Domain Decomposition of Constrained Optimal Control Problems for 2D Elliptic System on Networked Domains: Convergence and A Posteriori Error Estimates

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Summary. We consider optimal control problems for elliptic systems under control constraints on networked domains. In particular, we study such systems in a format that allows for applications in problems including membranes and Reissner-Mindlin plates on multi-link-domains, called networks. We first provide the models, derive first order optimality conditions in terms of variational equations and inequalities for a control-constrained linear-quadratic optimal control problem, and then introduce a non-overlapping iterative domain decomposition method, which is based on Robin-type interface updates at multiple joints (edges). We prove convergence of the iteration and derive a posteriori error estimates with respect to the iteration across the interfaces.

1 Introduction

Partial differential equations on networks or networked domains consisting of 1-d, 2-d and possibly 3-d sub-domains linked together at multiple joints, edges or faces, respectively, arise in many important applications, as in gas-, water-, traffic- or blood-flow in pipe-, channel-, road or artery networks, or in beam-plate structures, as well as in many micro-, meso- or macro-mechanical smart structures. The equations governing the processes on those multi-link domains are elliptic, parabolic and hyperbolic, dependent on the application. Problems on such networks are genuinely subject to sub-structuring by the way of non-overlapping domain decomposition methods. This remains true even for optimal control problems formulated for such partial differential equations on multi-link domains. While non-overlapping domain decompositions for unconstrained optimal control problems involving partial differential equations on such networks have been studied in depth in the monograph [5], such non-overlapping domain decompositions for problems with control constraints have not been discussed so far. This is the purpose of these notes. In order to keep matters simple but still provide some insight also in the modeling, we study elliptic systems only. The time-dependent case can also be handled, but is much more involved, see [5] for unconstrained problems. The ddm is in the spirit of [1]. See also [9] for a general reference, and [2] for vascular flow in artery networks, where Dirichlet-Neumann iterates are considered. See also [3] and [4] for ddm in the context of optimal control problems. Decomposition of optimality systems corresponding to state-constrained optimal control problems seem not to have been discussed so far. This is ongoing research of the author.

2 Elliptic Systems on 2-D Networks

As PDEs on networks are somewhat unusual, we take some effort in order to make the modeling and its scope more transparent. Unfortunately, this involves some notation. A *two-dimensional polygonal network* $\mathcal{P}$ in $\mathrm{I\!R}^N$ is a finite union of nonempty subsets $\mathcal{P}_i$, $i \in \mathcal{I}$, such that

(i) each $\mathcal{P}_i$ is a simply connected open polygonal subset of a plane Π_i in $\mathrm{I\!R}^N$;
(ii) $\bigcup_{i\in\mathcal{I}} \overline{\mathcal{P}}_i$ is connected;
(iii) for all $i, j \in \mathcal{I}$, $\overline{\mathcal{P}}_i \bigcap \overline{\mathcal{P}}_j$ is either empty, a common vertex, or a whole common side.

The reader is referred to [8], whose notation we adopt, for more details about such 2-d networks. For each $i \in \mathcal{I}$ we fix once and for all a system of coordinates in Π_i. We assume that the boundary $\partial\mathcal{P}_i$ of $\mathcal{P}_i$ is the union of a finite number of linear segments $\overline{\Gamma}_{ij}$, $j = 1, \ldots, N_i$. It is convenient to assume that Γ_{ij} is open in $\partial\mathcal{P}_i$. The collection of all Γ_{ij} are the *edges* of $\mathcal{P}$ and will be denoted by $\mathcal{E}$. An edge Γ_{ij} corresponding to an $e \in \mathcal{E}$ will be denoted by Γ_{ie} and the *index set* $\mathcal{I}_e$ of e is $\mathcal{I}_e = \{i|\, e = \Gamma_{ie}\}$. The *degree* of an edge is the cardinality of $\mathcal{I}_e$ and is denoted by $d(e)$. For each $i \in \mathcal{I}_e$ we will denote by ν_{ie} the unit outer normal to $\mathcal{P}_i$ along Γ_{ie}.

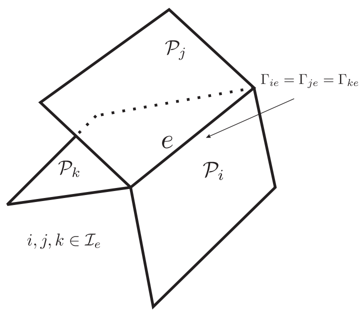

Fig. 1. A star-like multiple link-subdomain

The coordinates of ν_{ie} in the given coordinate system of $\mathcal{P}_i$ are denoted by (ν_{ie}^1, ν_{ie}^2). We partition the edges of $\mathcal{E}$ into two disjoint subsets $\mathcal{D}$ and $\mathcal{N}$, corresponding respectively to edges along which Dirichlet conditions hold and along which Neumann or transmission conditions hold. The Dirichlet edges are assumed to be *exterior edges*, that is, edges for which $d(e) = 1$. The Neumann edges consist of exterior edges $\mathcal{N}^{\text{ext}}$ and *interior edges* $\mathcal{N}^{\text{int}} := \mathcal{N} \backslash \mathcal{N}^{\text{ext}}$. Let $m \geq 1$ be a given integer. For a function $W : \mathcal{P} \mapsto \mathbb{R}^m$, W_i will denote the restriction of W to $\mathcal{P}_i$, that is $W_i : \mathcal{P}_i \mapsto \mathbb{R}^m : x \mapsto W(x)$. We introduce real $m \times m$ matrices $A_i^{\alpha\beta},\ B_i^\beta,\ C_i,\quad i \in \mathcal{I},\ \alpha, \beta = 1, 2$, where $A_i^{\alpha\beta} = (A_i^{\beta\alpha})^*,\quad C_i = C_i^*$ and where the * superscript denotes transpose. For sufficiently regular $W, \Phi : \mathcal{P} \mapsto \mathbb{R}^m$ we define the symmetric bilinear form

