Optimized SWR Algorithm for the Advection-Diffusion Equation

We now try to improve locally the transmission between the subdomains. Therefore we restrict ourselves to the case of two subdomains, and we develop the analysis in $\mathbb{R}^n \times (0,T)$.

5.1 Partial Differential Transmission Conditions

In this part, we write differential transmission conditions as follows. We replace in $r_\pm$ the square root by a polynomial of degree lower than or equal to 1, *i.e.* we set

$$\tilde{r}_\mp = \frac{a \pm \left(p + q\left(i(\omega + \mathbf{b}\cdot\boldsymbol{\zeta}) + \nu|\boldsymbol{\zeta}|^2\right)\right)}{2\nu},$$

with real parameters p and q to be chosen. This defines the new transmission operators

$$\widetilde{\mathcal{B}}_{ii+1} = \partial_x - \frac{a - p_{ii+1}}{2\nu} + q_{ii+1}(\partial_t + \mathbf{b}\cdot\boldsymbol{\nabla} - \nu\Delta_y),$$

$$\widetilde{\mathcal{B}}_{ii-1} = \partial_x - \frac{a + p_{ii-1}}{2\nu} - q_{ii-1}(\partial_t + \mathbf{b}\cdot\boldsymbol{\nabla} - \nu\Delta_y),$$

with real parameters p_{ij} and q_{ij} to be chosen. In the case where $q_{ij} = 0$, the transmission condition reduces to Robin transmission condition, which is already used as a preconditioner for domain decomposition in the steady case. We call it transmission condition of order 0, whereas when $q_{ij} \neq 0$, we talk about first order transmission condition.

5.2 Well-Posedness and Convergence of the Algorithm

We examine in details the Robin case. It was proved in [14] that in the case of two half-spaces with only one coefficient $p > 0$, the boundary value problems are well-posed in suitable Sobolev spaces, and the algorithm is convergent. The proof relies on Lebesgue Theorem in the overlapping case. In the non overlapping case, the proof uses energy estimates as in [5], and therefore holds for space varying advection. We prove below that this result holds in any reasonable geometry, as depicted in Figure 2.

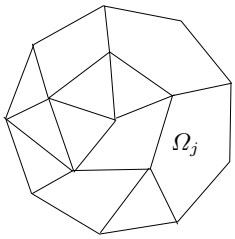

Fig. 2. Decomposition in space with nonoverlapping subdomains

The operators B_{jl} are given by

$$B_{jl} = \frac{\partial}{\partial_{n_j}} - \frac{\mathbf{a}\cdot\mathbf{n_j} - p_{jl}}{2\nu}, \tag{6}$$

where n_j is the normal to $\partial\Omega_j$, exterior to Ω_j.

Theorem 2. *In the nonoverlapping case, the domain decomposition algorithm* (1) *with transmission conditions* (6) *converges for any choice of the positive coefficients p_{jl} with $p_{jl} = p_{lj}$, provided each domain is visited infinitely many times.*

Proof. In order to shorten the proof, we work here with the heat equation. We write the equation for the error $e_j^k = u - u_j^k$ in Ω_j. We multiply it by e_j^k and integrate by parts:

$$\frac{1}{2}\frac{d}{dt}\|e_j^k\|^2_{\Omega_j} + \nu\|e_j^k\|^2_{\Omega_j} - \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \nu\frac{\partial e_j^k}{\partial_{n_j}}\, e_j^k ds = 0.$$

