Exact and Inexact FETI-DP Methods for Spectral Elements in Two Dimensions

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1 Introduction

High-order finite element methods based on spectral elements or *hp*-version finite elements improve the accuracy of the discrete solution by increasing the polynomial degree p of the basis functions as well as decreasing the element size h. The discrete systems generated by these high-order methods are much more ill-conditioned than the ones generated by standard low-order finite elements. In this paper, we will focus on spectral elements based on Gauss-Lobatto-Legendre (GLL) quadrature and construct nonoverlapping domain decomposition methods belonging to the family of Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) methods; see [4, 9, 7]. We will also consider inexact versions of the FETI-DP methods, i.e., irFETI-DP and iFETI-DP, see [8]. We will show that these methods are scalable and have a condition number depending only weakly on the polynomial degree.

2 Spectral Element Discretization of Second Order Elliptic Problems

Let $T_{\rm ref}$ be the reference square $(-1,1)^d$, $d = 2$, and let $Q_p(T_{\rm ref})$ be the set of polynomials on $T_{\rm ref}$ of degree $p \geq 1$ in each variable. We assume that the domain Ω can be decomposed into N_e nonoverlapping finite elements T_k of characteristic diameter h, $\overline{\Omega} = \bigcup_{k=1}^{N_e} \overline{T}_k$, each of which is an affine image of the reference square or cube, $T_k = \phi_k(T_{\rm ref})$, where ϕ_k is an affine mapping (more general maps could be considered as well). Later, we will group these elements into N nonoverlapping subdomains Ω_i of characteristic diameter H, forming themselves a coarse finite element partition of Ω, $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$, $\overline{\Omega}_i = \bigcup_{k=1}^{N_i} \overline{T}_k$. Hence, the fine element partition $\{T_k\}_{k=1}^{N_e}$ can be considered a refinement of the coarse subdomain partition $\{\Omega_i\}_{i=1}^{N}$, with matching finite element nodes on the boundaries of neighboring subdomains.

We consider linear, selfadjoint, elliptic problems on Ω, with zero Dirichlet boundary conditions on a part $\partial\Omega_D$ of the boundary $\partial\Omega$:

Find $u \in V = \{v \in H^1(\Omega) : \ v = 0 \ \text{ on } \ \partial\Omega_D\}$ such that

$$a(u,v) = \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, dx \quad = \quad \int_{\Omega} f v \, dx \qquad \forall \; v \in \; V. \tag{1}$$

Here, $\rho(x) > 0$ can be discontinuous, with very different values for different subdomains, but we assume this coefficient to vary only moderately within each subdomain Ω_i. In fact, without decreasing the generality of our results, we will only consider the piecewise constant case of $\rho(x) = \rho_i$, for $x \in \Omega_i$.

Conforming spectral element discretizations consist of continuous, piecewise polynomials of degree p in each element:

$$V^p = \{v \in V : v|_{T_i} \circ \phi_i \in Q_p(T_{\text{ref}}),\ i = 1, \ldots, N_e\}.$$

A convenient tensor product basis for V^p is constructed using Gauss-Lobatto-Legendre (GLL) quadrature points; see Figure 1. Let $\{\xi_i\}_{i=0}^{p}$ denote the set of GLL points

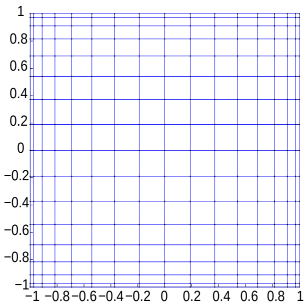

Fig. 1. Quadrilateral mesh defined by the Gauss-Lobatto-Legendre (GLL) quadrature points with $p = 16$ on one square element.

