Generalization of Lions' Nonoverlapping Domain Decomposition Method for Contact Problems

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Summary. We propose a Robin domain decomposition algorithm to approximate a frictionless Signorini contact problem between two elastic bodies. The present method is a generalization to variational inequality of Lions' nonoverlapping domain decomposition method. The Robin algorithm is a parallel one, in which we have to solve a contact problem on each domain.

1 Introduction

Contact problems take an important place in computational structural mechanics (see [8, 10, 13] and the references therein). Many numerical procedures have been proposed in the literature. They are based on standard numerical solvers for the solution of global problem in combination with a special implementation of the non-linear contact conditions (see [5, 4]).
The numerical treatment of such nonclassical contact problems leads to very large (due to the large ratio of degrees of freedom concerned by contact conditions) and ill-conditioned systems. Domain decomposition methods are good alternative to overcome this difficulties (see [2, 3, 15, 11, 14]).

The aim of this paper is to present and study an efficient iterative schemes based on domain decomposition techniques for a nonlinear problem modeling the frictionless contact of linear elastic bodies. The present method is a generalization to variational inequality of the method described in [17, 9]. It can be interpreted as a nonlinear Robin-Robin type preconditioner.

2 Weak Formulation of the Continuous Problem

Let us consider two elastic bodies, occupying two bounded domains Ω^α, $\alpha = 1, 2$, of the space $\mathbb{R}^2$. The boundary $\Gamma^\alpha = \partial\Omega^\alpha$ is assumed piecewise continuous, and composed of three complementary parts Γ_u^α, Γ_ℓ^α and Γ_c^α. The body $\overline{\Omega}^\alpha$ is fixed on the set Γ_u^α of positive measure. It is subject to surface traction forces $\Phi^\alpha \in (L^2(\Gamma_\ell^\alpha))^2$ and the body forces are denoted by $f^\alpha \in (L^2(\Omega^\alpha))^2$. In the initial configuration, both bodies have a common contact portion $\Gamma_c = \Gamma_c^1 = \Gamma_c^2$. We seek the displacement field $u = (u^1, u^2)$ (where the notation u^α stands for $u|_{\Omega^\alpha}$) and the stress tensor field $\sigma = (\sigma(u^1), \sigma(u^2))$ satisfying the following equations and conditions (1)-(3) for $\alpha = 1, 2$:

$$\begin{cases} div\,\sigma(u^\alpha) + f^\alpha &= 0 \text{ in } \Omega^\alpha, \\ \sigma(u^\alpha)n^\alpha - \Phi^\alpha &= 0 \text{ on } \Gamma_\ell^\alpha, \\ u^\alpha &= 0 \text{ on } \Gamma_u^\alpha. \end{cases} \tag{1}$$

The symbol div denotes the divergence operator of a tensor function and is defined as

$$div\,\sigma = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)_i.$$

The summation convention of repeated indices is adopted. The elastic constitutive law, is given by Hooke's law for homogeneous and isotropic solid:

$$\sigma(u^\alpha) = A^\alpha(x)\varepsilon(u^\alpha), \tag{2}$$

where $A^\alpha(x) = (a_{ijkh}^\alpha(x))_{1\le i,j,k,h\le 2} \in (L^\infty(\Omega^\alpha))^{16}$ is a fourth-order tensor satisfying the usual symmetry and ellipticity conditions in elasticity. The linearized strain tensor $\varepsilon(u^\alpha)$ is given by

$$\varepsilon(u^\alpha) = \frac{1}{2}\Big(\nabla u^\alpha + (\nabla u^\alpha)^T\Big).$$

We will use the usual notations for the normal and tangential components of displacement and stress vector on the contact zone Γ_c:

$$u_N^\alpha = u_i^\alpha n_i^\alpha, \quad [u_N] = u^1 n^1 + u^2 n^2,$$
$$\sigma_N^\alpha = \sigma_{ij}(u^\alpha) n_i^\alpha n_j^\alpha, \quad \sigma_T^\alpha = \sigma_{ij}(u^\alpha) n_j^\alpha - \sigma_N^\alpha n_i^\alpha,$$

where n^α is the unitary normal exterior to Ω^α.
The unilateral contact law on the interface Γ_c is given by:

$$[u_N] \le 0, \quad \sigma_N \le 0, \quad \sigma_N \cdot [u_N] = 0. \tag{3}$$

The contact is supposed frictionless so on Γ_c we get:

$$\sigma_T = 0.$$

In order to give the variational formulation corresponding to the problem (1)-(3), let us introduce the following spaces

