Reinforcement-Matrix Interaction Modeled by FETI Method

Jaroslav Kruis and Zdeněk Bittnar

Czech Technical University in Prague, Faculty of Civil Engineering, Department of Mechanics, Thákurova 7, 166 29 Prague, Czech Republic.
{jk,bittnar}@fsv.cvut.cz

1 Introduction

There are composite materials created from two constituents, composite matrix and reinforcement. The reinforcement is usually significantly stiffer than the composite matrix and proper orientation of the reinforcement leads to excellent overall properties of the composite materials. Interaction between the reinforcement and the composite matrix is very important. Perfect or imperfect bonding between the reinforcement and matrix may occur. The perfect bonding takes place only for lower level of applied loads. The perfect bonding occurs when there is no slip between interface points on fiber and points on composite matrix. In other words, interface points on fiber and matrix have the same displacements. Higher load levels cause debonding which decreases the overall stiffness of the composite. The debonding causes different displacements on the fiber and matrix. A special attention is devoted to the modeling of the interaction between the matrix and reinforcement because it can reduce properties of the composite.

The modeling of the interaction is based on pullout tests. The arrangement of such tests is the following. There is a composite matrix with one embedded fiber which is under tension. The growing force in the fiber causes debonding of matrix-fiber connection and fiber moves out from the matrix. Detailed description of pullout effects is relatively complicated and several simplified approaches are used. This contribution deals with the case with perfect bonding between reinforcement and matrix as well as debonding which is controlled by a linear relationship. The most general model with nonlinear debonding is not studied, but it is in the center of our attention.

This contribution deals with application of the FETI method to bonding or debonding problems. The perfect bonding can be directly described by the classical FETI method while the debonding can be modeled by slightly modified FETI method. The FETI method offers all necessary components for bonding/debonding problems.

2 Overview of the FETI Method

The FETI method was introduced by Farhat and Roux in 1991 in [2]. It is a non-overlapping domain decomposition method which enforces the continuity among subdomains by Lagrange multipliers. The FETI method or its variants have been applied to a broad class of two and three dimensional problems of the second and the fourth order. More details can be found e.g. in [6, 3, 4, 5, 1].

The FETI method will be shortly described on a problem of mechanical equilibrium of a solid body. The finite element method is used for the problem discretization. The equilibrium state minimizes the energy functional

$$\Pi(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{u}^T\mathbf{f}, \tag{1}$$

where $\mathbf{u}$ denotes the vector of unknown displacements, $\mathbf{K}$ denotes the stiffness matrix and $\mathbf{f}$ denotes the vector of prescribed forces.

Let the original domain be decomposed to m subdomains. Unknown displacements defined on the j-th subdomain are located in the vector $\mathbf{u}^j$. All unknown displacements are located in the vector

$$\mathbf{u}^T = \left((\mathbf{u}^1)^T, (\mathbf{u}^2)^T, \dots, (\mathbf{u}^m)^T\right) . \tag{2}$$

The stiffness matrix of the j-th subdomain is denoted $\mathbf{K}^j$ and the stiffness matrix of the whole problem has the form

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}^1 & & & \\ & \mathbf{K}^2 & & \\ & & \ddots & \\ & & & \mathbf{K}^m \end{pmatrix} . \tag{3}$$

The nodal loads of the j-th subdomain are located in the vector $\mathbf{f}^j$ and the load vector of the problem has the form

$$\mathbf{f}^T = \left((\mathbf{f}^1)^T, (\mathbf{f}^2)^T, \dots, (\mathbf{f}^m)^T\right) . \tag{4}$$

Continuity among subdomains has the form

$$\mathbf{B}\mathbf{u} = \mathbf{0} \tag{5}$$

where the boolean matrix $\mathbf{B}$ has the form

$$\mathbf{B} = \left(\mathbf{B}^1, \mathbf{B}^2, \dots, \mathbf{B}^m\right) . \tag{6}$$