on $[-1, 1]$ and σ_i the associated quadrature weights. Let $l_i(\cdot)$ be the Lagrange interpolating polynomial which vanishes at all the GLL nodes except ξ_i, where it equals one. The basis functions on the reference square are defined by a tensor product as $l_i(x_1) l_j(x_2),\ 0 \le i, j \le p$. This basis is nodal, since every element of $Q_p(T_{\text{ref}})$ can be written as $u(x_1, x_2) = \sum_{i=0}^{p} \sum_{j=0}^{p} u(\xi_i, \xi_j) l_i(x_1) l_j(x_2)$. Each integral of the continuous model (1) is replaced by GLL quadrature over each element

$$(u, v)_{p,\Omega} = \sum_{k=1}^{N_e} \sum_{i,j=0}^{p} (u \circ \phi_k)(\xi_i, \xi_j)(v \circ \phi_k)(\xi_i, \xi_j) |J_k| \sigma_i \sigma_j, \tag{2}$$

where $|J_k|$ is the determinant of the Jacobian of ϕ_k. This inner product is uniformly equivalent to the standard L_2-inner product on $Q_p(T_{\text{ref}})$. Applying these quadrature rules, we obtain the discrete elliptic problem:

$$\text{Find } u \in V^p \qquad \text{such that} \qquad a_p(u, v) = (f, v)_{p,\Omega} \quad \forall v \in V^p, \tag{3}$$

with discrete bilinear form $a_p(u, v) = \sum_{k=1}^{N_e} (\rho_k \nabla u, \nabla u)_{p,T_k}$ and each quadrature rule $(\cdot, \cdot)_{p,T_k}$ defined as in (2). Having chosen a basis for V^p, the discrete problem (3) is then turned into a linear system of algebraic equations $K_g u_g = f_g$, with K_g the globally assembled, symmetric, positive definite stiffness matrix; see [2] for more details.

3 The FETI-DP Algorithms

Let a domain $\Omega \subset \mathbb{R}^2$ be decomposed into N nonoverlapping subdomains Ω_i of diameter H, each of which is the union of finite elements with matching finite element nodes on the boundaries of neighboring subdomains across the interface $\Gamma := \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$, where $\partial\Omega_i, \partial\Omega_j$ are the boundaries of Ω_i, Ω_j, respectively. The interface Γ is the union of edges and vertices. We regard edges in 2D as open sets shared by two subdomains, and vertices as endpoints of edges; see, e.g., [11, Chapter 4.2]. For a more detailed definition of faces, edges, and vertices in 2D and 3D; see [9, Section 3] and [7, Section 2].

For each subdomain $\Omega_i,\ i = 1, \dots, N$, we assemble the local stiffness matrices $K^{(i)}$ and load vectors $f^{(i)}$. We denote the unknowns on each subdomain by $u^{(i)}$. We then partition the unknowns $u^{(i)}$ into primal variables $u_\Pi^{(i)}$ and nonprimal variables $u_B^{(i)}$. As we only treat two dimensional problems here, the primal variables $u_\Pi^{(i)}$ will be associated with vertex unknowns whereas the nonprimal variables are interior $(u_I^{(i)})$ and dual $(u_\Delta^{(i)})$ unknowns. We will enforce the continuity of the solution in the primal unknowns $u_\Pi^{(i)}$ by global subassembly of the subdomain stiffness matrices $K^{(i)}$. For all other interface variables $u_\Delta^{(i)}$, we will introduce Lagrange multipliers to enforce continuity. We partition the stiffness matrices according to the different sets of unknowns,

$$K^{(i)} = \begin{bmatrix} K_{BB}^{(i)} & K_{\Pi B}^{(i)\,T} \\ K_{\Pi B}^{(i)} & K_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad K_{BB}^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)\,T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} \end{bmatrix},$$

$$\text{and} \qquad f^{(i)} = [f_B^{(i)}\ f_\Pi^{(i)}], \quad f_B^{(i)} = [f_I^{(i)}\ f_\Delta^{(i)}].$$

3.1 The Exact FETI-DP Algorithm

We define the block matrices

$$K_{BB} = \operatorname{diag}_{i=1}^N(K_{BB}^{(i)}), \quad K_{\Pi B} = \operatorname{diag}_{i=1}^N(K_{\Pi B}^{(i)}), \quad K_{\Pi\Pi} = \operatorname{diag}_{i=1}^N(K_{\Pi\Pi}^{(i)}),$$

and right hand sides $f_B^T = [f_B^{(1)\,T}, \dots, f_B^{(N)\,T}],\ f_\Pi^T = [f_\Pi^{(1)\,T}, \dots, f_\Pi^{(N)\,T}]$.

