Scalable FETI Algorithms for Frictionless Contact Problems

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Summary. We review our FETI based domain decomposition algorithms for the solution of 2D and 3D frictionless contact problems of elasticity and related theoretical results. We consider both cases of restrained and unrestrained bodies. The scalability of the presented algorithms is demonstrated on the solution of 2D and 3D benchmarks.

1 Introduction

The Finite Element Tearing and Interconnecting (FETI) method was originally proposed by Farhat and Roux (see [15]) as a parallel solver for problems described by elliptic partial differential equations. The computational domain is decomposed (teared) into non-overlapping subdomains that are "glued" by Lagrange multipliers, so that, after eliminating the primal variables, the original problem is reduced to a small, relatively well-conditioned, possibly equality constrained quadratic programming problem that is solved iteratively. The time that is necessary for both the elimination and iterations can be reduced nearly proportionally to the number of the subdomains, so that the algorithm enjoys parallel scalability. Since then, many preconditioning methods were developed which guarantee also numerical scalability of the FETI methods (see, e.g., [18]). The equality constraints can be avoided by using the Dual-Primal FETI method (FETI–DP) introduced by Farhat et al., see [13]. The continuity of the primal solution at crosspoints is implemented directly into the formulation of the primal problem by considering one degree of freedom per variable at each crosspoint. Across the rest of the subdomain interfaces, the continuity of the primal solution is once again enforced by Lagrange multipliers. After eliminating the primal variables, the problem is again reduced to a small, unconstrained, relatively well conditioned, strictly convex quadratic programming problem that is solved iteratively. An attractive feature of FETI–DP is that the local problems are nonsingular. Moreover, the conditioning of the resulting quadratic programming problem may be further improved by preconditioning, see [17], and the method performs better than the original FETI method on the fourth order problems.

Though the FETI and FETI-DP domain decomposition methods were originally developed for solving efficiently large-scale linear systems of equations arising from the discretization of the problems defined on a single domain Ω, it was soon observed that they can be even more efficient for the solution of multidomain contact problems, see e.g. [12, 5], and [1]. The reason is that the duality in this case not only reduces the original discretized problem to a smaller and better conditioned problem, but it also transforms the more general inequalities describing non-penetration into the bound constraints that can be treated much more efficiently. Moreover, since the FETI method treats naturally such subdomains, this approach is well suited for the solution of semicoercive contact problems with "floating" subdomains. These observations were soon confirmed by numerical experiments ([12, 5], and [1]). Recently, using new results in development of quadratic programming (see [11, 4]), the experimental evidence was supported by theory (see [3, 6, 9, 10]). There are also references to some other development of scalable algorithms for contact problems. See also [16] or the paper by Krause in this proceedings.

In this paper, we review our work related to the development of scalable algorithms for the solution of multibody contact problems by FETI–DP based methods with a special stress on the solution of 3D problems. For the sake of simplicity, we consider only the frictionless problems of linear elasticity with the linearized, possibly non-matching non-interpenetration conditions implemented by mortars, but the results may be exploited also for the solution of the problems with friction or large deformations with more sophisticated implementation of the kinematic constraints, see e.g. [8].

2 FETI and Contact Problems

Assuming that the bodies are assembled from the subdomains $\Omega^{(s)}$, the equilibrium of the system may be described as a solution $\mathbf{u}$ of the problem

$$\min j(\mathbf{v}) \quad \text{subject to} \quad \sum_{s=1}^{N_s} \mathsf{B}_I^{(s)}\mathbf{v}^{(s)} \le \mathbf{g}_I \quad \text{and} \quad \sum_{s=1}^{N_s} \mathsf{B}_E^{(s)}\mathbf{v}^{(s)} = \mathbf{o}, \tag{1}$$

where $j(\mathbf{v})$ is the energy functional defined by

$$j(\mathbf{v}) = \sum_{s=1}^{N_s} \frac{1}{2}\mathbf{v}^{(s)T}\mathsf{K}^{(s)}\mathbf{v}^{(s)} - \mathbf{v}^{(s)T}\mathbf{f}^{(s)},$$

$\mathbf{v}^{(s)}$ and $\mathbf{f}^{(s)}$ denote the admissible subdomain displacements and the subdomain vector of prescribed forces, $\mathsf{K}^{(s)}$ is the subdomain stiffness matrix, $\mathsf{B}^{(s)}$ is a block of the matrix $\mathsf{B} = \left[\mathsf{B}_I^T, \mathsf{B}_E^T\right]^T$ that corresponds to $\Omega^{(s)}$, and $\mathbf{g}_I$ is a vector collecting the gaps between the bodies in the reference configuration. The matrix B_I and the vector $\mathbf{g}_I$ arise from the nodal or mortar description of non-penetration conditions, while B_E describes the "gluing" of the subdomains into the bodies.