The matrices $\mathbf{B}^j$ contain only entries equal to $1, -1, 0$. With the previously defined notation, the energy functional has the form

$$\Pi(\mathbf{u}, \boldsymbol{\lambda}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{u}^T\mathbf{f} + \boldsymbol{\lambda}^T\mathbf{B}\mathbf{u} \tag{7}$$

where the vector $\boldsymbol{\lambda}$ contains Lagrange multipliers. Stationary conditions of the energy functional have the form

$$\frac{\partial \Pi}{\partial \mathbf{u}} = \mathbf{K}\mathbf{u} - \mathbf{f} + \mathbf{B}^T \boldsymbol{\lambda} = \mathbf{0} \tag{8}$$

$$\frac{\partial \Pi}{\partial \boldsymbol{\lambda}} = \mathbf{B}\mathbf{u} = \mathbf{0}\,. \tag{9}$$

Equation (8) expresses the equilibrium condition while (9) expresses the continuity condition. The known feature of the FETI method is application of a pseudoinverse matrix in relationship for unknown displacements

$$\mathbf{u} = \mathbf{K}^+ \left(\mathbf{f} - \mathbf{B}^T \boldsymbol{\lambda}\right) + \mathbf{R}\boldsymbol{\alpha} \tag{10}$$

which stems from floating subdomains. The stiffness matrix of a floating subdomain is singular. The matrix $\mathbf{R}$ contains the rigid body modes of particular subdomains and the vector $\boldsymbol{\alpha}$ contains amplitudes that specify the contribution of the rigid body motions to the displacements. The pseudoinverse matrix and the rigid body motion matrix can be written in the form

$$\mathbf{K}^+ = \begin{pmatrix} (\mathbf{K}^1)^+ & & & \\ & (\mathbf{K}^2)^+ & & \\ & & \ddots & \\ & & & (\mathbf{K}^m)^+ \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \mathbf{R}^1 & & & \\ & \mathbf{R}^2 & & \\ & & \ddots & \\ & & & \mathbf{R}^m \end{pmatrix}. \tag{11}$$

Besides of utilization of the pseudoinverse matrix, a solvability condition in the form

$$\left(\mathbf{f} - \mathbf{B}^T \boldsymbol{\lambda}\right) \perp \ker \mathbf{K} = \mathbf{R} \tag{12}$$

has to be taken into account. Substitution of unknown displacements to the continuity condition leads to the form

$$\mathbf{B}\mathbf{K}^+\mathbf{B}^T \boldsymbol{\lambda} = \mathbf{B}\mathbf{K}^+\mathbf{f} + \mathbf{B}\mathbf{R}\boldsymbol{\alpha}\,. \tag{13}$$

The solvability condition can be written in the form

$$\mathbf{R}^T \left(\mathbf{f} - \mathbf{B}^T \boldsymbol{\lambda}\right) = \mathbf{0}\,. \tag{14}$$

Usual notation in the FETI method is the following

$$\mathbf{F} = \mathbf{B}\mathbf{K}^+\mathbf{B}^T \tag{15}$$

$$\mathbf{G} = -\mathbf{B}\mathbf{R} \tag{16}$$

$$\mathbf{d} = \mathbf{B}\mathbf{K}^+\mathbf{f} \tag{17}$$

$$\mathbf{e} = -\mathbf{R}^T\mathbf{f}\,. \tag{18}$$

The continuity and solvability conditions can be rewritten with the defined notation in the form

$$\begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{e} \end{pmatrix}. \tag{19}$$

The system of equations (19) is called the coarse or interface problem.

3 Modification of the Method

The classical FETI method uses the continuity condition (5) which enforces the same displacements at the interface nodes. If there is a reason for different displacements between two neighbor subdomains, the continuity condition transforms itself to a slip condition. The slip condition can be written in the form

$$\mathbf{B}\mathbf{u} = \mathbf{s}\,. \tag{20}$$

The vector $\mathbf{s}$ stores slips between interface nodes. For this moment, the slip is assumed to be prescribed and constant.