$$V^\alpha = \left\{ v^\alpha \in (H^1(\Omega^\alpha))^2,\ v = 0 \quad \text{on} \quad \Gamma_u^\alpha \right\}, \quad \text{and } V = V^1 \times V^2$$

equipped with the product norm $\|\cdot\|_V = \left(\sum_{\alpha=1}^2 \|\cdot\|^2_{(H^1(\Omega^\alpha))^2}\right)^{\frac{1}{2}}$,

$$\mathcal{H}^{\frac{1}{2}}(\Gamma_c) = \Big\{\varphi \in (L^2(\Gamma_c))^2;\ \exists v \in V^\alpha;\ \gamma v_{|\Gamma_c} = \varphi\Big\},$$
$$H^{\frac{1}{2}}(\Gamma_c) = \Big\{\varphi \in L^2(\Gamma_c);\ \exists v \in H^1(\Omega^\alpha);\ \gamma v_{|\Gamma_c} = \varphi\Big\},$$

where γ is the usual trace operator. Now, we denote by K the following non-empty closed convex subset of V:

$$K = \left\{ v = (v^1, v^2) \in V \ , \ [v_N] \leq 0 \text{ on } \Gamma_c \right\}.$$

The variational formulation of problem (1)-(3) is

$$\left\{ \begin{array}{l} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \geq L(v-u), \quad \forall v \in K, \end{array} \right. \tag{4}$$

where

$$a(u,v) = a^1(u,v) + a^2(u,v),$$

$$a^\alpha(u,v) = \int_{\Omega^\alpha} A^\alpha(x)\varepsilon(u^\alpha) \cdot \varepsilon(v^\alpha) dx, \tag{5}$$

and

$$L(v) = \sum_{\alpha=1}^{2} \int_{\Omega^\alpha} f^\alpha \cdot v^\alpha \, dx + \int_{\Gamma_\ell^\alpha} \Phi^\alpha \cdot v^\alpha d\sigma.$$

There exists a unique solution u to problem (4) (see [7, 13]).

3 Multibody Formulation and Algorithm

In the following, we will use some lift operators which allow us to build specific function from their values on Γ_c. For $\alpha = 1, 2$, let

$$\begin{aligned} R^\alpha : \mathcal{H}^{\frac{1}{2}}(\Gamma_c) &\longrightarrow V^\alpha \\ \varphi &\longrightarrow R^\alpha \varphi = v^\alpha, \end{aligned} \tag{6}$$

where v^α is the solution of

$$\left\{ \begin{array}{ll} a^\alpha(v^\alpha, w) = 0 & \forall w \in V^\alpha \text{ with } w = 0 \text{ on } \Gamma_c \\ v^\alpha = \varphi & \text{on } \Gamma_c. \end{array} \right.$$

The two-body contact problem (4) is approximated by an iterative procedure involving a contact problem for each body Ω^α with a rigid foundation described by a given initial gap g^α.
Given $g_0^\alpha \in \Gamma_c$, $\alpha = 1, 2$, for $m \geq 1$, we build the sequence of functions $(u_m^1)_{m \geq 0}$ and $(u_m^2)_{m \geq 0}$, by solving in parallel the following problems:

Step 1:
1. Solve the contact problem

$$\begin{aligned} -div(\sigma(u_m^1)) &= f^1 \quad \text{in } \Omega^1, \\ \sigma(u_m^1)n^1 &= \Phi^1 \quad \text{on } \Gamma_\ell^1, \\ u_m^1 &= 0 \quad \text{on } \Gamma_u^1, \\ \sigma_{T,m}^1 &= 0 \quad \text{on } \Gamma_c, \\ u_m^1 n^1 &\leq g_{m-1}^1 \quad \text{on } \Gamma_c, \end{aligned} \tag{7}$$

$$\begin{aligned}\sigma^1_{N,m} &\leq 0 \quad \text{on } \Gamma_c,\\ \sigma^1_{N,m}(u^1_m n^1 - g^1_{m-1}) &= 0 \quad \text{on } \Gamma_c,\end{aligned}$$

with initial gap $g^1_{m-1} = \alpha_{S_1}(\sigma^2_{N,m-1} - \sigma^1_{N,m-1}) - u^2_{m-1}n^2$.
2. Solve the contact problem

$$\begin{aligned}-div(\sigma(u^2_m)) &= f^2 \quad \text{in } \Omega^2,\\ \sigma(u^2_m)n^2 &= \Phi^2 \quad \text{on } \Gamma^2_\ell,\\ u^2_m &= 0 \quad \text{on } \Gamma^2_u,\\ \sigma^2_{T,m} &= 0 \quad \text{on } \Gamma_c,\\ u^2_m n^2 &\leq g^2_{m-1} \quad \text{on } \Gamma_c,\\ \sigma^2_{N,m} &\leq 0 \quad \text{on } \Gamma_c,\\ \sigma^2_{N,m}(u^2_m n^2 - g^2_{m-1}) &= 0 \quad \text{on } \Gamma_c,\end{aligned} \tag{8}$$