We rewrite the boundary term as

$$-\nu\int_{\Gamma_{jl}} \frac{\partial e}{\partial_{n_j}}\, e\, ds = \frac{1}{2p_{jl}}\left[\int_{\Gamma_{jl}}\left(\nu\frac{\partial e}{\partial_{n_j}} - \frac{p_{jl}}{2}\, e\right)^2 ds - \int_{\Gamma_{jl}}\left(\nu\frac{\partial e}{\partial_{n_j}} + \frac{p_{jl}}{2}\, e\right)^2 ds\right],$$

and obtain

$$\frac{1}{2}\frac{d}{dt}\|e_j^k\|^2_{\Omega_j} + \nu\|e_j^k\|^2_{\Omega_j} + \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \frac{1}{2p_{jl}}\left[\nu\frac{\partial e_j^k}{\partial_{n_j}} - \frac{p_{jl}}{2}\,e_j^k\right]^2 ds$$
$$= \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \frac{1}{2p_{jl}}\left[\nu\frac{\partial e_j^k}{\partial_{n_j}} + \frac{p_{jl}}{2}\,e_j^k\right]^2 ds.$$

We use on the right hand side the transmission condition:

$$\frac{1}{2}\frac{d}{dt}\|e_j^k\|^2_{\Omega_j} + \nu\|e_j^k\|^2_{\Omega_j} + \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \frac{1}{2p_{jl}}\left[\nu\frac{\partial e_j^k}{\partial_{n_j}} - \frac{p_{jl}}{2}\,e_j^k\right]^2 ds$$
$$= \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \frac{1}{2p_{jl}}\left[\nu\frac{\partial e_l^{k-1}}{\partial_{n_j}} + \frac{p_{jl}}{2}\,e_l^{k-1}\right]^2 ds$$
$$= \sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}} \frac{1}{2p_{jl}}\left[-\nu\frac{\partial e_l^{k-1}}{\partial_{n_l}} + \frac{p_{jl}}{2}\,e_l^{k-1}\right]^2 ds.$$

Summing up on all domains, we have on the left a boundary term at step k and on the right the same term at step $k-1$. Summing up on all steps k, we obtain

$$\frac{d}{dt}\sum_{k=0}^{K}\sum_{j=1}^{J}\|e_j^k\|^2_{\Omega_j} + 2\nu\sum_{k=0}^{K}\sum_{j=1}^{J}\|e_j^k\|^2_{\Omega_j} + \sum_{j=1}^{J}\sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}}\frac{1}{p_{jl}}\left[\nu\frac{\partial e_j^K}{\partial_{n_j}} - \frac{p_{jl}}{2}\,e_j^K\right]^2 ds$$
$$= \sum_{j=1}^{J}\sum_{l\in\mathcal{V}(j)}\int_{\Gamma_{jl}}\frac{1}{p_{jl}}\left[\nu\frac{\partial e_l^0}{\partial_{n_l}} - \frac{p_{jl}}{2}\,e_l^0\right]^2 ds.$$

which proves that $\sum_{j=1}^{J}\|e_j^k\|^2_{L^\infty(0,T;\Omega_j)} + 2\nu\sum_{j=1}^{J}\|e_j^k\|^2_{L^2(0,T;\Omega_j)}$ is bounded. By assumption, we have an infinite sequence e_j^k, and for every j, e_j^k tends to 0 as k tends to infinity.

There is no proof available for the overlapping domains in general geometry. Concerning the case $q \neq 0$, there is a proof of well-posedness and convergence for the layered case in [2]. The well-posedness in general geometry, even in the nonoverlapping case, has not yet been addressed.

5.3 Optimization of the Convergence Factor

In order to improve the performances of the method, it is important to optimize the convergence between two subdomains. In this case, the notations are much simpler, there are only two parameters p and q, and we write

$$\mathcal{F}e_j^{k+2}(0,\boldsymbol{\zeta},\omega) = \rho(\boldsymbol{\zeta},\omega,P,L)\mathcal{F}e_j^k(0,\boldsymbol{\zeta},\omega),$$

with the convergence factor defined for a polynomial $P \in \mathbf{P}_n$ by

$$\rho(\boldsymbol{\zeta}, \omega, P, L) = \left(\frac{P\left(i(\omega + \mathbf{b}\cdot\boldsymbol{\zeta}) + \nu|\boldsymbol{\zeta}|^2\right) - \sqrt{\delta(\boldsymbol{\zeta},\omega)}}{P\left(i(\omega + \mathbf{b}\cdot\boldsymbol{\zeta}) + \nu|\boldsymbol{\zeta}|^2\right) + \sqrt{\delta(\boldsymbol{\zeta},\omega)}} \right)^2 e^{-2\frac{L}{\nu}\sqrt{\delta(\boldsymbol{\zeta},\omega)}}.$$