Let the boundary unknowns be split to two disjunct parts. The boundary unknowns which satisfy the continuity condition are located in the vector $\mathbf{u}_c$, while the boundary unknowns which satisfy the slip condition are located in the vector $\mathbf{u}_s$. Similarly to the continuity condition in the FETI method, the vectors $\mathbf{u}_c$ and $\mathbf{u}_s$ can be written in the form

$$\mathbf{u}_c = \mathbf{B}_c\mathbf{u} \tag{21}$$

$$\mathbf{u}_s = \mathbf{B}_s\mathbf{u} \tag{22}$$

where $\mathbf{B}_c$ and $\mathbf{B}_s$ are the boolean matrices. Now, the continuity condition has the form

$$\mathbf{B}_c\mathbf{u} = \mathbf{0} \tag{23}$$

and the slip condition has the form

$$\mathbf{B}_s\mathbf{u} = \mathbf{s}\,. \tag{24}$$

The conditions (23) and (24) can be amalgamated to a new interface condition

$$\mathbf{B}\mathbf{u} = \begin{pmatrix} \mathbf{B}_c \\ \mathbf{B}_s \end{pmatrix} \mathbf{u} = \begin{pmatrix} \mathbf{0} \\ \mathbf{s} \end{pmatrix} = \mathbf{c}\,. \tag{25}$$

The energy functional can be rewritten to the form

$$\Pi = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{u}^T\mathbf{f} + \boldsymbol{\lambda}^T(\mathbf{B}\mathbf{u} - \mathbf{c})\,. \tag{26}$$

The stationary conditions have the form

$$\mathbf{K}\mathbf{u} - \mathbf{f} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0} \tag{27}$$

$$\mathbf{B}\mathbf{u} = \mathbf{c}\,. \tag{28}$$

As was mentioned before, the system of two stationary conditions is accompanied by the solvability condition (12). The expression of the vector $\mathbf{u}$ given in (10) remains the same and the interface condition has the form

$$\mathbf{B}\mathbf{K}^+\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{B}\mathbf{K}^+\mathbf{f} + \mathbf{B}\mathbf{R}\boldsymbol{\alpha} - \mathbf{c} \tag{29}$$

and the solvability condition has the form

$$\mathbf{R}^T\left(\mathbf{f} - \mathbf{B}^T\boldsymbol{\lambda}\right) = \mathbf{0}\,. \tag{30}$$

The coarse problem can be written with the help of notation (15) - (18) in the form

$$\begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{d} - \mathbf{c} \\ \mathbf{e} \end{pmatrix} . \tag{31}$$

The modified coarse problem (31) differs from the original coarse problem (19) by the vector of prescribed slips $\mathbf{c}$ on the right hand side.

The prescribed slip between two subdomains is not a common case. On the other hand, the slip often depends on shear stress. Discretized form of equations used in the coarse problem requires a discretized law between slip as a difference of two neighbor displacements and nodal forces as integrals of stresses along element edges. One of the simplest laws is the linear relationship

$$\mathbf{c} = \mathbf{H}\boldsymbol{\lambda} \tag{32}$$

where $\mathbf{H}$ denotes the compliance matrix. Substitution of (32) to the coarse problem (31) leads to the form

$$\begin{pmatrix} \mathbf{F} + \mathbf{H} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{d} \\ \mathbf{e} \end{pmatrix} . \tag{33}$$

It should be noted that the coarse system of equations (33) is usually solved by the modified conjugate gradient method. Details can be found in [3] and [5]. The only difference with respect to the system (19) is the compliance matrix $\mathbf{H}$. Only one step, the matrix-vector multiplication, of the modified conjugate gradient method should be changed. The compliance matrix may be a diagonal or nearly diagonal matrix.

4 Numerical Examples

Four cases of bonding/debonding behavior are computed by the classical and modified FETI method. There are always two subdomains. One subdomain represents the composite matrix and the other one represents the fiber. A perfect bonding is described directly by the classical FETI method. The usual continuity condition is used. The displacements of the fiber and composite matrix at a selected point are identical and the situation is depicted in Figure 1 (left).

An imperfect bonding is described by the modified FETI method with the constant compliance matrix $\mathbf{H}$. The displacements of a fiber are greater than the displacements of the composite matrix. The greater force is applied, the greater slip occurs. The situation is depicted in Figure 1 (right).

A perfect bonding followed by an imperfect bonding is modeled by the modified FETI method. At the beginning, the compliance matrix is zero which expresses infinitely large stiffness between subdomains. At a certain load level, debonding effect is assumed and the compliance matrix is redefined and it is a constant matrix in the following steps. The displacements of the fiber and matrix are the same at the beginning but then they are different. The situation is depicted in Figure 2 (left).

