Fast Domain Decomposition Algorithms for Discretizations of 3-*d* Elliptic Equations by Spectral Elements

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1 Introduction

In DD (domain decomposition) methods, the main contribution to the computational work is due to the two major components – solvers for local Dirichlet problems on subdomains of decomposition and local problems on their faces. Without loss of generality, we assume that the domains of FE's (finite elements) serve as subdomains of decomposition. At that, under the conditions of shape regularity, optimization of these components is reduced to obtaining fast preconditioners-solvers for the stiffness matrix of the p reference element and the Schur complement, related to its boundary.

Competitors for spectral FE's are *hierarchical* FE's, which have the tensor products of the integrated Legendre's polynomials for the form functions. As a starting point for optimization of major solvers for these two types of FE discretizations, primarily served the finite-difference preconditioners, suggested by [5], see also [8] for hierarchical and by [14] for spectral reference elements stiffness matrices. For internal stiffness matrices of hierarchical elements, a number of fast preconditioners-solvers have been justified theoretically by [6, 7, 2, 3] and thoroughly tested numerically. For spectral elements, to the best of the authors knowledge, there is known, the multilevel solver of [16], which efficiency was well approved numerically.

Hierarchical and spectral elements look differently. However, [11, 12] established an interrelation between them, showing that in computations they can be treated with a great measure of similarity. In particular, they considered optimal multilevel and DD types preconditioners-solvers for 2-d spectral elements, similar to those designed earlier for hierarchical elements. In this paper, first of all, we obtain fast multiresolution wavelet preconditioners-solvers for the internal FE and face subproblems, arising in DD algorithms for 3-d discretizations by spectral elements. The former realizes a technique alike the one implemented by [3] for hierarchical elements. The preconditioner of the same kind can be derived for the mass matrix, allowing in turn to obtain the face solver by K-interpolation. Inefficient prolongations from the interface boundary can also compromise optimality of DD algorithm. We approve the computationally fast prolongations by means of the inexact iterative solver for inner problems on FE's. With the mentioned three main fast DD components in hands, it is left to find a good preconditioner for the wire basket subproblem, having relatively small dimension $\mathcal{O}(\mathcal{R}p)$, where $\mathcal{R}$ is the number of finite elements. We use the one considered by [4] and other authors (see this paper for references), assuming that in a whole it is sufficiently fast. Our main conclusion is that the DD preconditioner-solver, with the pointed out components, has the relative condition number $\mathcal{O}((1+\log p)^2)$, while solving the system of algebraic equations with the DD preconditioner for the matrix requires $\mathcal{O}(N(1+\log p))$ arithmetic operations, where $N \simeq \mathcal{R}p^3$ is the order of the FE system.

We use notations: $\mathcal{Q}_{p,\mathbf{x}}$ – the space of polynomials of the order $p \geq 1$ in each variable of $\mathbf{x} = (x_1, x_2, .., x_d)$, d is the dimension; GLL and GLC nodes are the nodes of the Gauss-Lobatto-Legendre and Gauss-Lobatto-Chebyshev quadratures, respectively; signs $\prec$, $\succ$, $\asymp$ are used for the inequalities and equalities hold up to positive absolute constants; $\mathbf{A}^+$ – pseudo-inverse to a matrix $\mathbf{A}$; $\mathbf{A} \prec \mathbf{B}$ with nonnegative matrices $\mathbf{A}, \mathbf{B}$ implies $\mathbf{v}^\top \mathbf{A}\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}\mathbf{v}$ for any vector $\mathbf{v}$ and similarly for signs $\succ$, $\asymp$; $\tau_0 = (-1,1)^d$ is the reference cube. Notations $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ stand for the semi-norm and the norm in Sobolev's space $H^k(\Omega)$, $\overset{\circ}{H}{}^1(\Omega) = (v \in H^1(\Omega) : v|_{\partial\Omega} = 0)$. Since their similarity in our context, the both Lagrange elements with the GLL and GLC nodes are called *spectral*.

