A Moving Mesh Method for Time–dependent Problems Based on Schwarz Waveform Relaxation

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1 Introduction

It is well accepted that the efficient solution of complex PDEs frequently requires methods which are adaptive in both space and time. Adaptive mesh methods for PDEs may be classified into one or more of the following broad categories:

- r–refinement: moving a fixed number of mesh points to difficult regions of the physical domain,
- p–refinement: varying the order of the numerical method to adapt to local solution smoothness,
- h–refinement: mesh refinement and derefinement, depending upon the local level of resolution.

These methods are applied in either a static fashion, where refining/coarsening or redistributing grids is done at fixed times during a simulation *or* in a dynamic fashion, where the solution and mesh are computed simultaneously.

In this paper we are interested in a class of spatially adaptive moving mesh PDE methods introduced in [17, 11] and [12]. Traditionally, moving mesh methods have been implemented in a (moving) method of lines framework — discretizing spatially and then integrating in time using a stiff initial value problem (IVP) solver. This approach propagates all unknowns (mesh and physical solution) forward in time using identical time steps. It is quite common, however, for problems with moving interfaces or singular behavior to have solution components which evolve on disparate scales in both space and time.

Our purpose is to introduce and explore a natural coupling of domain decomposition, in the Schwarz waveform context, and the spatially adaptive moving mesh PDE methods. This will allow the mesh and physical solution to be evolved according to local space and time scales.

2 Moving Mesh Methods

We consider the solution of a PDE of the form

$$u_t = \mathcal{L}(u) \qquad 0 < x < 1, \quad t > 0,$$

subject to appropriate initial and boundary conditions, where $\mathcal{L}$ denotes a spatial differential operator. The assumption is that the solution of this PDE has features which are difficult to resolve using a uniform mesh in the physical coordinate x. We seek, for fixed t, a one–to–one coordinate transformation

$$x = x(\xi, t) : [0, 1] \to [0, 1], \quad \text{with } x(0, t) = 0,\ x(1, t) = 1$$

such that $u(x(\xi, t), t)$ is sufficiently smooth that a simple (typically uniform) mesh $\xi_i, i = 0, 1, \ldots, N$ can be used to resolve solution features in the computational domain $\xi \in [0, 1]$. The mesh in the physical coordinate x is then specified from the coordinate transformation by $x_i(t) = x(\xi_i, t),\ i = 0, 1, \ldots, N$.

One standard way to perform adaptivity in space is to use the equidistribution principle (EP), introduced by [3]. We assume for the moment that a monitor function, $M = M(t, x)$, measuring the difficulty or error in the numerical solution, is given. Typically, its dependence on t and x is through the physical solution $u = u(t, x)$. Then, equidistribution requires that the mesh points satisfy

$$\int_{x_{i-1}}^{x_i} M(t, \tilde{x})\, d\tilde{x} = \frac{1}{N} \int_0^1 M(t, \tilde{x}) d\tilde{x} \qquad \text{for } i = 1, ..., N,$$

or equivalently,

$$\int_0^{x(\xi_i, t)} M(t, \tilde{x})\, d\tilde{x} = \xi_i \int_0^1 M(t, \tilde{x}) d\tilde{x} \quad \text{for } i = 1, \ldots, N.$$

The continuous generalization of this is that

$$\int_0^{x(\xi, t)} M(t, \tilde{x})\, d\tilde{x} = \xi \theta(t), \tag{1}$$

where $\theta(t) \equiv \int_0^1 M(t, \tilde{x})\, d\tilde{x}$ (e.g., see [11]). It follows immediately from (1) that

$$\frac{\partial}{\partial \xi} \left\{ M(t, x(\xi, t)) \frac{\partial}{\partial \xi} x(\xi, t) \right\} = 0. \tag{2}$$