The convergence factor has two terms: the exponential one comes from the overlap, and kills the high frequencies in time or space, whereas the fractional term comes from the transmission condition and tends to 1 as frequencies in time or space become high. Since the overlap has to be small for computational reasons, and also since it is of interest to have a nonoverlapping strategy, we want to make the fractional part as small as possible. In a real computation, only frequencies supported by the grid are relevant, ω and $\boldsymbol{\zeta}$ live in the compact set

$$K = \{(\omega, \boldsymbol{\zeta}),\ \omega_m \le |\omega| \le \omega_M,\ \zeta_{j,m} \le |\zeta_j| \le \zeta_{j,M},\ j = 1, \cdots, n-1\}.$$

with $\omega_m = \pi/T$, $\omega_M = \pi/\Delta t$, where Δt is the time step, and similarly $\zeta_{j,m} = \pi/Y_j$, $\zeta_{j,M} = \pi/\Delta y_j$, where Y_j is the length in the y_j direction and Δy_j the mesh size. The optimization of the convergence factor is formulated as a best approximation problem in $\mathbf{P}_n$ for $n = 0$ or 1,

$$\sup_{(\boldsymbol{\zeta},\omega)\in K} |\rho(\boldsymbol{\zeta}, \omega, P_n^*)| = \inf_{P\in\mathbf{P}_n} \sup_{(\boldsymbol{\zeta},\omega)\in K} |\rho(\boldsymbol{\zeta}, \omega, P)|. \tag{7}$$

Problem (7) has been solved in a more general setting in [2], and exact formulas in the one-dimensional case, together with asymptotic results, have been given. A general result asserts that the infimum is strictly small than 1, the problem has a unique solution, and furthermore, the solution is a real polynomial, and is the solution of the best approximation problem set in the space of real polynomials. Here we study Problem (7) for $n = 0$, which corresponds to Robin transmission conditions, and for $L = 0$, which means without overlap. In particular we have $p^* > 0$. We are going now to characterize p^*.

We choose new variables $\tau = \omega + \mathbf{b}\cdot\boldsymbol{\zeta}$, $\eta = \sqrt{a^2 + 4\nu c + 4\nu|\boldsymbol{\zeta}|^2}$, and use $\xi = \Re\sqrt{\delta}$. With the new notations we have

$$|\rho(\boldsymbol{\zeta}, \omega, p)| = R(\tau, \eta, p) = \frac{(\xi - p)^2 + \xi^2 - \eta^2}{(\xi + p)^2 + \xi^2 - \eta^2},\ \xi(\tau, \eta) = \sqrt{(\eta^2 + \sqrt{\tau^2 + \eta^4}\,)/2}\ ,$$

and the best approximation problem for ρ has the same solution as the one for R in the subspace of $\mathbb{R}^2$, $D = [\eta_m, \eta_M] \times [\tau_m, \tau_M]$, with $\tau_m = 0$. A point M in the plane will be defined by its coordinates (η, τ) and we will call A_j the edges of D: $A_1 = (\eta_m, \tau_m)$, $A_2 = (\eta_M, \tau_m)$, $A_3 = (\eta_M, \tau_M)$, $A_4 = (\eta_m, \tau_M)$. We will interchangeably use $R(M, p)$ or $R(\tau, \eta, p)$.

