The Dilemma of Domain Decomposition Approaches in Fluid-Structure Interactions with Fully Enclosed Incompressible Fluids

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Many popular non-overlapping domain decomposition approaches to fluid-structure interaction (FSI) problems fail to work for an interesting subset of FSI problems, the interaction of highly deformable structures with incompressible but fully enclosed fluids. This is particularly true for coupling approaches based on Dirichlet-Neumann substructuring, both for weak and strong coupling schemes. The breakdown of simulation can be attributed to a lack of knowledge transfer – e.g. of the incompressibility constraint to the structure – between the fields. Another explanation is the absence of any unconstrained outflow boundary at the fluid field, that is the fluid domain is entirely enclosed by Dirichlet boundary conditions. Inflating of a balloon with a prescribed inflow rate constitutes a simple problem of that kind. To overcome the dilemma inherent to partitioned or domain decomposition approaches in these cases a small augmentation is proposed that consists of introducing a volume constraint on the structural system of equations. Additionally the customary applied relaxation of the interface displacements has to be abandoned in favor of the relaxation of coupling forces. These modifications applied to a particular strongly-coupled Dirichlet-Neumann partitioning scheme result in an efficient and robust approach that exhibits only little additional numerical effort. A numerical example with large changes of fluid volume shows the capabilities of the proposed scheme.

1 The Domain Decomposition Approach to FSI Problems

Various solution approaches for FSI problems have been suggested. Most of them are based on a Dirichlet-Neumann partitioning of the coupled problem into fluid and structural part. This constitutes a non-overlapping domain decomposition with fluid field and structural field acting as separate domains. The wet structural surface acts as the coupling interface Γ_{FSI}. These solution schemes require an iterative treatment of the coupling conditions and therefore considerable computational resources, however stability and accuracy are not sacrificed. Additionally these schemes can be

built based on available field solvers, which accounts for their constant popularity, see for instance [10, 4, 5, 7, 2, 1, 9, 8].

To sketch the FSI coupling algorithm the structural and fluid problems are abbreviated as follows

$$\mathbf{A}^S \mathbf{d}^S = \mathbf{f}^S \qquad \text{and} \qquad \mathbf{A}^F \mathbf{u}^F = \mathbf{f}^F \tag{1}$$

where both systems are understood to be nonlinear and the fluid system also needs to take the domain deformations into account.

In the following $(\cdot)_I$ and $(\cdot)_\Gamma$ denote variables or coefficients in the interior of a subdomain Ω^j and those coupled at the interface, respectively, while the absence of any subscript comprises degrees of freedom on the entire subdomain including interior and interface.

In every time step the following calculations have to be performed until convergence is reached. The variable i denotes the loop counter.

1. Transfer the latest structure displacements $\mathbf{d}^S_{\Gamma,i+1}$ to the fluid field, calculate the fluid domain deformation and determine the appropriate fluid velocities at the interface $\mathbf{u}^S_{\Gamma,i+1}$.
2. Solve the fluid equation for inner fluid velocities and all (inner and boundary) fluid pressures $\mathbf{u}^F_{I,i+1}$.

$$\mathbf{A}^F_{II}\mathbf{u}^F_{I,i+1} = \mathbf{f}^F_{I\,ext} - \mathbf{A}^F_{I\Gamma}\mathbf{u}^S_{\Gamma,i+1}\,. \tag{2}$$

3. Find the fluid forces $\mathbf{f}^F_{\Gamma,i+1}$ at the interface Γ_{FSI}.

$$\mathbf{f}^F_{\Gamma,i+1} = \mathbf{A}^F_{\Gamma I}\mathbf{u}^F_{I,i+1} + \mathbf{A}^F_{\Gamma\Gamma}\mathbf{u}^S_{\Gamma,i+1}\,. \tag{3}$$

4. Apply the fluid forces $\mathbf{f}^F_{\Gamma,i+1}$ to the structure. Solve the structure equations for the structural displacements.

$$\begin{bmatrix} \mathbf{A}^S_{\Gamma\Gamma} & \mathbf{A}^S_{\Gamma I} \\ \mathbf{A}^S_{I\Gamma} & \mathbf{A}^S_{II} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{d}}^S_{\Gamma,i+1} \\ \mathbf{d}^S_{I,i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^S_{\Gamma\,ext} - \mathbf{f}^F_{\Gamma,i} \\ \mathbf{f}^S_{I\,ext} \end{bmatrix}\,. \tag{4}$$

5. The calculation is finished when the difference between $\tilde{\mathbf{d}}^F_{\Gamma,i+1}$ and $\mathbf{d}^F_{\Gamma,i}$ is sufficiently small.
6. Relax the interface displacement using a suitable ω_i.

$$\mathbf{d}^S_{\Gamma,i+1} = \omega_i \tilde{\mathbf{d}}^S_{\Gamma,i+1} + (1-\omega_i)\mathbf{d}^S_{\Gamma,i}\,. \tag{5}$$

7. Update i and return to step 1.

Information on the appropriate choice of the relaxation coefficient ω_i can be found in [10, 5].