By assembly of the local subdomain matrices in the primal variables using the operator $R_\Pi^T = [R_\Pi^{(1)\,T}, \dots, R_\Pi^{(N)\,T}]$ with entries 0 or 1, we have the partially assembled global stiffness matrix $\widetilde{K}$ and right hand side $\tilde{f}$,

$$\widetilde{K} = \begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} \end{bmatrix} = \begin{bmatrix} I_B & 0 \\ 0 & R_\Pi^T \end{bmatrix} \begin{bmatrix} K_{BB} & K_{\Pi B}^T \\ K_{\Pi B} & K_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} I_B & 0 \\ 0 & R_\Pi \end{bmatrix},$$

$$\tilde{f} = \begin{bmatrix} f_B \\ \tilde{f}_\Pi \end{bmatrix} = \begin{bmatrix} I_B & 0 \\ 0 & R_\Pi^T \end{bmatrix} \begin{bmatrix} f_B \\ f_\Pi \end{bmatrix}.$$

Choosing a sufficient number of primal variables $u_\Pi^{(i)}$, i.e., all vertex unknowns, to constrain our solution, results in a symmetric, positive definite matrix $\widetilde{K}$.

To enforce continuity on the remaining interface variables $u_\Delta^{(i)}$ we introduce a jump operator B_B with entries $0, -1$ or 1 and Lagrange multipliers λ.

We can now formulate the FETI-DP saddle-point problem,

$$\begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T & B_B^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} & 0 \\ B_B & 0 & 0 \end{bmatrix} \begin{bmatrix} u_B \\ \tilde{u}_\Pi \\ \lambda \end{bmatrix} = \begin{bmatrix} f_B \\ \tilde{f}_\Pi \\ 0 \end{bmatrix}. \tag{4}$$

By eliminating u_B and u_Π from the system (4), we obtain an equation system

$$F\lambda = d, \quad \text{where} \tag{5}$$

$F = B_B K_{BB}^{-1} B_B^T + B_B K_{BB}^{-1} \widetilde{K}_{B\Pi} \widetilde{S}_{\Pi\Pi}^{-1} \widetilde{K}_{\Pi B} K_{BB}^{-1} B_B^T$ and $d = B_B K_{BB}^{-1} f_B - B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T \widetilde{S}_{\Pi\Pi}^{-1} (\tilde{f}_\Pi - \widetilde{K}_{\Pi B} K_{BB}^{-1} f_B)$. Let us define

$$K_{II} = \operatorname{diag}_{i=1}^N (K_{II}^{(i)}), \quad K_{\Delta I} = \operatorname{diag}_{i=1}^N (K_{\Delta I}^{(i)}), \quad K_{\Delta\Delta} = \operatorname{diag}_{i=1}^N (K_{\Delta\Delta}^{(i)}).$$

The theoretically almost optimal Dirichlet preconditioner M_{D} is then defined

$$\text{by} \quad M_{\mathrm{D}}^{-1} = B_{B,D} (R_\Delta^B)^T (K_{\Delta\Delta} - K_{\Delta I} K_{II}^{-1} K_{\Delta I}^T) R_\Delta^B B_{B,D}^T, \quad \text{where}$$

$R_\Delta^B = \operatorname{diag}_{i=1}^N (R_\Delta^{B\,(i)})$. The matrices $R_\Delta^{B\,(i)}$ are restriction operators with entries 0 or 1 which restrict the nonprimal degrees of freedom $u_B^{(i)}$ of a subdomain to the dual part $u_\Delta^{(i)}$. The matrices B_D are scaled variants of the jump operator B where the contribution from and to each interface node is scaled by the inverse of the multiplicity of the node. The multiplicity of a node is defined as the number of subdomains it belongs to. It is well known that for heterogeneous problems a more elaborate scaling is necessary, see, e.g., [9].