Note that (2) does not explicitly involve the node speed $\dot{x}$. This is generally introduced by relaxing the equation to require equidistribution at time $t + \tau$. A number of parabolic moving mesh PDEs (MMPDEs) are developed using somewhat subtle simplifying assumptions and their correspondence to various heuristically derived moving mesh methods is shown in [17] and [11, 12]. A particularly useful one is MMPDE5,

$$\dot{x} = \frac{1}{\tau M(t, x(\xi, t))} \frac{\partial}{\partial \xi} \left(M(t, x(\xi, t)) \frac{\partial x}{\partial \xi} \right). \qquad \text{(MMPDE5)}$$

The relaxation parameter τ is chosen in practice (cf. [10]) so that the mesh evolves at a rate commensurate with that of the solution $u(t, x)$.

A simple popular choice for $M(t, x)$ is the arclength-like monitor function

$$M(t, x) = \sqrt{1 + \frac{1}{\alpha} |u_x|^2}, \qquad (3)$$

based on the premise that the error in the numerical solution is large in regions where the solution has large gradients. It is recommended in [10] (also see [1] and [18]) that the intensity parameter α be chosen as

$$\alpha = \left[\int_0^1 |u_x| dx \right]^2,$$

expecting that about one–half of mesh points are concentrated in regions of large gradients. We note that there are other choices for the monitor function for certain classes of problems, cf. [2] and [15].

Using the coordinate transformation $x = x(\xi, t)$ to rewrite the physical PDE in quasi-Lagrangian form, a moving mesh method is obtained by solving the coupled system

$$\begin{aligned} \dot{u} - u_x \dot{x} &= \mathcal{L}(u), \\ \dot{x} &= \frac{1}{\tau M} \left(M x_\xi \right)_\xi, \end{aligned} \qquad (4)$$

where $\dot{u}$ is the total time derivative of u.

Initial and boundary conditions for the physical PDE come from the problem description. On a fixed interval the boundary conditions for the mesh can be specified as $\dot{x}_0 = \dot{x}_N = 0$. If the initial solution $u(x, 0)$ is smooth then it suffices to use a uniform mesh as the initial mesh. Otherwise, an adaptive initial mesh can be obtained by solving MMPDE5 for a monitor function computed based on the initial solution $u(x, 0)$ (cf. [13]).

A typical implementation (cf. [13]) to solve (4) involves spatial discretization and solution of a nonlinear system of ODEs with a stiff ODE solver like DASSL, see [16]. This becomes quite expensive in higher dimensions. Instead we use an alternating solution procedure where the mesh PDE is integrated over a time step for the new mesh and then the physical PDE(s) is integrated with available old and new meshes. The reader is referred to [14] for a detailed description of the alternating solution procedure.

3 The Schwarz Waveform Implementation

Schwarz waveform relaxation methods have garnered tremendous attention as a means of applying domain decomposition strategies to problems in both space and time. Convergence results for linear problems may be found in [7] and [4] and for nonlinear problems in [6]. There are several ways to implement Schwarz waveform relaxation and moving meshes together to design an effective solver. In [8] and [9] the classical Schwarz waveform algorithm is applied to the coupled system of mesh and physical PDEs. Specifically, if x_j and ξ_j denote the physical and computational meshes on each overlapping subdomain Ω_j and the physical solution on each subdomain is denoted by u_j then

$$\begin{aligned} \dot{u}_j^k - \frac{\partial u_j^k}{\partial x}\dot{x}_j^k &= \mathcal{L}(u_j^k) \\ \dot{x}_j^k &= \frac{1}{\tau M(t, x_j^k)}\frac{\partial}{\partial \xi}\left(M(t, x_j^k)\frac{\partial x_j^k}{\partial \xi}\right) \end{aligned} \tag{5}$$

is solved for $j = 1, \ldots, D$ and $k = 1, 2, \ldots$. The boundary values for u_j^k and x_j^k are obtained from the values $u_{j-1}^{k-1}, x_{j-1}^{k-1}$ and $u_{j+1}^{k-1}, x_{j+1}^{k-1}$ from the previous iteration on the respective boundaries of Ω_{j-1} and Ω_{j+1}. If this Schwarz iteration converges it will converge to the mono–domain solution for both the mesh and physical solution.

