A Multilevel Method for Discontinuous Galerkin Approximation of Three-dimensional Elliptic Problems

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Summary. We construct optimal order multilevel preconditioners for interior-penalty discontinuous Galerkin (DG) finite element discretizations of 3D elliptic boundary-value problems. A specific assembling process is proposed which allows us to characterize the hierarchical splitting locally. This is also the key for a local analysis of the angle between the resulting subspaces. Applying the corresponding two-level basis transformation recursively, a sequence of algebraic problems is generated. These discrete problems can be associated with coarse versions of DG approximations (of the solution to the original variational problem) on a hierarchy of geometrically nested meshes. The presented numerical results demonstrate the potential of this approach.

1 Introduction

Discontinuous Galerkin (DG) finite element (FE) methods have gained much interest in the last decade due to their suitability for *hp*-adaptive techniques. They offer several advantages, e.g. the ease of treatment of meshes with hanging nodes, elements of varying shape and size, polynomials of variable degree, parallelization, preservation of local conservation properties, etc. An excellent overview and a detailed analysis of DG methods for elliptic problems can be found in [1]. Unfortunately, DG discretizations result in excessive number of degrees of freedom (DOF) as compared to their counter-part, i.e. the standard FE methods. Developing efficient preconditioning techniques, which yield fast iterative solvers, thus becomes of significant importance.

Optimal-order preconditioners obtained from recursive application of two-level FE methods have been introduced and extensively analyzed in the context of conforming methods, see e.g., [2, 3, 4]. For DG discretizations geometric multigrid (MG) type preconditioners and solvers for the linear system of equations have been considered in [6, 9]. However, our approach falls into the category of algebraic multilevel techniques. The method is obtained from

recursive application of the two-level algorithm. A sequence of FE spaces is created using geometrically nested meshes. A specific splitting of the bilinear terms is proposed which results in an assembling process similar to that of the conforming methods. In this approach one avoids the projection onto a coarse (auxiliary) space [7, 13], where the auxiliary space is related to a standard Galerkin discretization, and instead, generates a sequence of algebraic problems associated with a hierarchy of coarse versions of DG approximations of the original problem.

The content of this paper is summarized as follows. In Section 2 we state our model problem and discuss the DG approximation. Discrete formulation and matrix assembly, based on the splitting proposed in [12], are parts of Section 3. In Section 4 we comment on the construction of a proper hierarchical basis transformation for the linear systems arising from DG discretization. The analysis of the angle between the induced subspaces is the subject of Section 5. Finally, numerical experiments are presented in Section 6.

2 Model Problem and DG Approximation

Consider a second order elliptic problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$:

$$-\nabla \cdot (\underline{A}(x)\nabla u) = \underline{f}(x) \qquad \text{in } \Omega, \tag{1a}$$

$$u(x) = u_D \qquad \text{on } \Gamma_D, \tag{1b}$$

$$\underline{A}\nabla u \cdot \mathbf{n} = u_N \qquad \text{on } \Gamma_N. \tag{1c}$$

Here $\mathbf{n}$ is the exterior unit normal vector to $\partial\Omega \equiv \Gamma$. The boundary is assumed to be decomposed into two disjoint parts Γ_D and Γ_N, and the boundary data u_D, u_N are smooth. For the DG formulation below we shall need the existence of the traces of u and $\underline{A}\nabla u \cdot \mathbf{n}$ on the faces in Ω, and the solution u is assumed to have the required regularity. It is assumed that $\underline{A}$ is a symmetric positive definite matrix such that

$$c_1 |\xi|^2 \leq \underline{A}\xi \cdot \xi \leq c_2 |\xi|^2 \qquad \forall \xi \in \mathbb{R}^3.$$

Let $\mathcal{T}_h$ be a non-overlapping partition of Ω into a finite number of elements e. For any $e \in \mathcal{T}_h$ we denote its diameter by h_e and the boundary by ∂e. Let $\mathcal{F} = \bar{e}^+ \cap \bar{e}^-$ be a common face of two adjacent elements e^+, and e^-. Further, let $h = \max_{e\in\mathcal{T}_h} h_e$ denote the characteristic mesh size of the whole partition. The set of all the internal faces is denoted by $\mathcal{F}_0$, and $\mathcal{F}_D$ and $\mathcal{F}_N$ contain the faces of finite elements that belong to Γ_D and Γ_N, respectively. Finally, $\mathcal{F}$ is the set of all the faces, i.e., $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_D \cup \mathcal{F}_N$. We assume that the partition is shape-regular. We allow finite elements to vary in size and shape for local mesh adaptation and the mesh is not required to be conforming, i.e. elements may possess hanging nodes. Further, the face measure h_f is constant on each face $\mathcal{F} \in \mathcal{F}$ such that $h_f = |\mathcal{F}|^{\frac{1}{2}}$, for $\mathcal{F} \in \mathcal{F}$.