2 Dilemma with Fully Enclosed, i.e. Dirichlet-Constraint, Fluid Domains

The Dirichlet-Neumann algorithm described above fails if there are prescribed velocities on all boundaries of the fluid domain. A fully Dirichlet-bounded fluid domain

can only be solved if (a) the prescribed velocities satisfy the mass balance of the incompressible fluid and (b) the pressure level is fixed by an additional constraint. Standard Dirichlet-Neumann algorithms fail on both conditions. Neither does the fluid domain deformation suggested by the structural solver match the fluid mass balance, nor are there means to transfer any pressure information form the structure to the fluid.

These two difficulties are closely related. The two fields are coupled much closer as compared to FSI problems with free outflow boundaries. Therefore any attempt to overcome the difficulties will result in an algorithm that is more expensive from a numerical point of view.

Several strategies might be pursued to arrive at a working coupling algorithm.

- The interface displacements, that is the structural solution, respect the incompressibility constraint of the fluid. Thus the introduction of a constraint to the structural equations is required. The fluid pressure level will need to be calculated from the structure solution. This approach is presented in detail in the following.
- Another point of departure is the pressure level coupling between structure and fluid. The natural way for the structure to determine the fluid pressure is to transfer interface forces from the structure to the fluid. It follows that the fluid has to prescribe the interface displacements on the structure, that is the Dirichlet-Neumann coupling is reversed to a Neumann-Dirichlet approach. The resulting algorithm, however, is numerically very sensitive and not suitable for general FSI problems. In addition it also runs into the old problem once one has to deal with incompressible solids, too. Details can be found in [3].
- Finally the whole problem is avoided if one can get rid of the incompressibility constraint, at least temporarily. However this also has been shown to be not a very robust or efficient approach. This idea has been pursued in [6] and will not be discussed here.

It is worth noting, that according to the insight discussed so far Dirichlet-Neumann approaches only work in standard examples because the fluid can temporarily escape through the Neumann boundary in staggered situations or during the field iterations in strong coupling schemes.

3 Augmented Dirichlet-Neumann Approach

3.1 Volume Constraint Applied to the Structural Equation

The augmentation of the structural solver to account for the mass balance of the enclosed fluid domain translates to a constraint of the interface displacements to enclose exactly the required volume. The required fluid volume V_c depends upon the Dirichlet boundary conditions of the fluid domain.

$$\begin{aligned} V_c = V^{n+1} &= V^n + \int_{\Gamma^F} \frac{1}{2}\Delta t \left(\mathbf{u}^{n+1}\cdot\mathbf{n} + \mathbf{u}^n\cdot\mathbf{n}\right)\, d\Gamma \\ &= V^n + \int_{\Gamma_{FSI}} \left(\mathbf{r}^{n+1}\cdot\mathbf{n} - \mathbf{r}^n\cdot\mathbf{n}\right)\, d\Gamma \end{aligned} \tag{6}$$

$$+ \int_{\Gamma_{in} \cup \Gamma_{out}} \frac{1}{2} \Delta t \left(\mathbf{u}^{n+1} \cdot \mathbf{n} + \mathbf{u}^{n} \cdot \mathbf{n} \right) \, d\Gamma$$

where $\mathbf{r}^{n+1}$ and $\mathbf{r}^{n}$ are the interface positions at the time t^{n+1} and t^{n}. The constraint $V_c - V = 0$ is introduced into the structural equation of motion by means of a Lagrangian multiplier λ. This Lagrangian multiplier represents the pressure increment required additional to the fluid pressure in order to satisfy the volume constraint on the structure. Thus the multiplier specifies the physical fluid pressure level.

If the fluid forces at the interface $\mathbf{f}^F$ are sufficient to maintain the required volume V_c, the pressure increment λ will be zero. This can be achieved by a coupling algorithm that transfers λ to the fluid partition and adds it to the pressure boundary condition which is used to determine the pressure level. This way the Lagrangian multiplier λ will tend to zero in the course of the coupling iteration.