$$a(W, \Phi) = \sum_{i \in \mathcal{I}} \int_{\mathcal{P}_i} [A_i^{\alpha\beta}(W_{i,\beta} + B_i^\beta W_i) \cdot (\Phi_{i,\alpha} + B_i^\alpha \Phi_i) + C_i W_i \cdot \Phi_i]\, dx, \quad (1)$$

where repeated lower case Greek indices are summed over 1,2. A subscript following a comma indicates differentiation with respect to the corresponding variable, e.g., $W_{i,\beta} = \partial W_i / \partial x_\beta$. The matrices $A_i^{\alpha\beta}$, B_i^β, C_i may depend on $(x_1, x_2) \in \mathcal{P}_i$ and $a(W, \Phi)$ is required to be $\mathcal{V}$-elliptic for an appropriate function space $\mathcal{V}$ specified below. We shall consider the variational problem

$$a(W, \Phi) = \langle F, \Phi \rangle_V, \quad \forall \Phi \in \mathcal{V},\ 0 < t < T, \quad (2)$$

where $\mathcal{V}$ is a certain space of test functions and F is a given, sufficiently regular function. The variational equation (2) obviously implies, in particular, that the W_i, $i \in \mathcal{I}$, formally satisfy the system of equations

$$-\frac{\partial}{\partial x_\alpha}[A_i^{\alpha\beta}(W_{i,\beta} + B_i^\beta W_i)] + (B_i^\alpha)^* A_i^{\alpha\beta}(W_{i,\beta} + B_i^\beta W_i) + C_i W_i = F_i \ \text{ in } \mathcal{P}_i,\ i \in \mathcal{I}. \quad (3)$$

To determine the space $\mathcal{V}$, we need to specify the conditions satisfied by W along the edges of $\mathcal{P}$. These conditions are of two types: *geometric edge conditions*, and *mechanical edge conditions*. As usual, the space $\mathcal{V}$ is then defined in terms of the geometric edge conditions. At a Dirichlet edge we set

$$W_i = 0 \ \text{ on } e \text{ when } \Gamma_{ie} \in \mathcal{D}. \quad (4)$$

Further, along each $e \in \mathcal{N}^{\text{int}}$ we impose the condition

$$Q_{ie} W_i = Q_{je} W_j \ \text{ on } e \text{ when } \Gamma_{ie} = \Gamma_{je},\ e \in \mathcal{N}^{\text{int}}, \quad (5)$$

where, for each $i \in \mathcal{I}_e$, Q_{ie} is a real, nontrivial $p_e \times m$ matrix of rank $p_e \leq m$ with p_e independent of $i \in \mathcal{I}_e$. If $p_e < m$ additional conditions *may* be imposed, such as

$$\Pi_{ie} W_i = 0 \ \text{ on } e,\ \forall\, i \in \mathcal{I}_e,\ e \in \mathcal{N}^{\text{int}}, \quad (6)$$

where Π_{ie} is the orthogonal projection onto the kernel of Q_{ie}. (Note that (5) is a condition on only the components $\Pi_{ie}^{\perp} W_i$, $i \in \mathcal{I}_e$, where $\Pi_{ie}^{\perp}$ is the orthogonal projection onto the orthogonal complement in $\mathbb{R}^m$ of the kernel of Q_{ie}.) For definiteness we always assume that (6) is imposed and leave to the reader the minor modifications that occur in the opposite case. Thus the geometric edge conditions are taken to be (4) - (6), and the space $\mathcal{V}$ of test functions consists of sufficiently regular functions $\Phi : \mathcal{P} \mapsto \mathbb{R}^m$ that satisfy the geometric edge conditions. Formal integration by parts in (2) and taking proper variations shows that, in addition to (3), W_i must satisfy

$$\nu_{ie}^{\alpha} A_i^{\alpha\beta}(W_{i,\beta} + B_i^{\beta} W_i) = 0 \text{ on } e \text{ when } \Gamma_{ie} \in \mathcal{N}^{\text{ext}}. \tag{7}$$

For each $\Gamma_{ie} \in \mathcal{N}^{\text{int}}$ write $\Phi_i = \Pi_{ie}\Phi_i + \Pi_{ie}^{\perp}\Phi_i$, and let Q_{ie}^{+} denote the generalized inverse of Q_{ie}, that is Q_{ie}^{+} is a $m \times p_e$ matrix such that $Q_{ie}Q_{ie}^{+} = I_{p_e}$, $Q_{ie}^{+}Q_{ie} = \Pi_{ie}^{\perp}$. Then we deduce that

$$\sum_{i \in \mathcal{I}_e} (Q_{ie}^{+})^* \nu_{ie}^{\alpha} A_i^{\alpha\beta}(W_{i,\beta} + B_i^{\beta} W_i) = 0 \text{ on } e \text{ if } e \in \mathcal{N}^{\text{int}}. \tag{8}$$

Conditions (7) and (8) are called the *mechanical edge conditions*. We also refer to (5) and (8) as the geometric and mechanical *transmission conditions*, respectively. To summarize, the edge conditions are comprised of the geometric edge conditions (4) - (6), and the mechanical edge conditions (7), (8). The geometric transmission conditions are (5) and (6), while the mechanical transmission conditions are given by (8).