On the partition $\mathcal{T}_h$ we define a broken Sobolev space:

$$\mathcal{V} := H^2(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_e \in H^2(e), \forall e \in \mathcal{T}_h\}.$$

Note that the functions in $\mathcal{V}$ **may not satisfy** any boundary condition. By

$$\mathcal{V}_h := \mathcal{V}_h(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_e \in P_r(e), \forall e \in \mathcal{T}_h\},$$

where P_r is the set of polynomials of degree $r \geq 1$, we define a finite dimensional subspace of $\mathcal{V}$. Obviously, $\mathcal{V}_h = \Pi_{e\in\mathcal{T}_h} P_r(e)$. For ease of notations in what follows, on $\mathcal{V}$ we introduce the following forms

$$(\underline{A}\nabla_h u_h, \nabla_h v_h)_{\mathcal{T}_h} := \sum_{e\in\mathcal{T}_h} \int_e \underline{A}\nabla_h u_h \cdot \nabla_h v_h dx, \quad \langle p, q\rangle_{\mathcal{F}^g} := \sum_{\mathcal{F}\in\mathcal{F}^g} \int_{\mathcal{F}} p \cdot q ds,$$

where $\mathcal{F}^g$ is one of the sets $\mathcal{F}$, $\mathcal{F}_0$, $\mathcal{F}_D$, $\mathcal{F}_N$ or any of their combinations.

Let us now recall the DG formulation for second order elliptic problems. In recent years a large number of DG FEM were developed for elliptic boundary value problems, for review see, e.g. [1] and the references therein. Below, we consider the standard interior penalty (IP) DG method, see, e.g., [1]. For the problem (1), the primal IP-DG formulation can be stated as follows:
Find $u_h \in \mathcal{V}$ such that

$$\mathcal{A}(u_h, v_h) = \mathcal{L}(v_h), \quad \forall v_h \in \mathcal{V}, \tag{2a}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and the linear form $\mathcal{L}(\cdot) : \mathcal{V} \to \mathbb{R}$ are defined by the relations

$$\begin{aligned}\mathcal{A}(u_h, v_h) =& (\underline{A}\nabla_h u_h, \nabla_h v_h)_{\mathcal{T}_h} + \alpha h_f^{-1} \langle [\![u_h]\!], [\![v_h]\!]\rangle_{\mathcal{F}_0\cup\mathcal{F}_D} \\ &- \langle \{\{\underline{A}\nabla_h u_h\}\}, [\![v_h]\!]\rangle_{\mathcal{F}_0\cup\mathcal{F}_D} - \langle [\![u_h]\!], \{\{\underline{A}\nabla_h v_h\}\}\rangle_{\mathcal{F}_0\cup\mathcal{F}_D}, \end{aligned} \tag{2b}$$

$$\mathcal{L}(v_h) = \int_\Omega f v_h dx + \alpha h_f^{-1} \langle u_D, v_h\rangle_{\mathcal{F}_D} - \langle u_D \mathbf{n}, \underline{A}\nabla_h v_h\rangle_{\mathcal{F}_D} + \langle u_N, v_h\rangle_{\mathcal{F}_N}. \tag{2c}$$

Here $\{\{\cdot\}\}$ and $[\![\cdot]\!]$ denote the trace operators for *average* and *jump*, respectively, and α is a parameter which is to be defined to guarantee the coercivity of the bilinear form $\mathcal{A}$, see e.g. [1, 11].

As usual, we assume that the Dirichlet boundary conditions are defined by a given function $u_D \in H^1(\Omega)$ in the sense that the trace of $u - u_D$ on Γ_D is zero. For the sake of simplicity, we also assume that u_D is such that the boundary condition can be exactly satisfied by the approximations used. For the coercivity, boundedness, and convergence properties of the bilinear form $\mathcal{A}$ the reader can refer, e.g., [1, 11].