2 Finite-Difference and Factorized Preconditioners for Stiffness Matrices of Spectral *p* Elements

The GLL nodes $\eta_i \in [-1,1]$ satisfy $(1-\eta_i^2)P_p'(\eta_i) = 0$, whereas the GLC nodes are extremal points of the Chebyshev polynomials: $\eta_i = \cos\left(\frac{\pi(p-i)}{p}\right)$, $i = 0, 1, .., p$. For $i \leq N$, the steps $\hbar_i := \eta_i - \eta_{i-1}$ of the both meshes have the asymptotic behavior $\hbar_i \asymp i/p^2$. The both orthogonal tensor product meshes with the nodes $\mathbf{x} = \boldsymbol{\eta_\alpha} = (\eta_{\alpha_1}, \eta_{\alpha_2}, .., \eta_{\alpha_d})$, $\boldsymbol{\alpha} \in \omega := \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, .., \alpha_d) : 0 \leq \alpha_1, \alpha_2, .., \alpha_d \leq p\}$, are termed in the paper *Gaussian*. We consider the stiffness matrices $\mathbf{A}_{\mathrm{sp}}$ of the respective Lagrange reference elements, induced by the Dirichlet integral

$$a_{\tau_0}(u,v) = \int_{\tau_0} \nabla u \cdot \nabla v \, d\mathbf{x} \, .$$

Let $\mathcal{H}(\tau_0)$ be the space of functions, continuous on $\overline{\tau}_0$ and belonging to $\mathcal{Q}_{1,\mathbf{x}}$ on each nest of the Gaussian mesh $x_k = \eta_i$, then $\boldsymbol{\mathcal{A}}_{\mathrm{sp}}$ denotes the preconditioner, which is the FE matrix, corresponding to this space and Dirichlet integral a_{τ_0}. As a preconditioner for $\mathbf{A}_{\mathrm{sp}}$ in 3-d, it can be used the simpler matrix

$$\mathbb{A}_\hbar = \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \boldsymbol{\Delta}_\hbar \, ,$$

where $\mathbb{D}_\hbar$ is the diagonal matrix $\mathbb{D}_\hbar = \mathrm{diag}\,[\widetilde{h}_i = \frac{1}{2}(\hbar_i + \hbar_{i+1})]_{i=0}^p$, with $\widetilde{h}_i = 0$ for $i = 0, p+1$, and $\boldsymbol{\Delta}_\hbar$ is the FE matrix of the bilinear form $(v', w')_{(-1,1)}$ on the space $\mathcal{H}(-1,1)$ of continuous and piece wise linear on the 1-d Gaussian mesh $x = \eta_i$.

We also introduce the mass matrix $\mathbf{M}_{\mathrm{sp}}$ of the spectral element, its FE preconditioner $\boldsymbol{\mathcal{M}}_{\mathrm{sp}}$, generated by the space $\mathcal{H}(\tau_0)$, and $\mathbb{M}_\hbar := \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar$.

Lemma 1. *Uniformly in p*

$$\mathbb{A}_\hbar, \boldsymbol{\mathcal{A}}_{\mathrm{sp}} \prec \mathbf{A}_{\mathrm{sp}} \prec \boldsymbol{\mathcal{A}}_{\mathrm{sp}}, \mathbb{A}_\hbar \, , \qquad \mathbb{M}_\hbar, \boldsymbol{\mathcal{M}}_{\mathrm{sp}} \prec \mathbf{M}_{\mathrm{sp}} \prec \boldsymbol{\mathcal{M}}_{\mathrm{sp}}, \mathbb{M}_\hbar \, .$$

Proof. The inequalities for $\boldsymbol{\mathcal{A}}_{\rm sp}$ in 1-d are due to [1], for the step to a greater dimension see, *e.g.,* [4]. With the inequalities for $\boldsymbol{\mathcal{A}}_{\rm sp}$ hold, the inequalities for $\mathbb{A}_\hbar$ are easy to obtain.