2.1 Examples

Example 1. (Scalar problems on networks) *Suppose that* $m = 1$. *In this case the matrices* $A_i^{\alpha\beta}$, B_i^{α}, C_i, Q_{ie} *reduce to scalars* $a_i^{\alpha\beta}$, b_i^{α}, c_i, q_{ie}, *where* $a_i^{\alpha\beta} = a_i^{\beta\alpha}$. *Set* $A_i = \big(a_i^{\alpha\beta}\big)$, $\quad b_i = col\big(b_i^{\alpha}\big)$. *The system* (3) *takes the form*

$$-\nabla \cdot (A_i \nabla W_i) + [-\nabla \cdot (A_i b_i) + b_i^* A_i b_i + c_i] W_i = F_i. \tag{9}$$

Suppose that all $q_{ie} = 1$. *The geometric edge conditions* (4), (5) *are then* $W_i = 0$ *on* e *when* $e \in \mathcal{D}$, $W_i = W_j$ *on* e *when* $e \in \mathcal{N}^{int}$ *while the mechanical edge conditions are*

$$\sum_{i \in \mathcal{I}_e} [\nu_{ie} \cdot (A_i \nabla W_i) + (\nu_{ie} \cdot A_i b_i) W_i] = 0 \quad on\ e\ when\ e \in \mathcal{N}.$$

Example 2. (Membrane networks in $\mathbb{R}^3$.) *In this case* $m = N = 3$. *For each* $i \in \mathcal{I}$ *set* $\eta_{i3} = \eta_{i1} \wedge \eta_{i2}$, *where* η_{i1}, η_{i2} *are the unit coordinate vectors in* Π_i. *Suppose that* $B_i = C_i = 0$, $Q_{ie} = I_3$, *where* I_3 *denotes the identity matrix with respect to the* $\{\eta_{ik}\}_{k=1}^3$ *basis. With respect to this basis the matrices* $A_i^{\alpha\beta}$ *are given by*

$$A_i^{11} = \begin{pmatrix} 2\mu_i + \lambda_i & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix}, \quad A_i^{22} = \begin{pmatrix} \mu_i & 0 & 0 \\ 0 & 2\mu_i + \lambda_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix},$$

$$A_i^{12} = \begin{pmatrix} 0 & \lambda_i & 0 \\ \mu_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_i^{21} = \begin{pmatrix} 0 & \mu_i & 0 \\ \lambda_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Write $W_i = \sum_{k=1}^{3} W_{ik}\eta_{ik}$, $w_i = W_{i\alpha}\eta_{i\alpha}$, $\varepsilon_{\alpha\beta}(w_i) = \frac{1}{2}(W_{i\alpha,\beta} + W_{i\beta,\alpha})$, $\sigma_i^{\alpha\beta}(w_i) = 2\mu_i\varepsilon_{\alpha\beta}(w_i) + \lambda_i\varepsilon_{\gamma\gamma}(w_i)\delta^{\alpha\beta}$. *The bilinear form* (1) *may be written*

$$a(W, \Phi) = \sum_{i\in\mathcal{I}} \int_{\mathcal{P}_i} [\sigma_i^{\alpha\beta}(w_i)\varepsilon_{\alpha\beta}(\phi_i) + \mu_i W_{i3,\alpha}\Phi_{i3,\alpha}]\, dx$$

where $\Phi_i = \sum_{k=1}^{3} \Phi_{ik}\eta_{ik} := \phi_i + \Phi_{i3}\eta_{i3}$. *The geometric edge conditions* (4), (5) *are as above, but now in a vectorial sense. The mechanical edge conditions are obtained as usual.*

The corresponding system models the small, static deformation of a network of homogeneous isotropic membranes $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ *in* $\mathbb{R}^3$ *of uniform density one and Lamé parameters* λ_i *and* μ_i *under distributed loads* F_i, $i \in \mathcal{I}$*;* $W_i(x_1, x_2)$ *represents the displacement of the material particle situated at* $(x_1, x_2) \in \mathcal{P}_i$ *in the reference configuration. The reader is referred to [6] where this model is introduced and analyzed.*

Also networks of Reissner-Mindlin plates can be considered in this framework see [6, 5]. Networks of thin shells, such as Naghdi-shells or Cosserat-shells do not seem to have been considered in the literature. Such networks are subject to further current investigations.

2.2 Existence and Uniqueness of Solutions

In this section existence and uniqueness of solutions of the variational equation (2) are considered. It is assumed that the elements of the matrices $A_i^{\alpha\beta}$, B_i^{β}, C_i are all in $L^\infty(\mathcal{P}_i)$. For a function $\Phi : \mathcal{P} \mapsto \mathbb{R}^m$ we denote by Φ_i the restriction of Φ to $\mathcal{P}_i$ and we set

$$\mathcal{H}^s(\mathcal{P}) = \{\Phi : \Phi_i \in \mathcal{H}^s(\mathcal{P}_i), \forall i \in \mathcal{I}\}$$

$$\|\Phi\|_{\mathcal{H}^s(\mathcal{P})} = \left(\sum_{i\in\mathcal{I}} \|\Phi_i\|^2_{\mathcal{H}^s(\mathcal{P}_i)}\right)^{1/2},$$

where $\mathcal{H}^s(\mathcal{P}_i)$ denotes the usual (vector) Sobolev space of order s on $\mathcal{P}_i$. Set $\mathcal{H} = \mathcal{H}^0(\mathcal{P})$ and define a closed subspace $\mathcal{V}$ of $\mathcal{H}^1(\mathcal{P})$ by

$$\mathcal{V} = \{\Phi \in \mathcal{H}^1(\mathcal{P}) |\ \Phi_i = 0 \text{ on } e \text{ when } \Gamma_{ie} \in \mathcal{D},\\ Q_{ie}\Phi_i = Q_{je}\Phi_j \text{ on } e \text{ when } \Gamma_{ie} = \Gamma_{je},\\ \Pi_{ie}\Phi_i = 0 \text{ on } e \text{ when } e \in \mathcal{N}^{\rm int},\ i \in \mathcal{I}\}.$$