3 Discrete Formulation and Matrix Assembly

The weak formulation (2) is transformed into a set of algebraic equations by approximating u_h and v_h using trilinear polynomials in each cubic element as

$$u_{e,h} = \sum_{j=0}^{7} \widetilde{u}_{e,j} \mathcal{N}_{e,j}(x), \quad v_{e,h} = \sum_{j=0}^{7} \widetilde{v}_{e,j} \mathcal{N}_{e,j}(x), \quad x \in e \subset \mathbb{R}^3. \tag{3}$$

Here $\widetilde{u}_{e,j} \in \mathbb{R}^8$ and $\widetilde{v}_{e,j} \in \mathbb{R}^8$ are the expansion coefficients of u_h and the test function v_h in the element e, respectively, and $\mathcal{N}_{e,j}$ are trilinear basis functions.

We now briefly show the computation of the element stiffness matrix. Consider a general element e with all its face internal. Let its neighboring elements, which share a face with this element, be denoted by e_1^+, e_2^+, e_3^+, e_4^+, e_5^+, and e_6^+. Here $\cdot^+$ represents the neighboring element and digits $1, \ldots, 6$ represent the face number with which the neighboring element is attached.

Using the definition of the trace operators $\{\{\cdot\}\}$ and $[\![\cdot]\!]$ and the specific splitting of the bilinear terms proposed in [12] the resulting elemental bilinear form reads

$$\begin{aligned}\mathcal{A}_e(u_h, v_h) = &\int_e \underline{A} \nabla_h u_h \cdot \nabla_h v_h dx - \frac{1}{2} \sum_{\mathrm{F}=1}^{6} \int_{\mathcal{F}_\mathrm{F}} \Big(\Big(v_e \mathbf{n}_e + v_{e_\mathrm{F}^+} \mathbf{n}_{e_\mathrm{F}^+} \Big) \cdot \underline{A} \nabla_h u_e \\ &+ \underline{A} \nabla_h v_e \cdot \Big(u_e \mathbf{n}_e + u_{e_\mathrm{F}^+} \mathbf{n}_{e_\mathrm{F}^+} \Big) \Big) ds \\ &+ \frac{\alpha h_f^{-1}}{2} \sum_{\mathrm{F}=1}^{6} \int_{\mathcal{F}_\mathrm{F}} \Big(v_e \mathbf{n}_e + v_{e_\mathrm{F}^+} \mathbf{n}_{e_\mathrm{F}^+} \Big) \cdot \Big(u_e \mathbf{n}_e + u_{e_\mathrm{F}^+} \mathbf{n}_{e_\mathrm{F}^+} \Big) ds. \end{aligned} \tag{4}$$

In this approach, the DOF of the element e are connected with only those DOF of its neighboring elements e_F^+ which are at the common face.

Now let $N = 8N_e$ denote the total number of DOF in the system. Using the polynomial approximation (3) into the weak form (2), with elemental bilinear form (4), we get the following linear system of equations

$$A\mathbf{x} = \mathbf{b}, \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^N$, $A \in \mathbb{R}^{N \times N}$ with N_e^2 blocks of size 8×8, and $\mathbf{b} \in \mathbb{R}^N$, denote the vector of expansion coefficients, the global stiffness matrix, and the right hand side data vector, respectively.

4 Generalized Hierarchical Basis

In this section we discuss the two-level hierarchical basis (HB) transformation which is used in the construction of the multilevel preconditioner. Let us consider a hierarchy of partitions $\mathcal{T}_{h_\ell} \subset \mathcal{T}_{h_{\ell-1}} \subset \ldots \subset \mathcal{T}_{h_1} \subset \mathcal{T}_{h_0}$ of Ω, where the notation $\mathcal{T}_{h_k} = \mathcal{T}_h \subset \mathcal{T}_H = \mathcal{T}_{h_{k-1}}$ points out the fact that for any element e of the fine(r) partition $\mathcal{T}_h$ there is an element E of the coarse(r) mesh partition $\mathcal{T}_H$ such that $e \subset E$. For the construction of the preconditioner of the linear system (5) resulting from the IP-DG approximation of the basic problem (1)

its DOF are partitioned into a *fine* and a *coarse* (sub-) set, indicated by the subscripts 1 and 2, respectively. The partitioning is induced by a regular mesh refinement at every level $(k-1) = 0, 1, \ldots, \ell - 1$. In other words, by halving the mesh size, i.e., $h = H/2$, each element is subdivided into eight elements of similar shape, herewith producing the mesh at levels $k = 1, 2, \ldots, \ell$. Hence, the linear system (5) can be represented in the 2×2 block form as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \tag{6}$$