Now we will introduce factored preconditioners. The rest of this section and Section 3 deal with matrices related to the internal unknowns. Usually they are supplied with the lower index I, but in many instances we omit this index. Without loss of generality it is assumed $p = 2N$.

The change of variables $\widetilde{\mathbf{v}} = \mathbf{C}\mathbf{v}$ by the diagonal matrix $\mathbf{C} = p^{-4}\,\mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2}$ (for 2-d $\mathbf{C} = p^{-2}\,\mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2}$) transforms $\mathbb{A}_{I,\hbar}$ as the matrix of a quadratic form into the matrix $\widetilde{\mathbb{A}}_{I,\hbar} := \mathbf{C}^{-1}\mathbb{A}_\hbar\mathbf{C}^{-1}$. Let us introduce also $(p-1)\times(p-1)$ matrices $\boldsymbol{\Delta}_{\rm sp} = \text{tridiag}\,[-1, 2, -1]$ and $\boldsymbol{\mathcal{D}}_{\rm sp} = \text{diag}\,[1, 4, .., N^2, (N-1)^2, (N-2)^2, .., 4, 1]$, and the $(p-1)^3 \times (p-1)^3$ matrix

$$\boldsymbol{\Lambda}_{I,\rm sp} = \boldsymbol{\Delta}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\Delta}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} + \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\mathcal{D}}_{\rm sp} \otimes \boldsymbol{\Delta}_{\rm sp}\,. \tag{1}$$

Lemma 2. *Matrices* $\widetilde{\mathbb{A}}_{I,\hbar}$, $\boldsymbol{\Lambda}_{I,\rm sp}$ *and simultaneously the matrix* $\boldsymbol{\Lambda}_{I,C} := \mathbf{C}\boldsymbol{\Lambda}_{I,\rm sp}\mathbf{C}$ *and the stiffness matrix* $\mathbf{A}_{I,\rm sp}$ *are spectrally equivalent uniformly in* p.

See [11, 12] for the proof.

Since matrix $\mathbf{C}$ is diagonal, the arithmetical costs of solving systems with matrices $\boldsymbol{\Lambda}_{I,C}$ and $\boldsymbol{\Lambda}_{I,\rm sp}$ are the same in the order. Matrix $\boldsymbol{\Lambda}_{I,\rm sp}$ looks exactly as the 7-point finite-difference approximation on the *uniform square mesh* of size $\hbar = 2/p = 1/N$ of the differential operator

$$L_{\rm sp}u = -\left[\phi^2(x_2)\phi^2(x_3)u_{,1,1} + \phi^2(x_1)\phi^2(x_3)u_{,2,2} + \phi^2(x_1)\phi^2(x_2)u_{,3,3}\right]\,,$$

$u|_{\partial\tau_0} = 0$, where $\phi(x) = \min(x+1, x-1)$, $x \in [-1, 1]$. Indeed, for $\phi_i := \phi(-1 + i\hbar)$ and $\mathbf{u} = (u_{\mathbf{i}})_{i_1,i_2,i_3=1}^{p-1}$ expanded by zero to the boundary nodes

$$\boldsymbol{\Lambda}_{I,\rm sp}\mathbf{u}|_{\mathbf{i}} = -\frac{1}{\hbar^2}\sum_{k=1,2,3}\phi_{i_l}^2\phi_{i_j}^2\left[u_{\mathbf{i}-\mathbf{e}_k} - 2u_{\mathbf{i}} + u_{\mathbf{i}+\mathbf{e}_k}\right], \quad 1 \le i_m \le (p-1)\,,$$

where $\mathbf{i} = (i_1, i_2, i_3)$, all numbers $k, l, j \in (1, 2, 3)$ are different, $\mathbf{e}_k = (\delta_{k,l})_{l=1}^3$, and $\delta_{k,l}$ are Kronecker's symbols.