with initial gap $g^2_{m-1} = \alpha_{S_2}(\sigma^1_{N,m-1} - \sigma^2_{N,m-1}) - u^1_{m-1}n^1$.
Step 2:
Relaxation

$$g^1_m = (1-\delta_m)g^1_{m-1} + \delta_m(\alpha_{S_2}(\sigma^1_{N,m} - \sigma^2_{N,m}) - u^2_m n^2) \ \text{ on } \Gamma_c, \tag{9}$$

$$g^2_m = (1-\delta_m)g^2_{m-1} + \delta_m(\alpha_{S_1}(\sigma^2_{N,m} - \sigma^1_{N,m}) - u^1_m n^1) \ \text{ on } \Gamma_c. \tag{10}$$

A key point in this algorithm is the choice of α_{S_i}, $i = 1, 2$:
- α_{S_i} is non-negative constant. It is the simplest choice but leads to an h-independent algorithm which is very sensible to boundary conditions.
- α_{S_i} is the Steklov-Poincaré operator defined on the interface Γ^α_c of Ω^α as introduced in [1]. This operator is not practical if the domains Ω^α are too large, but it has interesting features. Mainly, it can be defined for any geometry and for any elliptic operator, including three-dimensional anisotropic heterogeneous elasticity, and it is coercive positive selfadjoint operator. In practice, this choice consists to resolve two auxiliary problems before step 2. These auxiliary problems are written by:
$m \geq 0$, $\alpha = 1, 2$

$$\begin{aligned}-div(\sigma(w^\alpha_m)) &= 0 \quad \text{in } \Omega^\alpha,\\ \sigma(w^\alpha_m)n^\alpha &= 0 \quad \text{on } \Gamma^\alpha_\ell,\\ w^\alpha_m &= 0 \quad \text{on } \Gamma^\alpha_u,\\ \sigma(w^\alpha_m)n^\alpha &= \pm(\sigma(u^1_m)n^1 - \sigma(u^2_m)n^2) \text{ on } \Gamma_c.\end{aligned} \tag{11}$$

So the variational formulation of our algorithm takes the following form:
Given g^α_0, $\alpha = 1, 2, \in \mathcal{H}^{\frac{1}{2}}(\Gamma_c)$, for $m \geq 1$, we build the sequence of functions $(u^1_m)_{m\geq 0} \in V^1$ and $(u^2_m)_{m\geq 0} \in V^2$ by solving the following problems:

1^{st} step:

$$\begin{aligned}&\text{Find } u^\alpha_m \in V^\alpha_-(g^\alpha_{m-1}),\\ &a^\alpha(u^\alpha_m, v^\alpha - u^\alpha_m) \geq (f^\alpha, w^\alpha - u^\alpha_m)\, \forall v^\alpha \in V^\alpha_-(g^\alpha_{m-1})\end{aligned} \tag{12}$$

where

$$V_-^\alpha(\varphi) = \{v \in V^\alpha / \; vn^\alpha \leq -\varphi \quad \text{on} \quad \Gamma_c^\alpha\}\,.$$

2^{nd} step:

$$\begin{cases} \text{Find } w_m^1 \in V^1, \\ a^1(w_m^1, v) = -a^2(u_m^2, R^2\gamma(v)) + (f^2, R^2\gamma(v)) - a^1(u_m^1, v) + (f^1, v)\,\forall v \in V^1. \\ \\ \text{Find } w_m^2 \in V^2, \\ a^2(w_m^2, v) = a^1(u_m^1, R^1\gamma(v)) - (f^1, R^1\gamma(v)) + a^2(u_m^2, v) - (f^2, v)\,\forall v \in V^2. \end{cases} \tag{13}$$

3^{th} step:

$$\begin{cases} g_m^1 = (1-\delta_m) g_{m-1}^1 + \delta_m (w_m^2 n^2 - u_m^2 n^2) \;\text{ on } \Gamma_c, \\ \\ g_m^2 = (1-\delta_m) g_{m-1}^2 + \delta_m (w_m^1 n^1 - u_m^1 n^1) \;\text{ on } \Gamma_c. \end{cases} \tag{14}$$

We refer to [12] for the convergence results of the continuous algorithm (12)-(14) and its finite elements approximation.

Remark 1. Another choice of α_{S_i}, for $i = 1, 2$, is to create an artificial small "dream" domain having Γ_c as one of its faces on which we define the Steklov-Poincaré operator (see [16, 18]).