To simplify the presentation of basic ideas, we can describe the equilibrium in terms of the global stiffness matrix K_g, the vector of global displacements $\mathbf{u}_g$, and the vector of global loads $\mathbf{f}_g$. In the original FETI methods, FETI I and FETI II, we have

$$\mathsf{K}_g = \mathrm{diag}(\mathsf{K}^{(1)}, \ldots, \mathsf{K}^{(\mathrm{N_s})}), \quad \mathbf{u}_{\mathrm{g}} = \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N_s)} \end{bmatrix}, \quad \text{and} \quad \mathbf{f}_{\mathrm{g}} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N_s)} \end{bmatrix},$$

where $\mathsf{K}^{(s)}$ is a positive definite or positive semidefinite matrix.

A distinctive feature of the FETI-DP method is that the continuity of the components of the displacement field at some "corner" interface nodes is not enforced by the Lagrange multipliers, but is achieved by defining the corner unknowns only at the global level, while defining all other displacement unknowns at the subdomain level. If the subscripts c and r are chosen to designate all the degrees of freedom that correspond to the corners and remainders, respectively, then the subdomain and global stiffness matrices have the form

$$\mathsf{K}^{(s)} = \begin{bmatrix} \mathsf{K}^{(s)}_{rr} & \mathsf{K}^{(s)}_{rc} \\ \mathsf{K}^{(s)}_{cr} & \mathsf{K}^{(s)}_{cc} \end{bmatrix} \quad \text{and} \quad \mathsf{K}_{\mathrm{g}} = \begin{bmatrix} \mathsf{K}^{g}_{rr} & \mathsf{K}^{g}_{rc} \\ \mathsf{K}^{g}_{cr} & \mathsf{K}^{g}_{cc} \end{bmatrix}, \quad \mathsf{K}^{\mathrm{g}}_{\mathrm{rr}} = \mathrm{diag}(\mathsf{K}^{(1)}_{\mathrm{rr}}, \ldots, \mathsf{K}^{(\mathrm{N_s})}_{\mathrm{rr}}),$$

where $\mathsf{K}^{g}_{rr} = \mathrm{diag}(\mathsf{K}^{(1)}_{\mathrm{rr}}, \ldots, \mathsf{K}^{(\mathrm{N_s})}_{\mathrm{rr}})$ is nonsingular and K^{g}_{cc} is a positive definite or semidefinite small matrix.

Whichever variant of the domain decomposition we use, the energy function reads

$$j(\mathbf{v}_g) = \frac{1}{2}\mathbf{v}_g^T \mathsf{K}_g \mathbf{v}_g - \mathbf{f}_g^T \mathbf{v}_g$$

and the vector of global displacements $\mathbf{u}_g$ solves

$$\min j(\mathbf{v}_g) \quad \text{subject to} \quad \mathsf{B}_I \mathbf{v}_g \le \mathbf{g}_I \quad \text{and} \quad \mathsf{B}_E \mathbf{v}_g = \mathbf{o}. \tag{2}$$

Alternatively, the global equilibrium my be described by the Karush-Kuhn-Tucker conditions (e.g. [2])

$$\mathsf{K}_g \mathbf{u}_g = \mathbf{f}_g - \mathsf{B}^T \boldsymbol{\lambda}, \quad \boldsymbol{\lambda}_I \ge \mathbf{o}, \quad \boldsymbol{\lambda}_I^T (\mathsf{B}_I \mathbf{u} - \mathbf{g}_I) = \mathbf{o}, \tag{3}$$

where $\mathbf{g} = \left[\mathbf{g}_E^T, \mathbf{o}^T\right]^T$, and $\boldsymbol{\lambda} = \left[\boldsymbol{\lambda}_I^T, \boldsymbol{\lambda}_E^T\right]^T$ denotes the vector of Lagrange multipliers which may be interpreted as the reaction forces. The problem (3) differs from the linear problem by the non-negativity constraint on the components of reaction forces $\boldsymbol{\lambda}_I$ and by the complementarity condition.