But changing the pressure boundary condition of the fluid during the coupling iteration means changing the overall problem definition. It is generally advisable to avoid it. Instead the fluid pressure level can be fixed to a constant value in the fluid domain. Of course the resulting pressure increment λ will not vanish in this case. Instead it is added to the fluid pressure values $\overline{p}^F$ after the fluid calculation to obtain the final pressure solution $\overline{p}$:

$$\overline{p} = \overline{p}^F + \lambda \, . \tag{7}$$

3.2 Modified Dirichlet-Neumann Coupling with Volume Constraint

The iterative coupling algorithm with the volume constraint in the structural equation is a slight modification of the algorithm in section 1. Because the structural solver has to account for the volume condition of the fluid domain, the displacement of the interface cannot be altered once the structural solution is done. In particular the relaxation of the displacements is no longer possible. Instead, because relaxation is needed to enforce and accelerate convergence, one has to relax the fluid forces at the interface.

By means of the symbolic structural and fluid system (1) in every time step the following calculations have to be performed.

1. Solve for the structural displacements loaded with the fluid forces $\mathbf{f}^F_{\Gamma,i}$, but respect the volume constraint required by the fluid

$$\begin{bmatrix} \mathbf{A}^S_{\Gamma\Gamma} & \mathbf{A}^S_{\Gamma I} & -V_{,\mathbf{d}^S_\Gamma} \\ \mathbf{A}^S_{I\Gamma} & \mathbf{A}^S_{II} & 0 \\ -V_{,\mathbf{d}^S_\Gamma} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d}^S_{\Gamma,i+1} \\ \mathbf{d}^S_{I,i+1} \\ \lambda_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^S_{\Gamma\, ext} - \mathbf{f}^F_{\Gamma,i} - V_{,\mathbf{d}^S_\Gamma} \lambda_i \\ \mathbf{f}^S_{I\, ext} \\ V_c - V_{,\mathbf{d}^S_\Gamma} \mathbf{d}^S_{\Gamma,i} \end{bmatrix} . \tag{8}$$

2. Transfer the interface displacements $\mathbf{d}^S_{\Gamma,i+1}$ to the fluid and determine the interface velocities $\mathbf{u}^S_{\Gamma,i+1}$. Solve for inner fluid velocities and all fluid pressures $\mathbf{u}^F_{I,i+1}$

$$\mathbf{A}^F_{II} \mathbf{u}^F_{I,i+1} = \mathbf{f}^F_{I\, ext} - \mathbf{A}^F_{I\Gamma} \mathbf{u}^S_{\Gamma,i+1} \, . \tag{9}$$

3. Find the fluid forces at the interface Γ_{FSI}

$$\tilde{\mathbf{f}}^F_{\Gamma,i+1} = \mathbf{A}^F_{\Gamma I} \mathbf{u}^F_{I,i+1} + \mathbf{A}^F_{\Gamma\Gamma} \mathbf{u}^S_{\Gamma,i+1} \, . \tag{10}$$

4. Relax the fluid forces

$$\mathbf{f}^F_{\Gamma,i+1} = \omega_i \tilde{\mathbf{f}}^F_{\Gamma,i+1} + (1-\omega_i)\mathbf{f}^F_{\Gamma,i}\,. \tag{11}$$

The relaxation parameter ω_i can again be calculated by any of the methods suggested in [5].

The iteration finishes when the error of the fluid boundary force $\tilde{\mathbf{f}}^F_{\Gamma,i+1}$ is sufficiently small.

4 Example: Damped Structural Instability

As an example a bended fluid domain is calculated that is surrounded by two thin structures with neo-Hookean material and different stiffness. The system is shown in figure 1. The structures are fixed at their short edges, the long edges are free respectively interacting with the fluid.

At the fluid domain inflow velocities are prescribed with the left one a little less than the right in order to avoid perfect symmetry. The fluid is loaded with the body force $\mathbf{f}_y = -1N/m^2$ in y direction. The simulation is carried out utilizing the augmented Dirichlet-Neumann algorithm and a uniform time step size $\Delta t = 0.005s$.

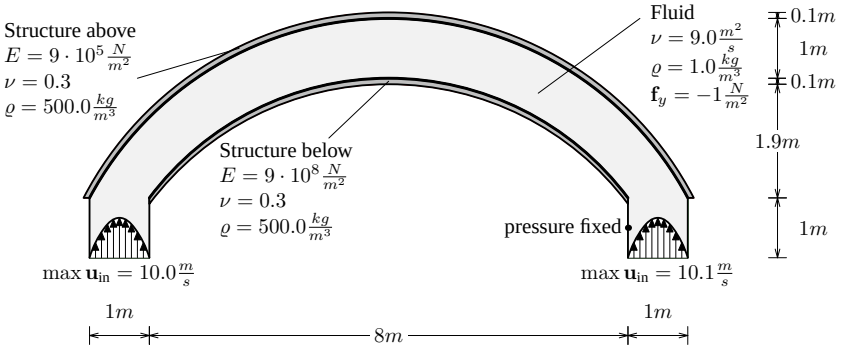

Fig. 1. A bended fluid domain with two inflow boundaries constraint by structures of different stiffness.