4 Numerical Experiments

In this section we describe some numerical results obtained with algorithm (12)-(14) for various relaxation parameter δ and various degrees of freedom $n = n_1 + n_2$ (d.o.f in $\Omega^1 \cup \Omega^2$) and m (d.o.f. on Γ_c). The computation is based on the iterative method of successive approximations. Each iterative step requires to solve two quadratics programming problems constrained by simple bounds. Our implementation uses recently developed algorithm of quadratic programming with proportioning and gradient projections [6].

The computation efficiency shall be assessed by

$$IT_{outer}/IT_{inner},$$

where IT_{outer} (resp. IT_{inner}) denotes the number of outer iterations (resp. the total number of conjugate gradient steps i.e the number of matrix-vector multiplications by Hessians).

The numerical implementations are performed in Scilab 2.7 on Pentium 4, 2.0 GHz with 256 MB RAM. We set $tol = 10^{-8}$ and we break down iterations, if their number is greater than eight hundred. For all experiments to be described below, the stopping criterion of Algorithm (12)-(14) is

$$\frac{\|g_m^1 - g_{m-1}^1\|}{\|g_m^1\|} + \frac{\|g_m^2 - g_{m-1}^2\|}{\|g_m^2\|} \leq tol,$$

where $||\cdot||$ denotes the Euclidean norm. The precisions in the inner iterations are adaptively adjusted by the precision achieved in the outer loop.

Let us consider the plane elastic bodies

$$\Omega^1 = (0,3) \times (1,2) \quad \text{and} \quad \Omega^2 = (0,3) \times (0,1)$$

made of an isotropic, homogeneous material characterized by Young's modulus $E_\alpha = 2.1\ 10^{11}$ and Poisson's ratio $\nu_\alpha = 0.277$. The decomposition of Γ^1 and Γ^2 read as:

$$\Gamma_u^1 = \{0\} \times (1,2),\ \Gamma_c^1 = (0,3) \times \{1\},\ \Gamma_l^1 = \Gamma^1 \setminus \overline{\Gamma_u^1 \cup \Gamma_c^1},$$

$$\Gamma_u^2 = \{0\} \times (0,1),\ \Gamma_c^2 = (0,3) \times \{1\},\ \Gamma_l^2 = \Gamma^2 \setminus \overline{\Gamma_u^2 \cup \Gamma_c^2}.$$

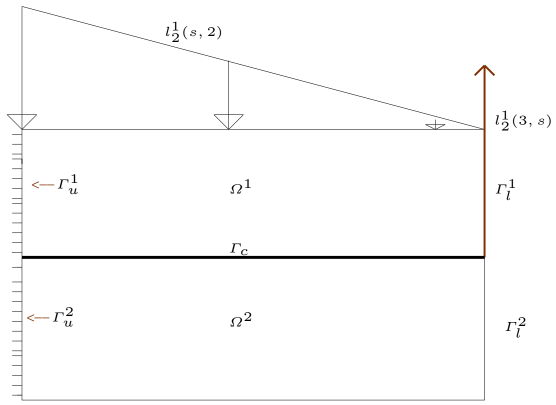

Fig. 1. Setting of the problem

The volume forces vanish for both bodies. The non-vanishing surface traction $\ell^1 = (l_1^1, l_2^1)$ (respectively, $\ell^2 = (l_1^2, l_2^2)$) act on Γ_l^1 (respectively, on Γ_l^2):

$$l_1^1(s,2) = 0, \quad l_2^1(s,2) = -3\,10^6 - 1\,10^6\, s, \quad s \in (0,3),$$
$$l_1^1(3,s) = 0, \quad l_2^1(3,s) = 2\,10^6, \quad s \in (1,2),$$

$$l_1^2(s,0) = 0, \quad l_2^2(s,0) = 0, \quad s \in (0,3),$$
$$l_1^2(3,s) = 0, \quad l_2^2(3,s) = 0, \quad s \in (0,1).$$

The Table 1 gives convergence of the algorithm (12)-(14) for different values of the relaxation parameter δ and various degrees of freedom (n and m). The results obtained show that the number of outer iterations (for an optimal value of δ=0.95) does not depend on the degrees of freedom n and m.

Table 1. Convergence of the algorithm

n/m	$\delta = 0.1$	$\delta = 0.5$	$\delta = 0.7$	$\delta = 0.95$	$\delta = 1$
12/3	287/903	76/243	62/208	47/160	–
36/6	285/899	79/272	66/237	49/179	–
288/16	270/878	74/282	79/295	45/188	–
816/24	296/957	92/332	93/340	47/204	–

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