We can use the left equation of (3) and the sparsity pattern of K_g to eliminate the displacements. We shall get the problem to find

$$\max \Theta(\boldsymbol{\lambda}) \quad \text{s.t.} \quad \boldsymbol{\lambda}_I \ge \mathbf{o} \quad \text{and} \quad \mathsf{R}^T (\mathbf{f}_g - \mathsf{B}^T \boldsymbol{\lambda}) = \mathbf{o}, \tag{4}$$

where

$$\Theta(\boldsymbol{\lambda}) = -\frac{1}{2}\lambda^T \mathsf{B}\mathsf{K}_g^{\dagger}\mathsf{B}^T \boldsymbol{\lambda} + \boldsymbol{\lambda}^T (\mathsf{B}\mathsf{K}_g^{\dagger}\mathbf{f}_g - \mathbf{g}) - \frac{1}{2}\mathbf{f}_g \mathsf{K}_g^{\dagger}\mathbf{f}_g, \tag{5}$$

$\mathsf{K}_g^{\dagger}$ denotes a generalized inverse that satisfies $\mathsf{K}_g \mathsf{K}_g^{\dagger} \mathsf{K}_g = \mathsf{K}_g$, and R denotes the full rank matrix whose columns span the kernel of K_g. Recalling the FETI notation

$$\mathsf{F} = \mathsf{B}\mathsf{K}_g^{\dagger}\mathsf{B}^T, \quad \mathbf{e} = \mathsf{R}^T\mathbf{f}_g, \quad \mathsf{G} = \mathsf{R}^T\mathsf{B}^T, \quad \mathsf{P} = \mathsf{G}^T(\mathsf{G}\mathsf{G}^T)^{\dagger}\mathsf{G}, \quad \mathbf{d} = \mathsf{B}\mathsf{K}_g^{\dagger}\mathbf{f}_g - \mathbf{g},$$

denoting $\mathsf{Q} = \mathsf{I} - \mathsf{P}$, and observing that $\mathsf{Q}\boldsymbol{\lambda} = \boldsymbol{\lambda}$ for any feasible $\boldsymbol{\lambda}$, we can modify (4) to

$$\min\ \theta(\boldsymbol{\lambda}) \quad \text{s.t.} \quad \boldsymbol{\lambda}_I \geq 0 \quad \text{and} \quad \mathsf{G}\boldsymbol{\lambda} = \mathbf{e}, \tag{6}$$

where

$$\theta(\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{\lambda}^T\mathsf{QFQ}\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\mathsf{Q}\ \mathbf{d}.$$

Alternatively, the Lagrange multipliers of the solution are determined by the KKT conditions for (4) which read

$$\mathsf{F}\boldsymbol{\lambda} - \mathbf{d} + \mathsf{G}^T\boldsymbol{\alpha} = \mathbf{o}, \quad \boldsymbol{\lambda}_I \geq \mathbf{o}, \quad \text{and} \quad \mathsf{G}\boldsymbol{\lambda} = \mathbf{e}. \tag{7}$$

For details concerning the matrices and parallel implementation, see e.g. in [8, 12], and [1].

3 Algorithms

We implemented two FETI based algorithms for the solution of contact problems, using the research software which is being developed in Stanford. The first one, FETI-DPC, is based on FETI-DP domain decomposition method. The algorithm uses the Newton-like method which solves the equilibrium equation (7) in Lagrange multipliers in the inner loop, while feasibility of each step is ensured in the outer loop by the primal and dual planning steps. The algorithm exploits standard FETI preconditioners, namely the Schur and lumped ones. The additional speedup of convergence is achieved by application Krylov type acceleration scheme. The algorithm exploits a globalization strategy in order to achieve monotonic global convergence.

The second algorithm is based on the TFETI domain decomposition (see [7]), a variant of the FETI-I domain decomposition method (see [14]), which treats all the boundary conditions by Lagrange multipliers, so that all the subdomains are floating, and their kernels are known a priori and can be used in construction of the natural coarse grid. It exploits our recently proposed algorithms MPRGP (Modified Proportioning with Reduced Gradient Projection) by Dostál and Schöberl (see [11]) and SMALBE (Semimonotonic Augmented Lagrangians for Bound and Equality constrained problems) (see [3, 4]). The SMALBE, a variant of augmented Lagrangian method with adaptive precision control for the solution of quadratic programming problems with bound and equality constraints, is applied to (6). It enforces the equality constraints by the Lagrange multipliers generated in the outer loop, while the auxiliary bound constrained problems are solved approximately in the inner loop by MPRGP, an active set based algorithm which uses the conjugate gradient method to explore the current face, the fixed steplength gradient projection to expand the active set, the adaptive precision control of auxiliary linear problems, and the reduced gradient with the optimal steplength to reduce the active set. The unique feature of SMALBE with the inner loop implemented by MPRGP when used to (6) is the rate of convergence in bounds on spectrum of the regular part of the Hessian of θ, so that using the classical results by Farhat, Mandel, and Roux (see [14]), the algorithm has been proved to be numerically scalable (see [6]).

4 Numerical Experiments

Algorithms described in this paper were tested and their results compared on two model contact problems.