We compare $\boldsymbol{\Lambda}_{I,\rm sp}$ with the finite-difference preconditioner for the hierarchical reference element, see, *e.g.,* $\boldsymbol{\Lambda}_{\rm e}$ in (2.5) of [11]. At $d = 2$, the related differential operators $L_{\rm sp}$ L, respectively, are similar, see for L (2.7), (2.8) in the same paper. In each quarter of τ_0, the differential expression for $L_{\rm sp}$ is the same as for L, defined on the square $(0, 1)^2$, up to the constant multiplier and rotation and translation of the axes. The same is true for finite-difference operators $\boldsymbol{\Lambda}_{I,\rm sp}$, $\boldsymbol{\Lambda}_{\rm e}$. At $d = 3$, the differential and finite-difference operators, related to the preconditioners for spectral and hierarchical elements, are different even in the order: $L_{\rm sp}$ is the differential operator of the 2-nd order, whereas L of the 4-th. However, multipliers $\boldsymbol{\mathcal{D}}_{\rm sp}$, $\boldsymbol{\Delta}_{\rm sp}$ and respectively $\boldsymbol{\Delta}$, $\boldsymbol{\mathcal{D}}$ in the representations of $\boldsymbol{\Lambda}_{I,\rm sp}$, $\boldsymbol{\Lambda}_{\rm e}$ by the sums of Kronecker's products are similar, see (1) above and (2.5) of [11]. Due to this, all known fast solvers for systems with the stiffness matrices of hierarchical reference elements can be adapted to systems with the stiffness matrices of spectral reference elements of any of the two types. We present two examples in the next section.

Instead of $\boldsymbol{\Lambda}_{I,\rm sp}$, one can as well use spectrally equivalent FE matrices, generated with the use of the 1-st order elements.

3 Fast Multilevel Wavelet Preconditioners-solvers for Interior of Reference Element and Face Problems

In order to obtain a fast preconditioner-solver for the internal stiffness matrix $\mathbf{A}_{I,\mathrm{sp}}$ of a spectral element, it is sufficient to design a fast solver for the preconditioner $\boldsymbol{\Lambda}_{I,\mathrm{sp}}$. For convenience, it is assumed $p = 2N,\ N = 2^{\ell_0 - 1}$.

For each $l = 1, 2, ..., \ell_0$, we introduce the uniform mesh $x_i^l,\ i = 0, 1, .., 2N_l$, $N_l = 2^{l-1}$, $x_0 = -1$, $x_{2N_l} = 1$ of the size $\hbar_l = 2^{1-l}$ and the space $\mathcal{V}_l(-1,1)$ of the continuous on $(-1,1)$ piece wise linear functions, vanishing at the ends of this interval. The dimension of $\mathcal{V}_l(-1,1)$ is $\mathcal{N}_l := p_l - 1 = 2^l - 1$ with $p_{\ell_0} = p$. Let $\phi_i^l \in \mathcal{V}_l(-1,1)$ be the the nodal basis function for the node x_i^l, so that $\phi_i^l(x_j^l) = \delta_{i,j}$ and $\mathcal{V}_l(-1,1) = \mathrm{span}\,\{\phi_i^l\}_{i=1}^{p_l - 1}$. For the Gram matrices

$$\boldsymbol{\Delta}_l = \hbar_l \left(\langle(\phi_i^l)', (\phi_j^l)'\rangle_{\omega=1}\right)_{i,j=1}^{p_l-1}, \quad \boldsymbol{\mathcal{M}}_l = \hbar_l^{-1} \left(\langle\phi_i^l, \phi_j^l\rangle_{\omega=\phi}\right)_{i,j=1}^{p_l-1}$$

with ϕ introduced in Section 2 and $\langle v, u\rangle_\omega := \int_{-1}^{1} \omega^2 v\, u\, dx$, we establish

$$\boldsymbol{\Delta}_{\ell_0} = \boldsymbol{\Delta}_{\mathrm{sp}}, \qquad \boldsymbol{\mathcal{M}} := \boldsymbol{\mathcal{M}}_{\ell_0} \asymp \boldsymbol{\mathcal{D}}_{\mathrm{sp}}.$$