The space $\mathcal{V}$ is densely and compactly embedded in $\mathcal{H}$. It is assumed that $a(\Phi, \Phi)$ is elliptic on $\mathcal{V}$: there are constants $k \geq 0$, $K > 0$, such that

$$a(\Phi, \Phi) + k\|\Phi\|^2_{\mathcal{H}} \geq K\|\Phi\|^2_{\mathcal{H}^1(\mathcal{P})}, \quad \forall \Phi \in \mathcal{V}. \tag{10}$$

Let $F = \{F_i\}_{i\in\mathcal{I}} \in \mathcal{H}$. It follows from standard variational theory and the Fredholm alternative that the variational equation (2) has a solution if and only if F is orthogonal in $\mathcal{H}$ to all solutions $W \in \mathcal{V}$ of $a(W, \Phi) = 0$, $\forall \Phi \in \mathcal{V}$. If it is known that $a(\Phi, \Phi) \geq 0$ for each $\Phi \in \mathcal{V}$, the last equation has only the trivial solution if, and only if, (10) holds with $k = 0$.

3 The Optimal Control Problem

We consider the following optimal control problem.

$$\begin{cases} \min\limits_{f\in\mathcal{U}_{ad}} \frac{1}{2}\int\limits_{\mathcal{P}} \|W - W_d\|^2 dx + \sum\limits_{i\in\mathcal{I}} \sum\limits_{e\in\mathcal{N}_i^{ext}} \frac{\kappa}{2} \int\limits_{\Gamma_{ie}} \|f_{ie}\|^2 d\Gamma, \quad \text{subject to} \\ -\frac{\partial}{\partial x_\alpha}\left[A_i^{\alpha\beta}\big(W_{i,\beta} + B_i^\beta W_i\big)\right] + (B_i^\alpha)^* A_i^{\alpha\beta}\big(W_{i,\beta} + B_i^\beta W_i\big) \\ \quad +C_i W_i = F_i \text{ in } \mathcal{P}_i, \\ W_i = 0 \text{ on } \Gamma_{ie} \text{ when } e \in \mathcal{D} \\ \nu_{ie}^\alpha A_i^{\alpha\beta}(W_{i,\beta} + B_i^\beta W_i) + \alpha_{ie} W_i = f_{ie} \text{ on } \Gamma_{ie} \text{ when } e \in \mathcal{N}^{\rm ext} \\ \Pi_{ie} W_i = 0 \text{ on } \Sigma_{ie},\ \forall i \in \mathcal{I}_e,\ e \in \mathcal{N}^{\rm int} \\ Q_{ie} W_i = Q_{je} W_j \text{ on } \Gamma_{ie} \text{ when } \Gamma_{ie} = \Gamma_{je},\ e \in \mathcal{N}^{\rm int} \\ \sum\limits_{i\in\mathcal{I}_e} (Q_{ie}^+)^* \nu_{ie}^\alpha A_i^{\alpha\beta}(W_{i,\beta} + B_i^\beta W_i) = 0 \text{ on } \Gamma_{ie} \text{ if } e \in \mathcal{N}^{\rm int}. \end{cases} \tag{11}$$

where

$$\mathcal{U} = \prod_{i\in\mathcal{I}} \prod_{e\in\mathcal{N}_i^{\rm ext}} L^2(\Gamma_{ie}) \tag{12}$$

and

$$\mathcal{U}_{ad} = \left\{f \in \mathcal{U} : f_{ie} \in U_{ie},\ i \in \mathcal{I},\ e \in \mathcal{N}_i^{\rm ext}\right\}, \tag{13}$$

where, in turn, the sets U_{ie} are all convex. Also notice that we added extra freedom on the external boundary, in order to allow for Robin-conditions. Instead of the strong form of this linear quadratic control constrained optimal control, we consider the weak formulation.

$$\begin{cases} \min\limits_{f\in\mathcal{U}_{ad}} \frac{1}{2}\int\limits_{\mathcal{P}} \|W - W_d\|^2 dx + \sum\limits_{i\in\mathcal{I}} \sum\limits_{e\in\mathcal{N}_i^{ext}} \frac{\kappa}{2} \int\limits_{\Gamma_{ie}} \|f_{ie}\|^2 d\Gamma, \quad \text{subject to} \\ a(W, \Phi) + b\big(W, \Phi\big) \\ \quad = \big(F, \Phi\big)_{\mathcal{H}} + \sum\limits_{i\in\mathcal{I}} \sum\limits_{e\in\mathcal{N}_i^{\rm ext}} \int_{\Gamma_{ie}} f_{ie} \cdot \Phi_i \, d\Gamma, \quad \forall \Phi \in \mathcal{V},\ 0 < t < T, \end{cases} \tag{14}$$

where

$$b(W,\Phi)=\sum_{i\in\mathcal{I}}\sum_{e\in\mathcal{N}_i^{\rm ext}}\int_{\Gamma_{ie}}\alpha_{ie}W_i\cdot\Phi_i\,d\Gamma. \tag{15}$$