In this paper we propose an alternate strategy. We apply a Schwarz iteration solver to the physical PDE and obtain the solution by using a moving mesh method on each subdomain, which allows one to use standard moving mesh software. Instead of solving the coupled mesh and physical PDEs on each subdomain, we use the approach mentioned in the previous section and alternately solve for the physical solution and the mesh.

As in the fixed mesh case, the rate of convergence of the classical Schwarz iteration is improved as the size of the overlap is increased, with the faster convergence being offset by the increased computational cost per iteration. Things are further complicated, however, by the desire to isolate difficult regions of the solution from regions where there is little activity. As the overlap is increased more subdomains become "active" requiring smaller time steps in a larger proportion of the physical domain.

4 Numerical Results

In this section we highlight some particular aspects of the moving Schwarz method described in the previous section with the viscous Burgers' equation, a standard test problem for moving mesh methods. Specifically we solve $u_t = \epsilon u_{xx} - \frac{1}{2}(u^2)_x, u(0,t) = 1, u(1,t) = 0$ and $u(x,0) = c - \frac{1}{2}\tanh((x - x_0)/4\epsilon)$. For our experiments we choose $c = 1/2$, $x_0 = 1/10$, and $\epsilon \ll 1$. The solution is a traveling front of thickness $O(\epsilon)$ which moves to the right from x_0 at speed c.

In Figure 1 we illustrate the mesh trajectories generated by a moving mesh method on one domain. The plot shows the time evolution of all mesh

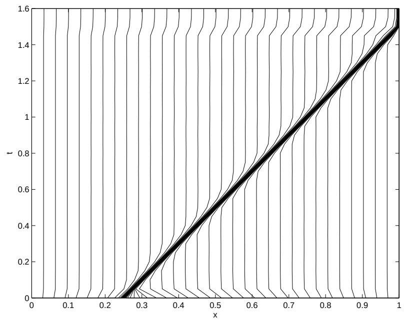

Fig. 1. Mesh trajectories generated on one domain.

points. Initially the mesh points are concentrated at the initial front location ($x_0 = 1/4$). As the solution evolves we see mesh points move in and out of the front location ensuring a sufficient resolution.

Figures 2 and 3 illustrate the meshes obtained using a two domain solution during the first two Schwarz iterations with 10% overlap. We see that the mesh points in subdomain one concentrate and follow the front until it passes into subdomain two. At that point the mesh in subdomain two, which was initially uniform, reacts and resolves the front until it reaches the right boundary. During the first Schwarz iteration the mesh points stay at the right boundary of subdomain one. The right boundary condition for subdomain one is incorrect during the first iteration and the solution presents itself as a layer. During the second iteration, however, the boundary condition issue has basically been resolved and the mesh in subdomain one returns to an essentially uniform state as the front moves into subdomain two.

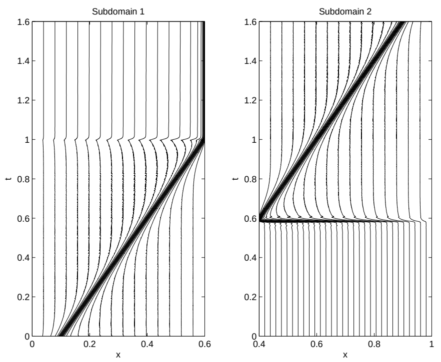

Fig. 2. Mesh trajectories generated on two domains during the first Schwarz iteration.