The representation $\mathcal{V}_l = \mathcal{V}_{l-1} \oplus \mathcal{W}_l$ results in the decomposition $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus ... \oplus \mathcal{W}_{\ell_0}$ with the notations $\mathcal{V} = \mathcal{V}_{\ell_0}$ and $\mathcal{W}_1 = \mathcal{V}_1$. Let $\{\psi_k^l\}_{k,l=1}^{p_{l-1},\,\ell_0}$ denote the *multiscale wavelet basis*, composed of some *single scale bases* $\{\psi_k^l\}_{k=1}^{p_{l-1}}$ in the spaces $\mathcal{W}_l = \mathrm{span}\,\{\psi_k^l\}_{k=1}^{p_{l-1}}$. It generates the matrices

$$\boldsymbol{\Delta}_{\mathrm{wlet}} = \left(\langle(\psi_i^k)', (\psi_j^l)'\rangle_1\right)_{i,j=1,\,k,l=1}^{p_{l-1},\ell_0}, \quad \boldsymbol{\mathcal{M}}_{\mathrm{wlet}} = \left(\langle\psi_i^k, \psi_j^l\rangle_\phi\right)_{i,j=1,\,k,l=1}^{p_{l-1},\ell_0},$$

$$\mathbb{D}_1 = \mathrm{diag}\,[\langle(\psi_i^l)', (\psi_i^l)',\rangle_1]_{i,l=1}^{p_{l-1},\ell_0}, \qquad \mathbb{D}_0 = \mathrm{diag}\,[\langle\psi_i^l, \psi_i^l,\rangle_\phi]_{i,l=1}^{p_{l-1},\ell_0}.$$

The transformation matrix from the multiscale wavelet basis to the FE basis $\{\phi_k^{l_0}\}_{k=1}^{p-1}$ is denoted by $\mathbf{Q}$. Thus, if $\mathbf{v}$ and $\mathbf{v}_{\mathrm{wlet}}$ are the vectors of the coefficients of a function from $\mathcal{V}(0,1)$, represented in these two bases, respectively, then $\mathbf{v} = \mathbf{Q}^\top \mathbf{v}_{\mathrm{wlet}}$.

Theorem 1. *There exist multiscale wavelet bases, such that* $\boldsymbol{\Delta}_{\mathrm{sp}}^{-1} \asymp \mathbf{Q}^\top \mathbb{D}_1^{-1} \mathbf{Q}$, $\boldsymbol{\mathcal{M}}_{\mathrm{sp}}^{-1} \asymp \mathbf{Q}^\top \mathbb{D}_0^{-1} \mathbf{Q}$, *and matrix-vector multiplications* $\mathbf{Q}\,\mathbf{w}$, $\mathbf{Q}^\top \mathbf{w}$ *require* $\mathcal{O}(p)$ *arithmetic operations.*

Proof. The proof is simpler than the proof of similar results in [3], because the weight ϕ is symmetric on (-1,1). The cited authors justified existence of multiscale wavelet bases with the required properties in the case of the space $\mathcal{V}(0,1) := \{\, v \in \mathcal{V}(-1,1) \,|\, v(x) = 0 \text{ at } x \notin (0,1)\,\}$ and the weight $\phi = x$.

Theorem 2. *Let* $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} = \mathbf{C}\,\mathbb{B}_{I,\mathrm{sp}}\mathbf{C}$ *and*

$$\mathbb{B}_{I,\mathrm{sp}}^{-1} = \begin{cases} (\mathbf{Q}^\top \otimes \mathbf{Q}^\top)[\mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_1 \otimes \mathbb{D}_0]^{-1}(\mathbf{Q} \otimes \mathbf{Q}), & d = 2, \\ (\mathbf{Q}^\top \otimes \mathbf{Q}^\top \otimes \mathbf{Q}^\top)[\mathbb{D}_0 \otimes \mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_0 \otimes \mathbb{D}_1 \otimes \mathbb{D}_0 + & \\ \qquad \mathbb{D}_0 \otimes \mathbb{D}_0 \otimes \mathbb{D}_1]^{-1}(\mathbf{Q} \otimes \mathbf{Q} \otimes \mathbf{Q}), \varrho & d = 3. \end{cases}$$