We may simplify the notation even further by introducing the inner product on the control space $\mathcal{U}$:

$$\langle f,\Phi\rangle_{\mathcal{U}}=\sum_{i\in\mathcal{I}}\sum_{e\in\mathcal{N}_i^{\rm ext}}\int_{\Gamma_{ie}}f_{ie}\cdot\Phi_i d\Gamma$$

$$\begin{cases}\min\limits_{f\in\mathcal{U}_{ad}}\frac{1}{2}\|W-W_d\|^2+\frac{\kappa}{2}\|f\|_{\mathcal{U}}^2, \quad \text{subject to}\\ a(W,\Phi)+b\big(W,\Phi\big)=\langle f,\Phi\rangle+(F,\Phi)_{\mathcal{H}}, \quad \forall\Phi\in\mathcal{V}.\end{cases} \tag{16}$$

Existence, uniqueness of optimal controls and the validity of the following first order optimality condition follow by standard arguments.

$$\begin{cases}a(W,\Phi)+b(W,\Phi)=\langle f,\Phi\rangle+(F,\Phi)_{\mathcal{H}} \quad \forall\Phi\in\mathcal{V}\\ a(P,\Psi)+b(P,\Psi)=(W-W_d,\Psi), \quad \forall\Psi\in\mathcal{V}\\ \langle P+\kappa f,v-f\rangle\ge 0, \quad \forall v\in\mathcal{U}_{ad}\end{cases} \tag{17}$$

4 Domain Decomposition

Some preliminary material is required in order to properly formulate the subsystems in the decomposition. Let $\mathcal{H}_i$, $\mathcal{V}_i$, and $a_i(W_i,V_i)$ be the spaces associated with the bilinear form

$$a_i(W_i,\Phi_i)=\int_{\mathcal{P}_i}[A_i^{\alpha\beta}(W_{i,\beta}+B_i^{\beta}W_i)\cdot(\Phi_{i,\alpha}+B_i^{\alpha}\Phi_i)+C_iW_i\cdot\Phi_i]\,dx, \tag{18}$$

It is assumed that $a_i(\Phi_i,\Phi_i)\ge K_i\|\Phi_i\|^2_{\mathcal{H}^1(\mathcal{P}_i)},\forall\Phi_i\in\mathcal{V}_i$, for some constant $K_i>0$. We may then define a norm on $\mathcal{V}_i$ equivalent to the induced $\mathcal{H}^1(\mathcal{P}_i)$ by setting $\|\Phi_i\|_{\mathcal{V}_i}=\sqrt{a_i(\Phi_i,\Phi_i)}$. We identify the dual of $\mathcal{H}_i$ with $\mathcal{H}_i$ and denote by $\mathcal{V}_i^*$ the dual space of $\mathcal{V}_i$ with respect to $\mathcal{H}_i$. We define the continuous bilinear functional b_i on $\mathcal{V}_i$ by

$$b_i(W_i,\Phi_i)=\sum_{e\in\mathcal{N}_i^{\rm ext}}\int_{\Gamma_{ie}}\alpha_{ie}W_i\cdot\Phi_i\,d\Gamma+\sum_{e\in\mathcal{N}_i^{\rm int}}\int_{\Gamma_{ie}}\beta_eQ_{ie}W_i\cdot Q_{ie}\Phi_i\,d\Gamma. \tag{19}$$

where β_e is a positive constant independent of $i\in\mathcal{I}_e$. For each $i\in\mathcal{I}$ we consider the following local problems for functions W_i defined on $\mathcal{P}_i$:

$$\begin{aligned}&a_i(W_i,\Phi_i)+b_i(W_i,\Phi_i)\\&\quad=\big(F_i,\Phi_i\big)_{\mathcal{H}_i}+\sum_{e\in\mathcal{N}_i^{\rm ext}}\int_{\Gamma_{ie}}f_{ie}\cdot\Phi_i\,d\Gamma+\sum_{e\in\mathcal{N}_i^{\rm int}}\int_{\Gamma_{ie}}(g_{ie}+\lambda_{ie}^n)\cdot Q_{ie}\Phi_i\,d\Gamma,\\&\hspace{20em}\forall\Phi\in\mathcal{V}_i,\end{aligned} \tag{20}$$

where the inter-facial input λ_{ie}^n is to be specified below. We are going to consider the following local control-constrained optimal control problem.

$$\begin{cases} \min\limits_{f_i,g_i} J(f_i,g_i) := \frac{1}{2}\|W_i - W_{di}\|^2 + \frac{\kappa}{2} \sum\limits_{e\in\mathcal{N}_i^{\rm ext}} \int\limits_{\Gamma_{ie}} |f_{ie}|^2 d\Gamma \\ \qquad + \sum\limits_{e\in\mathcal{N}_i^{\rm int}} \frac{1}{2\gamma_e} \int\limits_{\Gamma_{ie}} |g_{ie}|^2 + |\gamma_e Q_{ie} W_i + \mu_{ie}^n|^2 d\Gamma \\ \qquad\qquad \text{subject to (20)},\ f_{ie} \in \mathcal{U}_{ad,i}, \end{cases} \tag{21}$$

where $g_{ie} \in L^2(\Gamma_{ie})$ serve as *virtual controls*. By standard arguments, we obtain the local optimality system:

$$\begin{cases} a_i(W_i,\Phi_i) + b_i(W_i,\Phi_i) = \big(F_i,\Phi_i\big)_{\mathcal{H}_i} + \sum\limits_{e\in\mathcal{N}_i^{\rm ext}} \int\limits_{\Gamma_{ie}} f_{ie}\cdot\Phi_i\, d\Gamma \\ \quad + \sum\limits_{e\in\mathcal{N}_i^{\rm int}} \int\limits_{\Gamma_{ie}} (\lambda_{ie}^n - \gamma_e Q_{ie} P_i)\cdot Q_{ie}\Phi_i\, d\Gamma, \quad \forall \Phi_i \in \mathcal{V}_i, \\ a_i(P_i,\Phi_i) + b_i(P_i,\Phi_i) = (W_i - W_{d,i},\Phi) \\ \quad + \sum\limits_{e\in\mathcal{N}_i^{\rm int}} \int_{\Gamma_{ie}} (\mu_{ie}^n + \gamma_e Q_{ie} W_i)\cdot Q_{ie}\Phi_i\, d\Gamma, \quad \forall \Phi \in \mathcal{V}_i, \\ \quad \sum\limits_{e\in\mathcal{N}_i^{\rm ext}} \int\limits_{\Gamma_{ie}} (\kappa f_{ie} + P_i)\cdot(\hat{f}_i - f_{ie}) d\Gamma \ge 0, \quad \forall f_i \in \mathcal{U}_{ad}\,. \end{cases} \tag{22}$$