where $A_{21} = A_{12}^T$. By using the two-level transformation matrix

$$J = \begin{bmatrix} I_{11} & P_{12} \\ 0 & I_{22} \end{bmatrix}, \tag{7}$$

the system to be solved in the new basis has the representation

$$\widehat{A}\,\widehat{\mathbf{x}} = \widehat{\mathbf{b}}, \tag{8}$$

where $\widehat{A}$ and its submatrices $\widehat{A}_{11}$, $\widehat{A}_{12}$ $\widehat{A}_{21}$, $\widehat{A}_{22}$ are given by

$$\widehat{A} = J^T A\, J = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \tag{9a}$$

$$\widehat{A}_{11} = A_{11}, \quad \widehat{A}_{12} = A_{11}P_{12} + A_{12}, \quad \widehat{A}_{21} = P_{12}^T A_{11} + A_{21}, \tag{9b}$$

$$\widehat{A}_{22} = P_{12}^T A_{11} P_{12} + A_{21} P_{12} + P_{12}^T A_{12} + A_{22}. \tag{9c}$$

The vectors $\widehat{\mathbf{x}}$ and $\mathbf{b}$ are transformed according to $\mathbf{x} = J\,\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}} = J^T\mathbf{b}$, where

$$\mathbf{x}_1 = \widehat{\mathbf{x}}_1 + P_{12}\widehat{\mathbf{x}}_2, \quad \mathbf{x}_2 = \widehat{\mathbf{x}}_2, \tag{10a}$$

$$\widehat{\mathbf{b}}_1 = \mathbf{b}_1, \qquad \widehat{\mathbf{b}}_2 = P_{12}^T\mathbf{b}_1 + \mathbf{b}_2. \tag{10b}$$

If the interpolation matrix P_{12} in (7) is chosen in such a way that the matrix $\widehat{A}_{22}$ in (9c) corresponds to a coarse discretization of the original problem then P_{12} (and thus J) will constitute an HB transformation. In order to apply a local analysis, see the next Section, P_{12} is to be defined for a set of macro elements $\{E\}$ that covers the whole mesh. The general macro element we are using to define the local interpolation matrix P_E is simply the union of eight elements that share one vertex. The macro element accumulates 160 DOF, 32 of which define then an element on the next coarser level. The interpolation weights are simply taken to be 1/8 for the 8 interior fine DOF, 1/4 for the 48 fine DOF located at the face centers of the macro-element, and 1/2 for the 72 fine DOF associated with macro-element edges.

5 Local Estimation of the Constant in the CBS Inequality

It is known that the constant γ in the Cauchy-Bunyakowski-Schwarz (CBS) inequality, which is associated with the abstract angle between the two subspaces induced by the two-level hierarchical basis transformation, plays a key role in the derivation of optimal convergence rate estimates for two- and multilevel methods. Moreover, the value of the upper bound for $\gamma \in (0,1)$ is part of the construction of a proper stabilization polynomial in the linear algebraic multilevel iteration (AMLI) method, see [3, 4].

For the constant γ in the strengthened CBS inequality, the following relation holds

$$\gamma = \cos(\mathcal{V}_1,\, \mathcal{V}_2) = \sup_{u \in \mathcal{V}_1,\ v \in \mathcal{V}_2} \frac{\mathcal{A}(u,v)}{\sqrt{\mathcal{A}(u,u)\,\mathcal{A}(v,v)}}, \tag{11}$$

where $\mathcal{A}(\cdot,\cdot)$ is the bilinear form given by (2b). If $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ then γ is strictly less than one. As shown in [8], the constant γ can be estimated locally over each macro-element $E \in \mathcal{T}_H$, i.e. $\gamma \le \max\limits_{E} \gamma_E$, where

$$\gamma_E = \sup_{u \in \mathcal{V}_1(E),\ v \in \mathcal{V}_2(E)} \frac{\mathcal{A}_E(u,v)}{\sqrt{\mathcal{A}_E(u,u)\,\mathcal{A}_E(v,v)}}, \quad v \neq \text{const.}$$

The above mentioned spaces $\mathcal{V}_m(E),\ \ m = 1, 2$, contain the functions from $\mathcal{V}_m$ restricted to E and $\mathcal{A}_E(u,v)$ corresponds to $\mathcal{A}(u,v)$ restricted to the macro element E.