Then $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} \asymp \mathbf{A}_{I,\mathrm{sp}}$ *and, therefore,* $\mathrm{cond}\,[\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}^{-1}\mathbf{A}_{I,\mathrm{sp}}] \prec 1$. *The arithmetical cost of the operation* $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}^{-1}\mathbf{v}$ *for any vector* $\mathbf{v}$ *is* $\mathcal{O}(p^d)$.

Proof. In view of Lemmas 1 and 2, it is sufficient to prove the equivalence cond $[\mathbb{B}_{I,\mathrm{sp}}^{-1}\boldsymbol{\Lambda}_{I,\mathrm{sp}}] \asymp 1$. The last is the consequence of the mentioned above relationships $\boldsymbol{\Delta}_{\ell_0} = \boldsymbol{\Delta}_{\mathrm{sp}}$ and $\boldsymbol{\mathcal{M}} := \boldsymbol{\mathcal{M}}_{\ell_0} \asymp \boldsymbol{\mathcal{D}}_{\mathrm{sp}}$, Theorem 1, and the representations of the involved matrices by the corresponding sums of Kronecker products.

Another important problem for DD algorithms is development of fast solvers for internal problems on faces. As it is now known, see, *e.g.,* [15], at nonsignificant lost in the condition, it is reduced to the preconditioning of the matrix of the quadratic form ${}_{00}|\cdot|^2_{1/2,\tau_0}$, $\tau_0 = (-1,1)\times(-1,1)$, on the subspace of polynomials $\mathring{\mathcal{Q}}_{p,\mathbf{x}}$ of two variables $\mathbf{x} = (x_1, x_2)$, vanishing on the boundary $\partial\tau_0$. Here ${}_{00}|\cdot|_{1/2,\tau_0}$ is the norm in the space $H^{1/2}_{00}(\tau_0)$, with the square τ_0 representing a typical face of the 3-d reference cube.

Theorem 3. *Let $d_{0,i}, d_{1,i}$ be diagonal entries of the matrices $\mathbb{D}_0, \mathbb{D}_1$, respectively, and $\mathbb{D}_{1/2}$ be the diagonal matrix with the entries on the main diagonal*

$$d^{(1/2)}_{i,j} = d_{0,i}d_{0,j}\sqrt{d_{1,i}/d_{0,i} + d_{1,j}/d_{0,j}}\,.$$

Let also $\boldsymbol{\mathcal{S}}_0 = \mathbf{C}\,\mathbb{S}_0\,\mathbf{C}$ and $\mathbb{S}_0^{-1} = (\mathbf{Q}^\top\otimes\mathbf{Q}^\top)\,\mathbb{D}_{1/2}^{-1}\,(\mathbf{Q}\otimes\mathbf{Q})$. Then for all $v \in \mathring{\mathcal{Q}}_{p,\mathbf{x}}$ and vectors $\mathbf{v}$, representing v in the basis $\mathring{\mathcal{M}}_{2,p}$, the norms ${}_{00}|\,v\,|_{1/2,F_0}$ and $||\mathbf{v}||_{\boldsymbol{\mathcal{S}}_0}$ are equivalent uniformly in p.

Proof. For the square $\tau_0 = (0,1)^2$, we have the preconditioner $\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} = \mathbf{C}\,\mathbb{B}_{I,\mathrm{sp}}\mathbf{C}$ for the stiffness matrix $\mathbf{A}_{I,\mathrm{sp}}$. Similarly, we can define the preconditioner $\boldsymbol{\mathcal{M}}_{I,\mathrm{sp}} = \mathbf{C}\,\mathbb{M}_{I,\mathrm{sp}}\mathbf{C}$ for the internal mass matrix $\mathbf{M}_{I,\mathrm{sp}}$ with $\mathbb{M}^{-1}_{I,\mathrm{sp}} = (\mathbf{Q}^\top\otimes\mathbf{Q}^\top)[\mathbb{D}_0\otimes\mathbb{D}_0]^{-1}(\mathbf{Q}\otimes\mathbf{Q})$. The further proof is produced by Peetre's K-interpolation method.