The upper bounds τ_M, η_M are inversely proportional to the time and space steps. Depending on whether an explicit or implicit scheme is used, we can have Δt of the order of Δy or Δy^2, respectively. Therefore we assume here that $\tau_M = C\eta_M^\beta$, with $\beta = 1$ or 2.

Theorem 3. *For $n = 0$ and $L = 0$, problem (7) has a unique solution $p^* > 0$. If η_M is large and $\tau_M = C\eta_M^\beta$, it is given by*

$$p^* = \sqrt{\frac{\xi(A_1)\sqrt{\tau_M^2+\eta_M^4} - \xi(A_3)\sqrt{\tau_m^2+\eta_m^4}}{\xi(A_3)-\xi(A_1)}} \text{ if } \beta = 1, \text{ or } \beta = 2 \text{ and } C < C_0,$$
$$p^* = \sqrt{\eta_m^2 + 2\eta_m\xi(A_4)} \text{ if } \beta = 2 \text{ and } C > C_0.$$

Proof. We proceed in several steps.

Lemma 1. *For any positive p, the maximum of $(\tau, \eta) \mapsto R(\tau, \eta, p)$ on D is reached on one of the edges of D.*

The analytic function ρ can reach extrema only on the boundary of the domain. The partial derivatives of R with respect to τ and η delimit regions in the plane, see Figure 3, from which we infer that an extremum on any segment of the boundary, distinct from the edges, can only be a minimum.

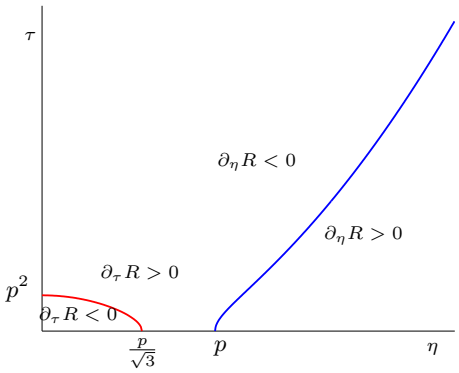

Fig. 3. Regions delimited by the zeros of the partial derivatives of R

Lemma 2. *If either η_M or τ_M is large, there exists a unique $p_* > 0$, such that $R(A_1, p_*) = R(A_3, p_*)$, and it is given by*

$$p_* = \sqrt{\frac{\xi(A_1)\sqrt{\tau_M^2+\eta_M^4} - \xi(A_3)\sqrt{\tau_m^2+\eta_m^4}}{\xi(A_3)-\xi(A_1)}}.$$

Further, there exists a unique $p_{**} > 0$, such that $R(A_1, p_{**}) = R(A_4, p_{**})$, and it is given by

$$p_{**} = \sqrt{\eta_m^2 + 2\eta_m\xi(A_4)}.$$

This can be seen by writing for any points M_1, M_2 in D,

$$R(M_1, p) - R(M_2, p) = \frac{4p(\xi(M_2)-\xi(M_1))}{D(M_1,M_2)}\left[Q(M_1, M_2) - p^2\right],$$
$$Q(M_1, M_2) = 2\xi(M_2)\xi(M_1) + \frac{\xi(M_2)\eta_1^2 - \xi(M_1)\eta_2^2}{\xi(M_2)-\xi(M_1)},$$

with a positive denominator $D(M_1, M_2)$, and discussing the sign of Q.