The constant inflow increases the fluid pressure so that first mainly the soft flexible structure above the fluid domain deforms to make room for the fluid. When a critical pressure value is reached the structure below the fluid collapses, however the instability is damped by the fluid volume constraint. That is why the deformation and the corresponding pressure decrease occur rather slowly. (Since this example is given just in order to demonstrate the augmented Dirichlet-Neumann approach, possible cavitation effects are not considered.) Afterward the system is in motion, the pressure varies rapidly in this phase. The pressure level development, that is the pressure increment λ calculated by the structural solver, is depicted in figure 2.

Figure 3 shows absolute velocities at different time steps.

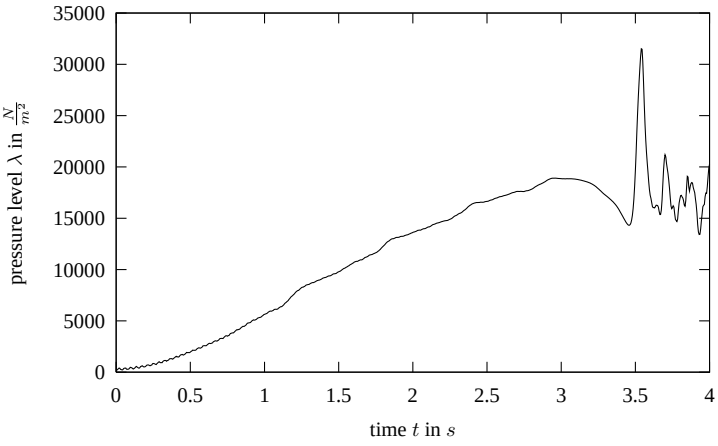

Fig. 2. Pressure level of the bended fluid domain.

5 Conclusion

The dilemma of non-overlapping domain decomposition approaches to FSI problems has been analyzed. Different solution strategies were considered. A small modification to an established iterative solution scheme has been proposed that consists of introducing the incompressibility constraint to the structural solver and results in a reliable and accurate algorithm.

This one condition seriously damages the bandwidth of the system matrix, it couples all displacements on the wet surface. Additionally the positive definiteness of the matrix is lost. Thus the approach is rather expensive from a numerical point of view. A common solver alternative in such a situation would be to use a staggered scheme on the structural side. However, the additional costs pertain the structural solver only. And because the fluid solution costs are dominating in most FSI calculations the proposed algorithm presents a viable approach for many FSI simulations which require Dirichlet constraints on all fluid boundaries.

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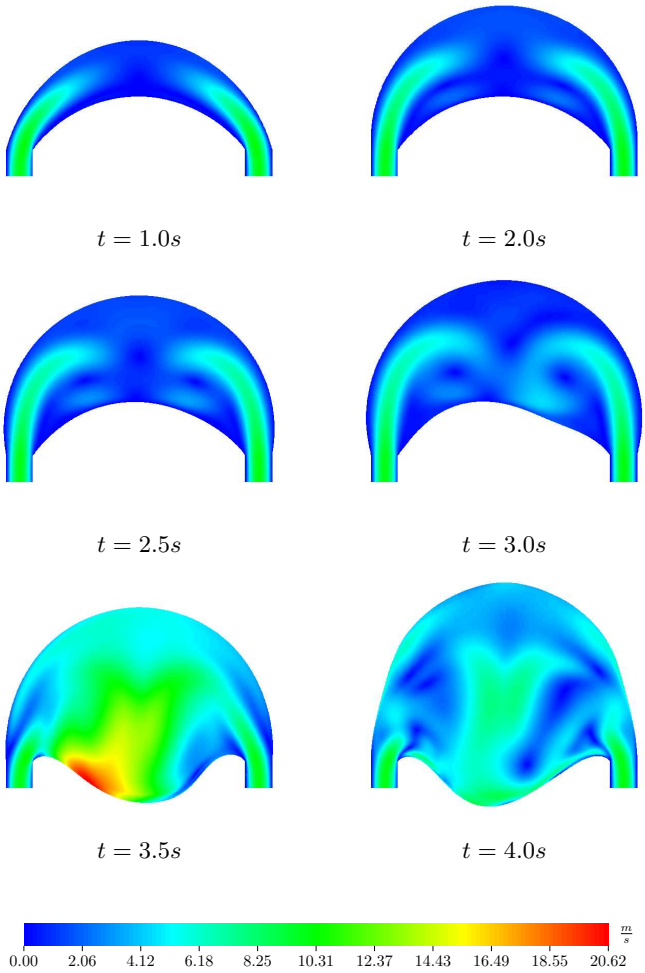

Fig. 3. Fluid velocity $|\mathbf{u}|$. The structural part is not shown.

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