Presented fast solvers for the internal and face problems can be easily generalized on the "orthotropic" spectral elements with the shape polynomials having different orders along different axes.

4 Domain Decomposition Algorithm for Discretizations by Spectral Elements

Let we have to solve the problem

$$a_\Omega(u,v) := \int_\Omega \varrho(\mathbf{x})\nabla u\cdot\nabla v\,d\mathbf{x} = (f,v)_\Omega\,, \qquad \forall\, v \in \mathring{H}^1(\Omega)\,,$$

in the domain $\overline{\Omega} = \cup_{r=1}^{\mathcal{R}}\overline{\tau}_r$, which is an assemblage of compatible and in general curvilinear finite elements occupying domains τ_r. We assume that the finite elements are specified by means of non degenerate mappings $\mathbf{x} = \mathcal{X}^{(r)}(\mathbf{y}) : \overline{\tau}_0 \to \overline{\tau}_r$ satisfying the generalized conditions of the angular quasiuniformity, see, *e.g.*, [10]. The coefficient ϱ in the DD algorithm under consideration may be piece wise constant, but for brevity we imply $\varrho(\mathbf{x}) \equiv 1$. For the system $\mathbf{Ku} = \mathbf{f}$ of FE equations, we apply PCG (Preconditioned Conjugate Gradient Method) with the DD preconditioner

$$\boldsymbol{\mathcal{K}}^{-1} = \boldsymbol{\mathcal{K}}_I^{\,+} + \mathbf{P}_{V_B\to V}\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{P}^\top_{V_B\to V}\,, \quad \boldsymbol{\mathcal{S}}_B^{-1} = \boldsymbol{\mathcal{S}}_F^{\,+} + \mathbf{P}_{V_W\to V_B}(\boldsymbol{\mathcal{S}}_W^B)^{-1}\mathbf{P}^\top_{V_W\to V_B}\,,$$

of the same structure as in [9, 10]. The involved in the preconditioner matrices are defined as follows.

i) $\boldsymbol{\mathcal{K}}_I = \text{diag}\,[h_1\boldsymbol{\mathcal{B}}_{I,\text{sp}}, h_2\boldsymbol{\mathcal{B}}_{I,\text{sp}}, \ldots, h_{\mathcal{R}}\boldsymbol{\mathcal{B}}_{I,\text{sp}}]$ is the block diagonal preconditioner for the internal Dirichlet problems on FE's, where $\boldsymbol{\mathcal{B}}_{I,\text{sp}}$ is the multiresolution wavelet preconditioner-solver found in Theorem 2 and h_r is the characteristic size of a finite element τ_r.

ii) $\boldsymbol{\mathcal{S}}_F = \text{diag}\,[\kappa_1\boldsymbol{\mathcal{S}}_0, \kappa_2\boldsymbol{\mathcal{S}}_0, \ldots, \kappa_Q\boldsymbol{\mathcal{S}}_0]$ is the block diagonal preconditioner for the internal problems on faces of finite elements, where $\boldsymbol{\mathcal{S}}_0$ is the multiresolution wavelet preconditioner for one face, defined in Theorem 3, Q is the number of different faces $F_k \subset \Omega$, and κ_k are multipliers. Let for a face F_k of the discretization, $r_1(k)$ and $r_2(k)$ are the numbers of two elements $\overline{\tau}_{r_1(k)}$ and $\overline{\tau}_{r_2(k)}$, sharing the face F_k. Then $\kappa_k = (h_{r_1(k)} + h_{r_2(k)})$.