Evidently, the global two-level stiffness matrix $\widehat{A}$ can be assembled from the macro-element two-level stiffness matrices $\widehat{A}_E$, which are obtained from assembling the element matrices for all elements e contained in E in the (local) hierarchical basis. In simplified notation this can be written as

$$\widehat{A} = J^T A\, J = \sum_{E \in \mathcal{T}_H} \widehat{A}_E = \sum_{E \in \mathcal{T}_H} J_E^T A_E\, J_E.$$

Like the global matrix, the local matrices are also of the following 2×2 block form

$$\widehat{A}_E = \begin{bmatrix} \widehat{A}_{E,11} & \widehat{A}_{E,12} \\ \widehat{A}_{E,21} & \widehat{A}_{E,22} \end{bmatrix} = J_E^T \begin{bmatrix} A_{E,11} & A_{E,12} \\ A_{E,21} & A_{E,22} \end{bmatrix} J_E, \tag{12}$$

where the local macro-element two-level transformation matrix J_E is defined by

$$J_E = \begin{bmatrix} I & P_E \\ 0 & I \end{bmatrix}, \tag{13}$$

and the transformation-invariant (local) Schur complement is given by

$$S_E = \widehat{A}_{E,22} - \widehat{A}_{E,21}\widehat{A}_{E,11}^{-1}\widehat{A}_{E,12} = A_{E,22} - A_{E,21}A_{E,11}^{-1}A_{E,12}. \tag{14}$$

In the present context the choice of P_E is based on simple averaging, see [11]. We know from the general framework of two-level block (incomplete) factorization methods that it suffices to compute the minimal eigenvalue $\lambda_{E;\min}$ of the generalized eigenproblem (cf. [8, Theorem 6])

$$S_E\mathbf{v}_{E,2} = \lambda_E\widehat{A}_{E,22}\mathbf{v}_{E,2}, \quad \mathbf{v}_{E,2} \perp (1,1,\ldots,1)^T, \tag{15}$$

in order to conclude the following upper bound for the constant γ in (11):

$$\gamma^2 \leq \max_{E\in\mathcal{T}_H}\gamma_E^2 = \max_{E\in\mathcal{T}_H}(1-\lambda_{E;\min}). \tag{16}$$

This relation then implies condition number estimates for the corresponding two-level preconditioner (of additive and multiplicative type), see, e.g., [2].

The analysis of multilevel methods obtained by recursive application of the two-level preconditioner necessitates the establishment of this kind of (local) bounds for each coarsening step since the two-level hierarchical basis transformation is also applied recursively. This requires the knowledge of the related (macro) element matrices on all coarse levels. For the hierarchical basis transformation, as described in detail in [11], we have a very simple recursion relation for the element matrices. This recursion relation shows that the sequence of (global) coarse-grid matrices can be associated with coarse-discretizations of the original problem but with an exponentially increasing sequence of stabilization parameters $\alpha^{(j)}$. In the following Lemma and Theorem we state the relation between the element matrices at successive levels and provide a local estimate for the CBS constant. For the proof the reader is referred to [11].

Lemma 1. *Let $\widehat{A}^{(\ell)} := (J^{(\ell)})^T A\, J^{(\ell)}$ denote the stiffness matrix from (5) in hierarchical basis, where $A = \sum_{e\in\mathcal{T}_h} A_e$ and $A_e = A_e(\alpha) =: A_e^{(0)}(\alpha)$ $\forall e \in \mathcal{T}_h$ denotes the element matrix. Let us further assume that A_e has the same representation over all the elements of the domain. Then, if one neglects the correction matrices related to the boundary conditions, the coarse-grid problem at level $(\ell - j)$, $j = 1,\ldots,\ell$, (involving the matrices $J^{(\ell)}, J^{(\ell-1)},\ldots,J^{(\ell-j+1)}$) is characterized by the element matrix*

$$A_e^{(j)}(\alpha) = A_e(\alpha^{(j)}) = A_e(2^j\,\alpha). \tag{17}$$

In other words, the stabilization parameter α after j applications of the HB transformation equals $2^j\,\alpha$.