Lemma 3. *For large η_M we have:*

1. *If $\beta = 1$, or $\beta = 2$ and $C < C_0$, $\sup_{M\in D} R(M, p_*) = R(A_1, p_*) = R(A_3, p_*)$,*

$$p_* \sim C_*\sqrt{\eta_m \eta_M}, \quad \sup_{M\in D} R(M, p_*) \sim 1 - 4\frac{\eta_m}{p_*},$$

$$C_* = 1 \text{ if } \beta = 1, \quad C_* = \left(\frac{2(C^2+1)}{1+\sqrt{(C^2+1)}}\right)^{1/4} \text{ if } \beta = 2.$$

2. *If $\beta = 2$ and $C > C_0$, $\sup_{M\in D} R(M, p_{**}) = R(A_1, p_{**}) = R(A_4, p_{**})$,*

$$p_{**} \sim (2C)^{1/4}\sqrt{\eta_m \eta_M}, \quad \sup_{M\in D} R(M, p_*) \sim 1 - 4\frac{\eta_m}{p_*}.$$

C_0 is the only positive root of the equation $\dfrac{C^2+1}{1+\sqrt{(C^2+1)}} = C$.

These results are obtained by comparing the asymptotic values of R at the edges.

Lemma 4. *The values p_* and p_{**} in Lemma 3 are in each case a strict local minimum for the function $p \mapsto \sup_{(\tau,\eta)\in D} R(\tau, \eta, p)$.*

Proof. For any positive p, we define $\bar{R}(p) = \sup_{(\tau,\eta)\in D} R(\tau, \eta, p)$, and $\mu(p) = \frac{1+\bar{R}(p)}{1-\bar{R}(p)}$. We write

$$R(\tau, \eta, p) - \bar{R}(p) = (1 - \bar{R}(p))\frac{\bar{Q}(\tau, \eta, p, \mu)}{(\xi + p)^2 + \xi^2 - \eta^2}$$

with $\bar{Q}(\tau, \eta, p, \mu) = 2\xi^2 - 2\mu p\xi - \eta^2 + p^2$. In the sequel, we will consider $\bar{Q}$ as a polynomial in the independent variables τ, η, p and μ. Defining $\mu_* = \mu(p_*)$ we have

$$\bar{Q}(A_1, p_*, \mu_*) = \bar{Q}(A_2, p_*, \mu_*) = 0,$$

and $\bar{Q}(M, p_*, \mu_*) \le 0$ for any M in D. Now for p_* to be a strict local minimum, it is sufficient that there exists no variation $(\delta p, \delta\mu)$ with $\delta\mu < 0$, such that $\bar{Q}(A_j, p_* + \delta p, \mu_* + \delta\mu) < 0$. By the Taylor formula, it is equivalent to proving that for $j = 1$ and $j = 3$, we can not have $\delta p(p_* - \mu_*\xi_j) - p_*\xi_j\delta\mu < 0$ for $\delta\mu < 0$. From the asymptotic behavior, we see that for $j = 1$, it gives $\delta p - \delta\mu < 0$, and for $j = 3$, it gives $-\delta p - \delta\mu < 0$, which together contradicts the fact that $\delta\mu < 0$. The arguments hold in all cases.

Another general result in [2] asserts that any strict local minimum is a global minimum. Therefore p_* is a global minimum, and equal to p^*, which concludes the proof of the theorem in the first case. The second proof is similar.

5.4 Numerical Results

To show that the optimization process is indeed important, we draw in Figure 4, for eight subdomains in one dimension, the convergence rates of the algorithm with Dirichlet transmission conditions compared to the same algorithm with the new optimized transmission conditions of zeroth or first order. The convergence curves are similar in the 2D case [14].

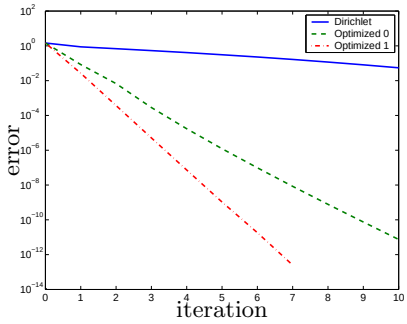

Fig. 4. Convergence rates

6 Conclusion

We have presented the main features of the Optimized Schwarz Waveform Relaxation algorithms, specifically for the advection-diffusion equation. Extension to nonlinear equations, and application to real problems in ocean modeling or combustion or waste disposal simulations will be the next step.

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