iii) The preconditioner $\boldsymbol{\mathcal{S}}_W^B$ for the wire basket subproblem. We borrow it, as well as the prolongation $\mathbf{P}_{V_W \to V_B}$, from [4], see also [15]. Let us note that the solving procedure for the system with the matrix $\boldsymbol{\mathcal{S}}_W^B$ is described in these papers up to solution of the sparse subsystem of the order $\mathcal{O}(\mathcal{R}) \times \mathcal{O}(\mathcal{R})$. We assume that there is a solver for this subsystem with the arithmetical cost not greater $\mathcal{O}(\mathcal{R}p^3)$.

iv) The matrix $\mathbf{P}_{V_B \to V}$ performs prolongations from the interelement boundary on the computational domain $\overline{\Omega}$. Its restriction to each FE is the master prolongation $\mathbb{P}_0$ defined for the reference element. For $\forall\,\mathbf{v}_B$, living on $\partial\tau_0$, we set $\mathbb{P}_0\mathbf{v}_B := \mathbf{u}$ with the subvectors $\mathbf{u}_I, \mathbf{u}_B$, where $\mathbf{u}_B := \mathbf{v}_B$ and $\mathbf{u}_I := \overline{\mathbf{v}}_I + \mathbb{P}_{\text{it}}(\mathbf{v}_B - \overline{\mathbf{v}}_B)$, where $\overline{\mathbf{v}}, \overline{\mathbf{v}}_I, \overline{\mathbf{v}}_B$ have for its entries the mean value on $\partial\tau_0$ of the polynomial $v \in \mathcal{O}_{p,\text{x}},\ v \leftrightarrow \mathbf{v}_B$. The matrix $\mathbb{P}_{\text{it}}$ is implicitly defined by the fixed number $k_0 \asymp (1+\log p)$ of the iterations $\mathbf{w}_I^{k+1} = \mathbf{w}_I^k - \sigma_{k+1}\boldsymbol{\mathcal{B}}_{I,\text{sp}}^{-1}[\boldsymbol{\mathcal{A}}_{I,\text{sp}}\mathbf{w}_I^k - \boldsymbol{\mathcal{A}}_{IB,\text{sp}}(\mathbf{v}_B - \overline{\mathbf{v}}_B)],\ \mathbf{w}_I^0 = \mathbf{0}$, with Chebyshev iteration parameters σ_k, so that $\mathbf{u}_I = \overline{\mathbf{v}}_I + \mathbf{w}^{k_0}$. Above, indices I, B are used for the subvectors, living on τ_0 and $\partial\tau_0$, respectively, and for the corresponding blocks of matrices, so that $\boldsymbol{\mathcal{A}}_{I,\text{sp}}, \boldsymbol{\mathcal{A}}_{IB,\text{sp}}$ are the blocks of $\boldsymbol{\mathcal{A}}_{I,\text{sp}}$, which in the iteration process can be replaced by the blocks $\mathbb{A}_{\hbar,I}, \mathbb{A}_{\hbar,IB}$ of $\mathbb{A}_{\hbar}$.

Theorem 4. *The DD preconditioner-solver* $\boldsymbol{\mathcal{K}}$ *provides the condition number* $\text{cond}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \le c(1+\log p)^2$, *whereas for any* $\mathbf{f}$ *the arithmetical cost of the operation* $\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}$ *is* $\mathcal{O}(p^3(1+\log p)\mathcal{R})$.

See [13] for the proof. Changes in the definition of $\boldsymbol{\mathcal{K}}$ allowing to retain Theorem 4 in the case of variable $\varrho,\ \varrho \asymp \overline{\varrho}$, where $\overline{\varrho} > 0$ is any element wise constant function, are obvious. Parallelization, robustness and h-adaptivity properties of the designed DD solver are exactly the same as for the DD solver in the case of hierarchical elements presented in [9], see also [13]. However, p-adaptivity is less flexible due to the Lagrange interpolation nature of spectral elements.

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