Theorem 1. *Consider the HB macro-element matrix $\widehat{A}_E^{(j)}(\alpha)$ associated with the eight elements defining the macro-element as a cube with side $2\,h_{\ell-j}$ where the element matrix $A_e^{(j)}(\alpha)$ from Lemma 1 is used in the standard way to assemble $A_E^{(j)}(\alpha)$. Then for the eigenvalues of (15) we have the lower bound*

$$\lambda_{E;\min}^{(j)} = \lambda_{E;\min}(\alpha^{(j)}) \geq \frac{1}{16}\left(1 - \frac{1}{\sqrt{2}\,\alpha^{(j)}}\right) = \frac{1}{16}\left(1 - \frac{1}{\sqrt{2}\,2^j\,\alpha}\right) \qquad (18)$$

for all $\alpha \geq 3/2$*, and thus the following relation holds for* γ_E

$$\gamma_E \leq \sqrt{\frac{15}{16} + \frac{1}{16\sqrt{2}\,\alpha^{(j)}}}. \qquad (19)$$

Remark 1. The bound (19) in Theorem 1 tells us that the condition number of the multiplicative preconditioner (with exact inversion of the A_{11}-block) can be stabilized using Chebyshev polynomials of degree five. However, as illustrated in the numerical examples below, this goal can also be achieved by employing four inner generalized conjugate gradient iterations.

6 Numerical Results

In this section we present numerical results which demonstrate the capabilities of the method. The computations are performed on Sun Fire V40z workstation with 4 AMD Opteron 852 CPUs (2.6GHz) with 32 GB RAM. For approximating u in all the examples we use trilinear elements i.e. linear shape functions for each of the variables x, y, and z. The stabilization parameter α is taken as 10. The pivot block in the multilevel preconditioner is approximated using incomplete LU (ILU) factorization based on a drop tolerance tol [12, 14]. For both the examples below we take Ω as a unit cube $(0,1) \times (0,1) \times (0,1)$.

Example 1. Consider the Poisson problem with homogeneous Dirichlet boundary conditions and choose f such that the analytic solution of the problem is given by $u = x\,(1-x)\,y\,(1-y)\,z\,(1-z)\exp(2x+2y+2z)$. The tolerance tol is taken as 10^{-2}.

Example 2. Consider the model problem (1) with homogeneous Dirichlet boundary conditions, $f = 1$, and the coefficient $\underline{A}$ as follows:

$$\underline{A} = \left\{ \begin{array}{ll} 1 \text{ in } & (I_1 \times I_1 \times I_1) \bigcup (I_2 \times I_2 \times I_1) \bigcup (I_2 \times I_1 \times I_2) \bigcup (I_1 \times I_2 \times I_2) \\ \varepsilon & \text{elsewhere} \end{array} \right\},$$

where $I_1 = (0, 0.5]$ and $I_2 = (0.5, 1)$, and $\varepsilon = \{0.1, 0.01, 0.001\}$. In this example the tolerance tol is chosen heuristically by relating it to the parameter ε as $\varepsilon \times 10^{-2}$.

For solving the linear system arising from various examples with varying h we employ the nonlinear algebraic multilevel iteration method (NLAMLI), see [5, 10, 12]. The stabilization of the condition number is achieved by using some fixed small number ν of inner generalized conjugate gradient (GCG) iterations. Here we choose $\nu = 4$ in all computations. The starting vector for the outer iteration is the zero vector and the stopping criteria is $\|r^{(n_{\text{it}})}\|/\|r^{(0)}\| \leq \delta =$

10^{-6}, where n_{it} is the number of iterations we report in the tables below. The coarsest mesh in all computations is of size 4×4×4 and has 512 DOF. The finer meshes for $1/h = 8, 16, 32, 64$ consist of $4096, \ldots, 2097152$ DOF, respectively.

Table 1. Numerical results

$1/h$	n_{it}	ρ	sec
8	27	0.59	0.47
16	27	0.60	4.76
32	27	0.60	44.27
64	27	0.60	422.09

(a) Example 1

	$\varepsilon = 0.1$		$\varepsilon = 0.01$		$\varepsilon = 0.001$	
$1/h$	n_{it}	ρ	n_{it}	ρ	n_{it}	ρ
8	25	0.57	25	0.56	25	0.56
16	28	0.61	28	0.60	28	0.61
32	30	0.62	29	0.62	30	0.62
64	30	0.62	29	0.62	30	0.63

(b) Example 2

In Table 1(a) we present the number of iterations, the average convergence factor ρ and the total CPU time (including the time for the construction of the preconditioner) for Example 1. We observe that the number of iterations is constant and the CPU time is proportional to the problem size which shows that the overall solution process is of optimal order of computational complexity. The same holds for Example 2, cf. Table 1(b). These results also indicate the robustness of the preconditioner with respect to the jumps in the coefficient $\underline{A}$.

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