We proceed to define update rules for $\lambda_{ie}^n, \mu_{ie}^n$ at the interfaces. To simplify the presentation, we introduce a 'scattering'-type mapping S_e for a given interior joint e:

$$S_e(u)_i := \frac{2}{d_e} \sum_{j\in\mathcal{I}_e} u_j - u_i, \quad i \in \mathcal{I}_e\,.$$

We obviously have $S_e^2 = Id$. We set

$$\begin{cases} \lambda_{ie}^{n+1} = S_e(2\beta_e Q_{\cdot e} W_\cdot^n + 2\gamma_e Q_{\cdot e} P_\cdot^n)_i - S_e(\lambda_\cdot^n)_i, \quad i \in \mathcal{I}_e\,, \\ \mu_{ie}^{n+1} = S_e(2\beta_e Q_{\cdot e} P_\cdot^n - 2\gamma_e Q_{\cdot e} W_\cdot^n)_i - S_e(\mu_\cdot^n)_i, \quad i \in \mathcal{I}_e\,. \end{cases} \tag{23}$$

If we assume convergence of the sequences $\lambda_{ie}^n, \mu_{ie}^n, W_i^n, P_i^n$, and if we use the properties of S_e in summing up the equations in (23) we obtain

$$\sum_i a_i(W_i,\Phi_i) = \sum_i \big(F_i,\Phi_i\big)_{\mathcal{H}_i} + \sum_i \sum_{e\in\mathcal{N}_i^{\rm ext}} \int_{\Gamma_{ie}} f_{ie}\cdot\Phi_i\, d\Gamma, \tag{24}$$

for all $\Phi = \Phi|_{\mathcal{P}_i} \in \mathcal{V}$, and similarly for P_i. Thus the limiting elements $W_i, P_i,\ i \in \mathcal{I}$ satisfy the global optimality system. Therefore, by the domain decomposition method above we have decoupled the global optimality system into local optimality systems, which are the necessary condition for local optimal control problems. In other words, we decouple the globally defined optimal control problem into local ones of similar structure.

5 Convergence

We introduce the errors

$$\begin{cases} \widetilde{W}_i^n = W_i^n - W_i|_{\mathcal{P}_i} \\ \widetilde{P}_i^n = P_i^n - P_i|_{\mathcal{P}_i} \\ \tilde{f}_{ie}^n = f_{ie}^n - f_{ie}, \end{cases} \tag{25}$$

where W_i^n, P_i^n and W_i, P_i solve the iterated system and the global one, respectively. By linearity, $\widetilde{W}_i^n, \widetilde{P}_i^n$ solve the systems

$$\begin{cases} a_i(\widetilde{W}_i^n, \Phi_i) + b_i(\widetilde{W}_i^n, \Phi_i) = \sum\limits_{e\in\mathcal{N}_i^{\text{ext}}} \int_{\Gamma_{ie}} \tilde{f}_{ie}^n \cdot \Phi_i \, d\Gamma \\ \quad + \sum\limits_{e\in\mathcal{N}_i^{\text{int}}} \int_{\Gamma_{ie}} (\tilde{\lambda}_{ie}^n - \gamma_e Q_{ie} \widetilde{P}_i^n) \cdot Q_{ie} \Phi_i \, d\Gamma, \forall \Phi \in \mathcal{V}_i, \\ a_i(\widetilde{P}_i^n, \Phi_i) + b_i(\widetilde{P}_i^n, \Phi_i) = (\widetilde{W}_i, \Phi_i) \\ \quad + \sum\limits_{e\in\mathcal{N}_i^{\text{int}}} \int_{\Gamma_{ie}} (\tilde{\mu_{ie}}^n + \gamma_e Q_{ie} \widetilde{W}_i^n) \cdot Q_{ie} \Phi_i \, d\Gamma, \forall \Phi \in \mathcal{V}_i, \end{cases} \tag{26}$$

and the variational inequality

$$\sum_{e\in\mathcal{N}_i^{\text{ext}}} \int_{\Gamma_{ie}} (\kappa f_{ie}^n + P_i^n) \cdot (\hat{f}_{ie} - f_{ie}^n) d\Gamma \geq 0, \quad \forall \hat{f}_{ie} \in \mathcal{U}_{ad,i}, \tag{27}$$

and a similar one for f_i. Upon choosing proper functions $\hat{f}_{ie}$ we obtain the inequality

$$\sum_{e\in\mathcal{N}_i^{\text{ext}}} \int_{\Gamma_{ie}} \widetilde{P}_i^n \cdot \tilde{f}_{ie}^n d\Gamma \leq -\kappa \sum_{e\in\mathcal{N}_i^{\text{ext}}} \int_{\Gamma_{ie}} |\tilde{f}_{ie}^n|^2 d\Gamma \,. \tag{28}$$