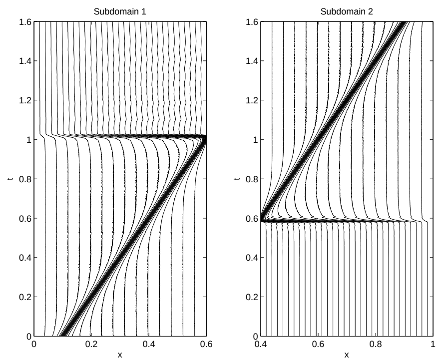

Fig. 3. Mesh trajectories generated on two domains during the second Schwarz iteration.

In Figures 4, 5, and 6 we show solutions of Burgers' equation using the moving Schwarz method on two subdomains with a 10% overlap. Each Figure shows the solution of the physical PDE on the left and the pointwise error on the right. The left plot of each figure shows the computed solution on subdomain one marked with circles and the solution on subdomain two marked with diamonds. The error plots are annotated in the same way. At $t = 0.8$ during the first Schwarz iteration (Figure 4) the solutions on each subdomain agree, at least qualitatively, with the one domain solution. By $t = 1.4$ (Figure 5), the front has moved across the subdomain boundary and the solution on subdomain one is not correct. This is to be expected since boundary data for subdomain one is incorrect during the first iteration. During the third Schwarz iteration (Figure 6), however, the solutions on both subdomains now agree with the one domain solution to within discretization error.

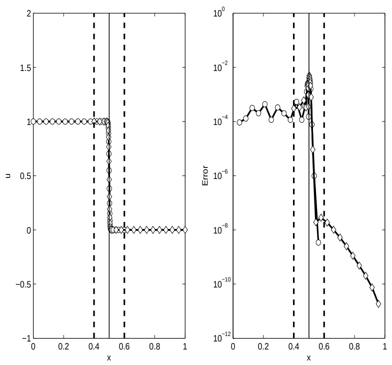

Fig. 4. Solution (left) and pointwise error (right) after Schwarz iteration 1 at $t = 0.8$.

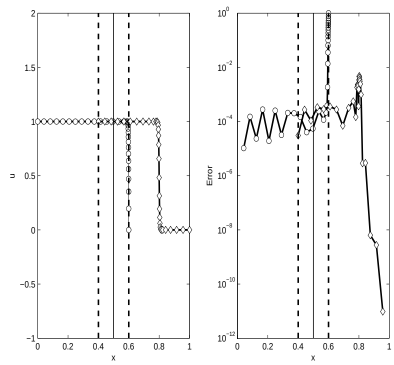

Fig. 5. Solution (left) and pointwise error (right) after Schwarz iteration 1 at $t = 1.4$.

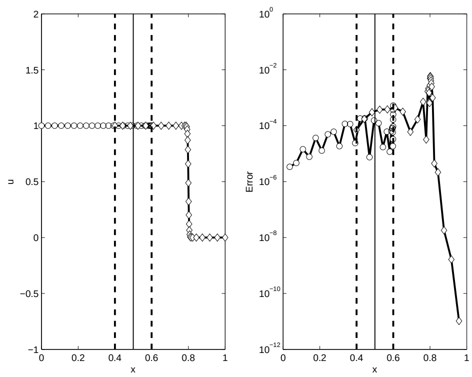

Fig. 6. Solution (left) and pointwise error (right) after Schwarz iteration 3 at $t = 1.4$.

In our experiments, total cpu time is increased as the overlap is increased. Convergence of the Schwarz iteration is rapid for small ϵ (the regime of interest for moving mesh methods), requiring only two or three iterations to reach discretization error. This is consistent with the theoretical results of [6]. Increasing the overlap for this model problem only serves to make each subdomain active for a larger portion of the time interval. Any improvement in the convergence rate is more than offset by the increased cpu time on each subdomain as the overlap is increased.