The original or standard, exact FETI-DP method is the method of conjugate gradients applied to the symmetric, positive definite system (5) using the preconditioner M_{D}^{-1}.

3.2 Inexact FETI-DP Algorithms

We will denote (4) as $\mathcal{A}x = \mathcal{F}$,

$$\text{where} \quad \mathcal{A} = \begin{bmatrix} K_{BB} & \widetilde{K}_{\Pi B}^T & B_B^T \\ \widetilde{K}_{\Pi B} & \widetilde{K}_{\Pi\Pi} & 0 \\ B_B & 0 & 0 \end{bmatrix}, \; x = \begin{bmatrix} u_B \\ \tilde{u}_\Pi \\ \lambda \end{bmatrix}, \; \mathcal{F} = \begin{bmatrix} f_B \\ \tilde{f}_\Pi \\ 0 \end{bmatrix}.$$

We also write this equation

$$\begin{bmatrix} \widetilde{K} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{f} \\ 0 \end{bmatrix}, \tag{6}$$

where $B = \begin{bmatrix} B_B & 0 \end{bmatrix}$, $u^T = [u_B^T \; \tilde{u}_\Pi^T]$, $\tilde{f}^T = [f_B^T \; \tilde{f}_\Pi^T]$. Eliminating u_B by one step of block elimination, we obtain the reduced system

$$\begin{bmatrix} \widetilde{S}_{\Pi\Pi} & -\widetilde{K}_{\Pi B} K_{BB}^{-1} B_B^T \\ -B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T & -B_B K_{BB}^{-1} B_B^T \end{bmatrix} \begin{bmatrix} \tilde{u}_\Pi \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{f}_\Pi - \widetilde{K}_{\Pi B} K_{BB}^{-1} f_B \\ -B_B K_{BB}^{-1} f_B \end{bmatrix}, \tag{7}$$

where $\widetilde{S}_{\Pi\Pi} = \widetilde{K}_{\Pi\Pi} - \widetilde{K}_{\Pi B} K_{BB}^{-1} \widetilde{K}_{\Pi B}^T$. For (7), we will also use the notation

$$\mathcal{A}_r x_r = \mathcal{F}_r, \quad \text{where} \quad x_r^T := [\tilde{u}_\Pi^T \;\; \lambda^T], \quad \text{and}$$

$$\mathcal{A}_r = \begin{bmatrix} \widetilde{S}_{\Pi\Pi} & -\widetilde{K}_{\Pi B} K_{BB}^{-1} B_B^T \\ -B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T & -B_B K_{BB}^{-1} B_B^T \end{bmatrix}, \quad \mathcal{F}_r := \begin{bmatrix} \tilde{f}_\Pi - \widetilde{K}_{\Pi B} K_{BB}^{-1} f_B \\ -B_B K_{BB}^{-1} f_B \end{bmatrix}.$$

The inexact FETI-DP methods are given by solving the saddle point problems (4) and (6) iteratively, using block triangular preconditioners and a suitable Krylov subspace method. For the saddle point problems (6) and (7), we introduce the block triangular preconditioners $\widehat{\mathcal{B}}_L$ and $\widehat{\mathcal{B}}_{r,L}$, respectively, as

$$\widehat{\mathcal{B}}_L^{-1} = \begin{bmatrix} \widehat{K}^{-1} & 0 \\ M^{-1} B \widehat{K}^{-1} & -M^{-1} \end{bmatrix}, \; \widehat{\mathcal{B}}_{r,L}^{-1} = \begin{bmatrix} \widehat{S}_{\Pi\Pi}^{-1} & 0 \\ -M^{-1} B_B K_{BB}^{-1} \widetilde{K}_{\Pi B}^T \widehat{S}_{\Pi\Pi}^{-1} & -M^{-1} \end{bmatrix},$$

where $\widehat{K}^{-1}$ and $\widehat{S}_{\Pi\Pi}^{-1}$ are assumed to be spectrally equivalent preconditioners for $\widetilde{K}$ and $\widetilde{S}_{\Pi\Pi}$, respectively, with bounds independent of the discretization parameters h, H. The matrix block M^{-1} is assumed to be a good preconditioner for the FETI-DP system matrix F and can be chosen as the Dirichlet preconditioner M_{D}^{-1} or any spectrally equivalent preconditioner. Our inexact FETI-DP methods are now given by using a Krylov space method for nonsymmetric systems, e.g., GMRES, to solve the preconditioned systems