We now introduce the space $\mathcal{X}$ at the interfaces

$$\mathcal{X} = \prod_{i\in\mathcal{I}} \prod_{e\in\mathcal{N}_i^{\text{int}}} L^2(\Gamma_{ie}), \; X = (\lambda_{ie}, \mu_{ie}), \; i \in \mathcal{I}, e \in \mathcal{N}_i^{\text{int}}$$

together with the norm

$$\|X\|_{\mathcal{X}}^2 = \sum_{i\in\mathcal{I}} \sum_{e\in\mathcal{N}_i^{\text{int}}} \frac{1}{2\gamma_e} \int_{\Gamma_{ie}} |\tilde{\lambda}_{ie}|^2 + |\tilde{\mu}_{ie}|^2 d\Gamma \,.$$

The iteration map is now defined in the space $\mathcal{X}$ as follows:

$$\left\{ \begin{array}{ll} \mathcal{T}: & \mathcal{X} \to \mathcal{X} \\ \mathcal{T}X: & \{(S_e(2\beta_e Q_{\cdot e} W_{\cdot} + 2\gamma_e Q_{\cdot e} P_{\cdot})_i - S_e(\lambda_{\cdot e})_i); \\ & S_e((2\beta_e Q_{\cdot e} P_{\cdot} - 2\gamma_e Q_{\cdot e} W_{\cdot})_i - S_e(\mu_{\cdot e})_i) \end{array} \right\} . \tag{29}$$

We now consider the errors $\widetilde{X}^n$ and the norms of the iterates. Indeed, for the sake of simplicity, we assume that $\gamma_e, \beta_e, \alpha_e$ are independent of e. After considerable calculus, we arrive at

$$\begin{aligned}\|\mathcal{T}(\tilde{X})^n\|^2_{\mathcal{X}} = \|\tilde{X}^n\|^2_{\mathcal{X}} \quad & -\frac{2}{\gamma}\beta\sum_i\Big[a_i(\widetilde{P}_i^n,\widetilde{P}_i^n)+a_i(\widetilde{W}_i^n,\widetilde{W}_i^n)\Big]\\ & -\frac{2}{\gamma}\sum_{e\in\mathcal{N}_i^{\mathrm{ext}}}\int_{\Gamma_{ie}}|\widetilde{W}_i^n|^2+|\widetilde{P}_i^n|^2 d\Gamma\\ & +\frac{2}{\gamma}\sum_{e\in\mathcal{N}_i^{\mathrm{ext}}}\int_{\Gamma_{ie}}\Big[\beta\tilde{f}_{ie}^n\cdot\widetilde{W}_i^n+\gamma\tilde{f}_{ie}^n\cdot\widetilde{P}_i^n\Big]d\Gamma\\ & +\frac{2}{\gamma}\beta\sum_i(\widetilde{W}_i^n,\widetilde{P}_i^n)-2(\widetilde{W}_i^n,\widetilde{W}_i^n)\,.\end{aligned} \tag{30}$$

We distinguish two cases: $\beta = 0$ and $\beta > 0$. In the first case we obtain using (28) the inequality

$$\|\mathcal{T}(\tilde{X})_i^n\|^2_{\mathcal{X}} \le \|\tilde{X}\|^2_{\mathcal{X}} - 2\kappa\sum_{e\in\mathcal{N}_i^{\mathrm{ext}}}\int_{\Gamma_{ie}}|\tilde{f}_{ie}|^2 d\Gamma - 2\sum_i\|\widetilde{W}_i^n|^2_{\mathcal{H}_i}\,. \tag{31}$$

Iterating (31) to zero we obtain the following result.

Theorem 1. *Let the parameters in* (30)*be independent of e, and let in particular* $\beta_e = \beta = 0,\ \forall e \in \mathcal{N}^{int}$ *and* $\alpha_e = \alpha = 0,\ \forall e \in \mathcal{N}^{ext}$ *then*

$$\begin{cases} i.) & \{\tilde{X}^n\}_n \quad \text{is bounded}\\ ii.) & \widetilde{W}_i^n \to 0 \quad \text{strongly in } L^2(\mathcal{P}_i)\\ iii.) & \tilde{f}_{ie}^n \to 0 \quad \text{strongly in } L^2(\Gamma_{ie})\,.\end{cases} \tag{32}$$

While this result can be refined by exploiting the first statement further, it gives convergence in the L^2-sense, only. In order to obtain convergence in stronger norms also for the adjoint variable, we need to take positive Robin-boundary- and interface parameters α, β into account. We thus estimate (30) in that situation as follows.

$$\begin{aligned}\|\mathcal{T}\tilde{X}^n\|^2_{\mathcal{X}} \le \|\tilde{X}^n\|^2_{\mathcal{X}} & \\ -\frac{2\beta}{\gamma}\sum_i\Big\{ & a_i(\widetilde{P}_i^n,\widetilde{P}_i^n)+a(\widetilde{W}_i^n,\widetilde{W}_i^n)\\ & +(\frac{\gamma}{\beta}-\frac{1}{2\epsilon})\|\widetilde{W}_i^n\|^2-\frac{\epsilon}{2}\|\widetilde{P}_i^n\|^2\Big\}\end{aligned} \tag{33}$$

$$-\frac{b}{\gamma}\sum_{e\in\mathcal{N}_i^{\text{ext}}}(2\alpha-\frac{1}{\epsilon})\int\limits_{\Gamma_{ie}}|\widetilde{W}_i^n|^2 d\Gamma-\frac{2\alpha\beta}{\gamma}\sum_{e\in\mathcal{N}_i^{\text{ext}}}|\widetilde{P}_i^n|^2 d\Gamma$$
$$-\frac{\beta}{\gamma}\sum_{e\in\mathcal{N}_i^{\text{ext}}}(\frac{2\kappa\gamma}{\beta}-\epsilon)\int\limits_{e\in\mathcal{N}_i^{\text{ext}}}|\tilde{f}_{ie}^n|^2 d\Gamma\,.$$