The last example shows a similar problem to the previous one. The compliance matrix $\mathbf{H}$ is not assumed constant but the compliances are growing from zero values up to a certain level. It means, that the stiffness is decreasing from infinitely large value to some finite value. The greater force acts, the higher compliance is attained and the greater slip between the fiber and the composite matrix occurs. The situation is depicted in Figure 2 (right).

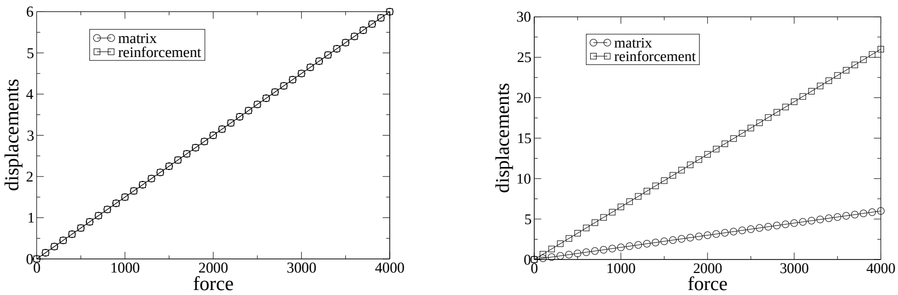

Fig. 1. Perfect bonding (left). Imperfect bonding (debonding) (right).

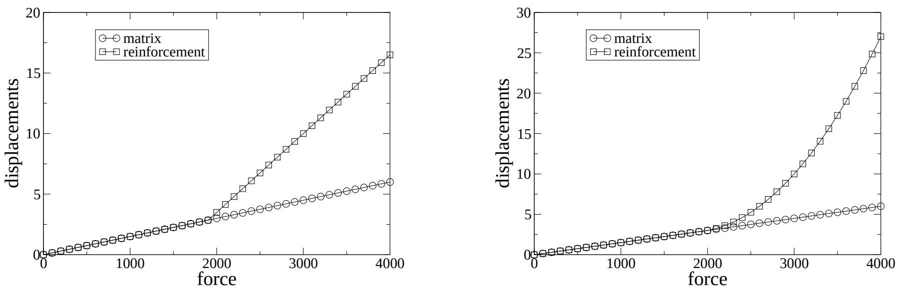

Fig. 2. Imperfect bonding: with delay (left), with changing compliance (right).

5 Conclusions

A slight modification of the FETI method is proposed for problems with the imperfect bonding between the composite matrix and reinforcement. The perfect bonding is modeled by the classical FETI method. Application of a constant compliance matrix leads to linear debonding while a variable compliance matrix can describe nonlinear debonding effects. The advantage of the proposed modification stems from the structure of the compliance matrix which can be nearly diagonal and therefore computationally cheap. The second advantage stems from possible parallelization. Each fiber, generally each piece of reinforcement, as well as the composite matrix can be assigned to one processor and thus large problems may be solved efficiently.

Acknowledgement. Financial support for this work was provided by the Ministry of Education, Youth and Sports of the Czech Republic, project VZ 03 CEZ MSM 6840770003. The financial support is gratefully acknowledged.

References

[1] M. Bhardwaj, D. Day, C. Farhat, M. Lesoinne, K. Pierson, and D. Rixen. Application of the FETI method to ASCI problems—scalability results on 1000 processors and discussion of highly heterogeneous problems. *Internat. J. Numer. Methods Engrg.*, 47:513–535, 2000.

[2] C. Farhat and F. X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Methods Engrg.*, 32:1205–1227, 1991.

[3] C. Farhat and F. X. Roux. Implicit parallel processing in structural mechanics. *Comput. Mech. Adv.*, 2:1–124, 1994.

[4] J. Kruis. *Domain Decomposition Methods for Distributed Computing.* Saxe-Coburg Publications, Kippen, Stirling, Scotland, 2006.

[5] D. J. Rixen, C. Farhat, R. Tezaur, and J. Mandel. Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver. *Internat. J. Numer. Methods Engrg.*, 46:501–533, 1999.

[6] A. Toselli and O. Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, Germany, 2005.