$$\widehat{\mathcal{B}}_L^{-1} \mathcal{A} x = \widehat{\mathcal{B}}_L^{-1} \mathcal{F}, \quad \text{and} \quad \widehat{\mathcal{B}}_{r,L}^{-1} \mathcal{A}_r x_r = \widehat{\mathcal{B}}_{r,L}^{-1} \mathcal{F}_r,$$

respectively. The first will be denoted iFETI-DP and the latter irFETI-DP. Let us note that we can also use a positive definite reformulation of the two preconditioned systems, which allows the use of conjugate gradients, see [8] for further details.

4 Convergence Estimates

As shown in [11] for the two main families of overlapping Schwarz methods (Ch. 7.3) and iterative substructuring methods of wirebasket and Neumann-Neumann type (Ch. 7.4), the main domain decomposition results obtained for finite element discretizations of scalar elliptic problems can be transferred to the spectral element case; see [11, Ch. 7] for further details. The same tools can be used here to obtain the following estimate, see [10, 6] for further details.

Theorem 1. *The minimum eigenvalue of the FETI-DP operator is bounded from below by 1 and the maximum eigenvalue is bounded from above by* $C\left(1+\log\left(p\,\frac{H}{h}\right)\right)^2$, *with* $C > 0$ *independent of* p, h, H *and the values of the coefficients* ρ_i *of the elliptic operator.*

Similar convergence estimates hold for the inexact versions of FETI-DP, i.e., i(r)FETI-DP, if spectrally equivalent preconditioners are used instead of the direct solvers and GMRES instead of cg; see [8].

5 Numerical Results

We first investigate the growth of the condition number for an increasing number of subdomains. We expect to see the largest eigenvalue, and thus also the condition number, approaching a constant value, independent of coefficient jumps but dependent on the polynomial degree. We have used PETSc, the Portable Extensible Toolkit for Scientific Computing, see [1], for the parallel results in this section. In Table 1 we see the expected behavior for different polynomial degrees and fixed $H/h = 1$. From these results we choose to use a number of $N \geq 256$ subdomains in our experiments to study the asymptotic behavior of the condition number. In Table 2 we choose a sufficient number of subdomains and increase the polynomial degree from 2 to 32. We see that the condition number grows only slowly. In Table 2, we have also shown the CPU timings and iteration counts of irFETI-DP, additionally to the ones of FETI-DP. For irFETI-DP, we have used GMRES as Krylov subspace method and BoomerAMG [5] to precondition the FETI-DP coarse problem. BoomerAMG is a highly scalable distributed memory parallel algebraic multigrid solver and preconditioner; it is part of the high performance preconditioner library hypre [3]. From the table we see that also for spectral elements irFETI-DP compares very well with standard FETI-DP.

We report on the parallel scalability for 2 to 16 processors in Table 4 for FETI-DP and irFETI-DP. Both methods show basically the same performance and same scalability. Nevertheless, we expect irFETI-DP to be superior if coarse problems much larger than the ones here need to be solved. This will be the case for large numbers of subdomains, especially in 3D.

Table 1. One spectral element (p=2–32) per subdomain, N=4–576 subdomains, homogeneous problem and a problem with jumps, random right hand side, rtol=10^{-10}.