Theorem 2. *Let the parameters $\alpha_e=\alpha,\beta_e=\beta,\gamma_e=\gamma$ with $\alpha,\beta,\gamma>0$ such that $\frac{\gamma}{\beta}$ is sufficiently large. Then the iterates in* (33) *satisfy*

$$\begin{cases} i.) \quad a_i(\widetilde{P}_i^n,\widetilde{P}_i^n)\to 0 \quad \forall i\\ ii.)\quad a_i(\widetilde{W}_i^n,\widetilde{W}_i^n)\to 0 \quad \forall i\\ iii.)\ \widetilde{P}_i^n|_{\Gamma_{ie}}\to 0 \quad \text{in } L^2(\Gamma_{ie}),\ i\in\mathcal{N}^{ext}\\ iv.)\ \tilde{f}_{ie}^n\to 0 \quad \text{in } L^2(\Gamma_{ie}),\ i\in\mathcal{N}^{ext}\,. \end{cases} \tag{34}$$

6 A Posteriori Error Estimates

We are going to derive a posteriori error estimates with respect to the domain iteration, similar to those developed in [5] for unconstrained problems and single domains, as well as for time-dependent problems and time-and-space domain decompositions. The a posteriori error estimates derived in this section refer to the transmission conditions across multiple joints, only. A posteriori error estimates for problems without control and serial in-plane interfaces have first been described by [7]. To keep matters simple, we consider $\alpha_e=\beta_e=0$ and $\gamma_e=\gamma$. We consider the following error measure

$$\sum_i\Big\{a_i(\widetilde{W}_i^{n+1},v_i)+a_i(\widetilde{W}_i^n,y_i)+a_i(\widetilde{P}_i^{n+1},u_i)+a_i(\widetilde{P}_i^n,z_i)\Big\}=$$
$$\sum_{e\in\mathcal{N}^{\text{ext}}}\sum_{i\in\mathcal{I}_e}\int\limits_{\Gamma_{ie}}[\tilde{f}_{ie}^{n+1}\cdot v_i d+\tilde{f}_i^n\cdot y_i d]d\Gamma-\sum_i[(\widetilde{W}_i^{n+1},u_i)+(\widetilde{W}_i^n,z_i)]$$
$$+\sum_{e\in\mathcal{N}^{\text{int}}}\sum_{i\in\mathcal{I}_e}\int\limits_{\Gamma_{ie}}\Big[\gamma Q_{ie}\widetilde{P}_i^n-\tilde{\lambda}_{ie}^n\Big]\cdot[S_e(Q_{\cdot e}v_\cdot)_i-Q_{ie}y_i]\,d\Gamma$$
$$+\sum_{e\in\mathcal{N}^{\text{int}}}\sum_{i\in\mathcal{I}_e}\int\limits_{\Gamma_{ie}}\Big[\gamma Q_{ie}\widetilde{W}_i^n+\tilde{\mu}_{ie}^n\Big]\cdot[Q_{ie}z_i-S_e(Q_{\cdot e}u_\cdot)_i]\,d\Gamma$$
$$+\sum_{e\in\mathcal{N}^{\text{int}}}\sum_{i\in\mathcal{I}_e}\int\limits_{\Gamma_{ie}}\Big[S_e(\gamma Q_{\cdot e}\widetilde{P}_\cdot^n)_i-\gamma Q_{\cdot e}\widetilde{P}_i^{n+1})Q_{ie}v_i\Big]\,d\Gamma$$
$$+\sum_{e\in\mathcal{N}^{\text{int}}}\sum_{i\in\mathcal{I}_e}\int\limits_{\Gamma_{ie}}\Big[\gamma Q_{\cdot e}\widetilde{W}_i^{n+1}-S_e(\gamma Q_{\cdot e}\widetilde{W}_\cdot^n)_i\Big]\cdot Q_{ie}u_i d\Gamma\,.$$

We first choose $v_i=\widetilde{W}_i^{n+1},y_i=\widetilde{W}_i^n,u_i=\widetilde{P}_i^{n+1},z_i=\widetilde{P}_i^n$ and then $v_i=\widetilde{P}_i^{n+1},y_i=\widetilde{P}_i^n,u_i=-\widetilde{W}_i^{n+1},z_i=-\widetilde{W}_i^n$. Then, after substantial calculations and estimations we obtain the following error estimate, where the details may be found in a forthcoming publication.

Theorem 3. *Let $\beta_e = \alpha_e = 0$ for all e. There exists a positive number $C(\kappa, \gamma, \Omega)$ such that the total error satisfies the a posteriori error estimate*

$$\sum_i \left\{ \|\widetilde{W}_i^{n+1}\|_{\mathcal{V}_i} + \|\widetilde{W}_i^{n}\|_{\mathcal{V}_i} + \|\widetilde{P}_i^{n+1}\|_{\mathcal{V}_i} + \|\widetilde{P}_i^{n}\|_{\mathcal{V}_i} \right\}$$
$$\leq C(\kappa, \gamma, \Omega) \sum_{e \in \mathcal{N}_i^{int}} \sum_{i \in \mathcal{I}_e} \big\{ \|S_e(Q_{\cdot e} W_i^{n+1})_i - Q_{ie} W_i^n\|_{L^2(\Gamma_{ie})}$$
$$+ \|S_e(Q_{\cdot e} P_i^{n+1})_i - Q_{ie} P_i^n\|_{L^2(\Gamma_{ie})} \big\} \ .$$

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