5 Conclusions

In this paper we propose a moving mesh Schwarz waveform relaxation method. In this approach, classical Schwarz waveform relaxation is applied to the physical PDE and a moving mesh method is used to facilitate the solution on each subdomain. In this way a solution is obtained which benefits both from the domain decomposition approach and the ability to dynamically refine meshes within each subdomain. A careful comparison with previous approaches [9] is ongoing. The benefits of such an approach are likely to be fully realized in two or more space dimensions. This is certainly the subject of current work and interest. The effects of higher order transmission conditions (cf. [5]) are also being studied in this context.

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References

[1] G. Beckett and J. A. Mackenzie. Uniformly convergent high order finite element solutions of a singularly perturbed reaction-diffusion equation using mesh equidistribution. *Appl. Numer. Math.*, 39(1):31–45, 2001.

[2] C.J. Budd, W. Huang, and R.D. Russell. Moving mesh methods for problems with blow-up. *SIAM J. Sci. Comput.*, 17(2):305–327, 1996.

[3] C. de Boor. Good approximation by splines with variable knots. II. In *Conference on the Numerical Solution of Differential Equations (Univ. Dundee, Dundee, 1973)*, pages 12–20. Lecture Notes in Math., Vol. 363. Springer, Berlin, 1974.

[4] M. J. Gander and A. M. Stuart. Space–time continuous analysis of waveform relaxation for the heat equation. *SIAM J. Sci. Comput.*, 19(6):2014–2031, 1998.

[5] M.J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731, 2006.

[6] M.J. Gander and C. Rohde. Overlapping Schwarz waveform relaxation for convection-dominated nonlinear conservation laws. *SIAM J. Sci. Comput.*, 27(2):415–439, 2005.
[7] E. Giladi and H.B. Keller. Space-time domain decomposition for parabolic problems. *Numer. Math.*, 93(2):279–313, 2002.
[8] R.D. Haynes. *The Numerical Solution of Differential Equations: Grid Selection for Boundary Value Problems and Adaptive Time Integration Strategies*. PhD thesis, Simon Fraser University, Burnaby, B.C. V5A 1S6, 2003.
[9] R.D. Haynes and R.D. Russell. A Schwarz waveform moving mesh method. *SIAM J. Sci. Comput.*, 2007. To Appear.
[10] W. Huang. Practical aspects of formulation and solution of moving mesh partial differential equations. *J. Comput. Phys.*, 171(2):753–775, 2001.
[11] W. Huang, Y. Ren, and R.D. Russell. Moving mesh methods based on moving mesh partial differential equations. *J. Comput. Phys.*, 113(2):279–290, 1994.
[12] W. Huang, Y. Ren, and R.D. Russell. Moving mesh partial differential equations (MMPDES) based on the equidistribution principle. *SIAM J. Numer. Anal.*, 31(3):709–730, 1994.
[13] W. Huang and R.D. Russell. A moving collocation method for solving time dependent partial differential equations. *Appl. Numer. Math.*, 20(1-2):101–116, 1996.
[14] W. Huang and R.D. Russell. Moving mesh strategy based on a gradient flow equation for two-dimensional problems. *SIAM J. Sci. Comput.*, 20(3):998–1015, 1999.
[15] W. Huang and W. Sun. Variational mesh adaptation. II. Error estimates and monitor functions. *J. Comput. Phys.*, 184(2):619–648, 2003.
[16] L.R. Petzold. A description of DASSL: a differential/algebraic system solver. In *Scientific computing (Montreal, Que., 1982)*, IMACS Trans. Sci. Comput., I, pages 65–68. IMACS, New Brunswick, NJ, 1983.
[17] Y. Ren and R.D. Russell. Moving mesh techniques based upon equidistribution, and their stability. *SIAM J. Sci. Statist. Comput.*, 13(6):1265–1286, 1992.
[18] J. M. Stockie, J. A. Mackenzie, and R. D. Russell. A moving mesh method for one-dimensional hyperbolic conservation laws. *SIAM J. Sci. Comput.*, 22(5):1791–1813, 2000.