	FETI-DP											
	$\rho_{ij}=1$			$\rho_{ij}=10^{(i-j)/4}$			$\rho_{ij}=1$			$\rho_{ij}=10^{(i-j)/4}$		
N	It	λmax	λmin	It	λmax	λmin	It	λmax	λmin	It	λmax	λmin
	p=2			p=2			p=8			p=8		
4	**2**	1.05	1	**2**	1.05	1	**4**	1.89	1	**4**	1.89	1
16	**6**	1.45	1.0026	**6**	1.46	1.0018	**12**	4.38	1.0007	**12**	4.37	1.0004
64	**8**	1.61	1.0014	**8**	1.61	1.0013	**16**	4.86	1.0013	**18**	4.86	1.0009
256	**8**	1.64	1.0028	**8**	1.62	1.0013	**17**	5.00	1.0014	**19**	4.97	1.0008
576	**8**	1.66	1.0032	**8**	1.63	1.0016	**17**	5.01	1.0015	**20**	4.98	1.0009
	p=3			p=3			p=16			p=16		
4	**3**	1.21	1	**3**	1.21	1	**5**	2.57	1	**5**	2.57	1
16	**8**	2.10	1.0007	**8**	2.10	1.0004	**14**	6.65	1.0009	**15**	6.63	1.0008
64	**11**	2.32	1.0006	**11**	2.31	1.0006	**21**	7.42	1.0013	**21**	7.38	1.0008
256	**11**	2.37	1.0006	**12**	2.36	1.0004	**21**	7.58	1.0017	**25**	7.53	1.0009
576	**11**	2.38	1.0006	**13**	2.35	1.0006	**21**	7.62	1.0016	**26**	7.55	1.0006
	p=4			p=4			p=32			p=32		
4	**3**	1.37	1	**3**	1.37	1	**6**	3.42	1	**6**	3.42	1
16	**9**	2.65	1.0018	**10**	2.65	1.0008	**16**	9.48	1.0012	**17**	9.44	1.0009
64	**12**	2.95	1.0022	**13**	2.94	1.0011	**25**	10.58	1.0012	**25**	10.52	1.0009
256	**13**	3.01	1.0020	**14**	3.00	1.0013	**25**	10.81	1.0017	**31**	10.74	1.0008
576	**13**	3.03	1.0020	**15**	3.00	1.0005	**25**	10.86	1.0018	**33**	10.77	1.0005

Table 2. Homogeneous problem ($\rho = 1$). Increasing polynomial degree (p=2–32). Fixed subdomain sizes (H/h=1,2,4). FETI-DP and inexact reduced FETI-DP (irFETI-DP, GMRES). irFETI-DP uses one iteration of BoomerAMG with parallel Gauss-Seidel smoothing to precondition the coarse problem, rtol=10^{-7}.

			FETI-DP				irFETI-DP		
H/h	N	p	It	λmax	λmin	Time (16 Proc)	It	Time (16 Proc)	dof
1	4096	2	**7**	1.66	1.0074	2s	**7**	2s	16 129
		4	**10**	3.05	1.0217	4s	**9**	3s	65 025
		8	**13**	5.03	1.0067	6s	**11**	4s	261 121
		12	**15**	6.48	1.0260	11s	**13**	8s	588 289
		16	**16**	7.64	1.0121	23s	**14**	16s	1 046 529
		20	**17**	8.62	1.0114	53s	**14**	37s	1 635 841
		24	**18**	9.46	1.0138	94s	**16**	81s	2 356 225
		28	**18**	10.21	1.0183	155s	**16**	130s	3 207 681
		32	**19**	10.89	1.0227	256s	**17**	228s	4 190 209
2	1024	2	**9**	2.35	1.0020	1s	**8**	1s	16 129
		4	**12**	4.03	1.0146	2s	**11**	2s	65 025
		8	**15**	6.31	1.0232	4s	**12**	3s	261 121
		12	**17**	7.93	1.0177	10s	**15**	7s	588 289
		16	**18**	9.21	1.0133	23s	**17**	20s	1 046 529
		20	**19**	10.28	1.0186	43s	**17**	38s	1 635 841
		24	**20**	11.21	1.0247	83s	**18**	76s	2 356 225
		28	**21**	12.03	1.0294	164s	**18**	146s	3 207 681
		32	**22**	12.76	1.0230	276s	**18**	244s	4 190 209
4	256	2	**11**	3.18	1.0150	1s	**11**	1s	16 129
		4	**14**	5.14	1.0146	1s	**14**	1s	65 025
		8	**18**	7.70	1.0230	4s	**17**	4s	261 121
		12	**19**	9.49	1.0143	9s	**18**	9s	588 289
		16	**20**	10.89	1.0223	21s	**20**	20s	1 046 529
		20	**21**	12.05	1.0267	45s	**20**	42s	1 635 841
		24	**22**	13.05	1.0253	86s	**21**	84s	2 356 225
		28	**23**	13.94	1.0188	170s	**22**	164s	3 207 681
		32	**23**	14.73	1.0191	328s	**21**	280s	4 190 209