The first 2D problem involves 6 rectangles in mutual contact as it is depicted in Figure 1 (left). The left rectangles are fixed on the left side (blue arrows) while the right ones are free and they are loaded (red arrows represent forces in opposite direction) such a way that the problem has unique solution. Each rectangle were further decomposed to the 4 subrectangles and therefore the original problem were decomposed to 24 subdomains (Figure 1 (middle)). The performance of the algorithms FETI-DPC and SMALBE is compared in Table 1. Outer iterations are used only in the case of SMALBE method while the number of subiterations is used only in methods FETI-DP. The number of dual plannings and primal plannings of FETI-DPC methods corresponds to the number of expansion and proportioning steps in the case of SMALBE method. Therefore they share the same column for each methods. The numbers on the left side of the slashes represent the number of iterations for 6 subdomains problem and the numbers on the right sides represent the number of iterations for 24 subdomains problem. The resulting deformation with distribution of the stresses are depicted in Figure 1 (right).

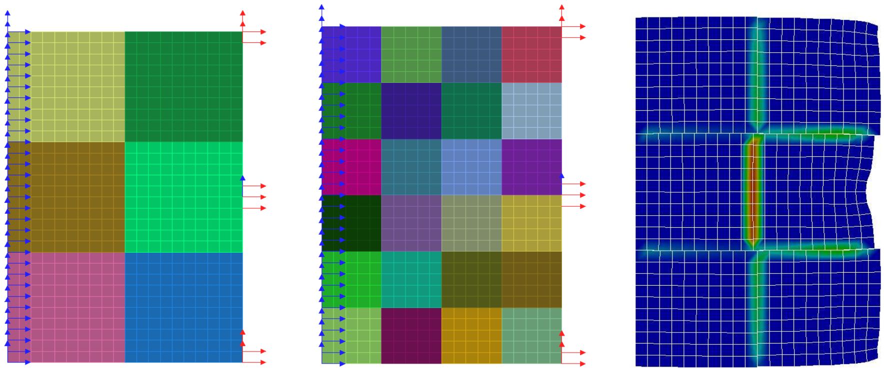

Fig. 1. 2D problem: decomposition in 6 subdomains (left), in 24 subdomains (middle), and computed stress distribution (right)

Table 1. Algorithms performance for 2D semicoercive problem with 6 and 24 subdomains.

	Outer iter.	Main iter.	subiter.	Primal plan. (Exp. step)	Dual plan. (Proport.)
FETI-DPC	-	17/32	0/0	2/2	0/0
SMALBE	1/21	9/68	-	0/18	1/3

The second 3D model problem consists of two bricks in mutual contact. The bottom brick is fixed in all degrees of freedom while the upper one is fixed only in such a way, that only vertical rigid body movement is allowed. This situation is depicted in Figure 2 (left and middle). The forces are chosen so that not all constraints are active on the contact interface as in Figure 2 (right). We have analyzed two cases. The first one, with matching grid on the contact interface prescribes node-to-node contact conditions. The second one allows non-matching grids and the mortar elements were used for assembling of contact conditions. The resulting performance of algorithms is collected in the Table 2. Columns in this table have the same meaning as in 2D case.

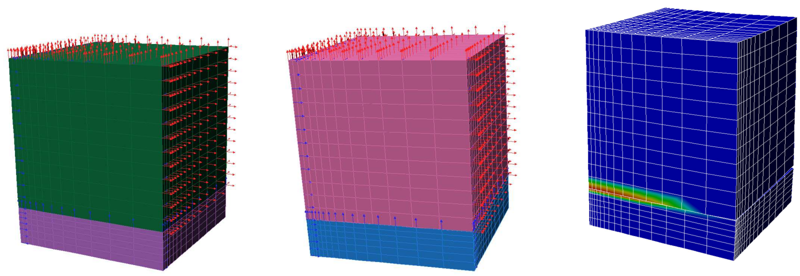

Fig. 2. 3D problem with matching grids (left), with non-matching grids (middle), and computed solution using non-matching grids (right)

Table 2. Algorithms performance for 3D problem with matching/non-matching grid on contact interface.

	Outer iter.	Main iter.	subiter.	Primal plan. (Exp. step)	Dual plan. (Proport.)
FETI-DPC	-	24/26	11/10	7/8	0/0
SMALBE	13/10	29/29	-	20/20	0/0

5 Comments and Conclusions

The FETI method turned out to be a powerful engine for the solution of contact problems of elasticity. Results of numerical experiments comply with recent theoretical results and indicate high efficiency of the methods presented here. Future research will include adaptation of the standard preconditioning strategies to the solution of inequality constraint problems, problems with friction (see e.g. [8]), and dynamic contact problems.

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