Table 3. Fixed polynomial degree (p=32), fixed subdomain sizes (H/h=1), increasing number of subdomains, $\rho = 1$, random right hand side, rtol=10^{-7}. Inexact FETI-DP for the block matrices using BoomerAMG and GMRES, local problem/coarse problem/Dirichlet preconditioner : (in)exact/(in)exact/(in)exact.

		iFETI-DP				FETI-DP		
p	N	It (i/i/i)	It (i/i/e)	It (i/e/e)	It (e/e/e)	It	$\lambda_{\min}$	$\lambda_{\max}$
32	4	13	13	13	6	6	3.42	1.0000
	16	22	21	20	16	17	9.48	1.0012
	64	30	30	29	24	25	10.57	1.0012
	100	30	30	30	24	24	10.69	1.0018
	144	30	29	30	24	25	10.75	1.0016

Table 4. Parallel scalability for p=20, N=256, H/h=4, rtol=10^{-7}.

	FETI-DP		irFETI-DP	
Proc	It	Time	It	Time
2	22	337s	20	309s
4	22	172s	20	156s
8	22	89s	20	82s
16	22	45s	20	42s

References

[1] S. Balay, K. Buschelman, V. Eijkhout, W.D. Gropp, D. Kaushik, M.G. Knepley, L.C. McInnes, B.F. Smith, and H. Zhang. PETSc users manual. Technical Report ANL-95/11 - Revision 2.1.5, Argonne National Laboratory, 2004.

[2] C. Bernardi and Y. Maday. *Spectral Methods.* In Handbook of Numerical Analysis, Volume V: Techniques of Scientific Computing (Part 2), P. G. Ciarlet and J.-L. Lions, editors. North-Holland, 1997.

[3] R.D. Falgout, J.E. Jones, and U.M. Yang. The design and implementation of hypre, a library of parallel high performance preconditioners. In A.M. Bruaset, P. Bjorstad, and A. Tveito, editors, *Numerical solution of Partial Differential Equations on Parallel Computers, Lect. Notes Comput. Sci. Eng.*, volume 51, pages 267–294. Springer-Verlag, 2006.

[4] C. Farhat, M. Lesoinne, P. Le Tallec, K. Pierson, and D. Rixen. FETI-DP: a dual-primal unified FETI method – part I. A faster alternative to the two-level FETI method. *Internat. J. Numer. Methods Engrg.*, 50(7):1523–1544, 2001.

[5] V.E. Henson and U.M. Yang. Boomeramg: A parallel algebraic multigrid solver and preconditioner. *Appl. Numer. Math.*, 41:155–177, 2002.

[6] A. Klawonn, L.F. Pavarino, and O. Rheinbach. Spectral element FETI-DP and BDDC preconditioners with multi-element subdomains and inexact solvers in the plane. Technical report, February 2007.

[7] A. Klawonn and O. Rheinbach. A parallel implementation of Dual-Primal FETI methods for three dimensional linear elasticity using a transformation of basis. *SIAM J. Sci. Comput.*, 28:1886–1906, 2006.

[8] A. Klawonn and O. Rheinbach. Inexact FETI-DP methods. *Inter. J. Numer. Methods Engrg.*, 69:284–307, 2007.

[9] A. Klawonn and O.B. Widlund. Dual-Primal FETI Methods for Linear Elasticity. *Comm. Pure Appl. Math.*, 59:1523–1572, 2006.

[10] L.F. Pavarino. BDDC and FETI-DP preconditioners for spectral element discretizations. *Comput. Meth. Appl. Mech. Engrg.*, 196 (8):1380 – 1388, 2007.

[11] A. Toselli and O.B. Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin Heidelberg New York, 2005.