Mortar and Discontinuous Galerkin Methods Based on Weighted Interior Penalties

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1 Introduction

A wide class of discontinuous Galerkin (DG) methods, the so called interior penalty methods, arise from the idea that inter-element continuity could be attained by mimicking the techniques previously developed for weakly enforcing suitable boundary conditions for PDE's, see [7]. Although the DG methods are usually defined by means of the so called numerical fluxes between neighboring mesh cells, see [1], for most of the interior penalty methods for second order elliptic problems it is possible to correlate the expression of the numerical fluxes with a corresponding set of local interface conditions that are weakly enforced on each inter-element boundary. Such conditions are suitable to couple elliptic PDE's with smooth coefficients and it seems that a little attention is paid to the case of problems with discontinuous data or to the limit case where the viscosity vanishes in some parts of the computational domain.

In this paper, we discuss the derivation of a DG method arising from a set of generalized interface conditions, considered in [4], which are adapted to couple both elliptic and hyperbolic problems. In order to obtain such method, it is necessary to modify the definition of the numerical fluxes, replacing the standard arithmetic mean with suitably weighted averages where the weights depend on the coefficients of the problem. Even though the underlying ideas could be equivalently applied both to mortars and DG methods, we privilege here the discussion of the latter case, since the former has already been considered in [2].

In the framework of mortar finite-element methods, different authors have highlighted the possibility of using an average with weights that differ from one half, see [8, 5]. These works present several mortaring techniques to match conforming finite elements on possibly non conforming computational meshes. However, these works do not consider any connection between the averaging weights and the coefficients of the problem. More recently, Burman and Zunino [2] have introduced this dependence for an advection-diffusion-reaction problem with discontinuous viscosity, and they have shown that the application of the harmonic mean on the edges where the viscosity is discontinuous improves the stability of the numerical scheme. In this work, we aim to generalize the definition of such method and to apply it to the DG case. After introducing the model problem and some notation, particular attention will be devoted here to illustrate how the definition of the scheme and the corresponding numerical fluxes obey to the requirement of obtaining a method which is well posed and robust not only in the elliptic regimen, but also in the presence of a locally vanishing viscosity. A complete a-priori error analysis is not addressed here, but we illustrate the behavior of the method by means of some numerical tests.

2 Derivation of the Numerical Method

We aim to find u, the solution of the following boundary value problem,

$$\begin{cases} \nabla\cdot(-\epsilon\nabla u + \beta u) + \mu u = f & \text{in } \Omega \subset \mathbb{R}^d,\ d = 2,3, \\ \left[\frac{1}{2}\big(|\beta\cdot n| - \beta\cdot n\big) + \chi_{\partial\Omega}(\epsilon)\right]u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a polygonal domain, n is the outer normal unit vector with respect to $\partial\Omega$ and $\chi_{\partial\Omega}(\epsilon) \geq 0$, satisfying $\chi_{\partial\Omega}(0) = 0$, will be made precise later. Here $\mu \in L^\infty(\Omega)$ is a positive function and $\beta \in [W^{1,\infty}(\Omega)]^d$ is a vector function such that $\mu + \frac{1}{2}(\nabla\cdot\beta) \geq \mu_0 > 0$, $f \in L^2(\Omega)$ and ϵ is a nonnegative function in $L^\infty(\Omega)$. The well posedness of problem (1), with $\epsilon \in W^1_\infty(\Omega)$, is addressed in [6] and references therein.

For the numerical approximation of problem (1) we consider a shape regular triangulation T_h of the domain Ω, we denote with K an element in T_h and with $n_{\partial K}$ its outward unit normal. We define a totally discontinuous approximation space,

$$V_h := \{v_h \in L^2(\Omega);\ \forall K \in T_h, v_h|_K \in \mathbb{P}^k\},\ \text{with } k > 0.$$

Let Γ_e be the set of the element edges $e \subset \partial K$ such that $e \cap \partial\Omega = \emptyset$ and let n_e be the unit normal vector associated to e. Nothing is said hereafter depends on the arbitrariness on the sign of n_e. We denote with $\Gamma_{\partial\Omega}$ the collection of the edges on $\partial\Omega$. For all $e \in \Gamma_e \cup \Gamma_{\partial\Omega}$ let h_e be the size of the edge. For any $v_h \in V_h$ we define,

$$v_h^{\mp}(x) := \lim_{\delta\to 0^+} v_h(x \mp \delta n_e) \text{ for a.e. } x \in e,\ \text{with } e \in \Gamma_e.$$

When not otherwise indicated, the v_h^- value is implied. Similar definitions apply to all fields that are two-valued on the internal interfaces. The jump over interfaces is defined as $[\![v_h(x)]\!] := v_h^-(x) - v_h^+(x)$. We denote the arithmetic mean with $\{v_h(x)\} := \frac{1}{2}(v_h^-(x) + v_h^+(x))$. We also introduce the weighted averages for any $e \in \Gamma_e$ and a.e. $x \in e$,

$$\{v_h(x)\}_w := w_e^-(x)v_h^-(x) + w_e^+(x)v_h^+(x),$$
$$\{v_h(x)\}^w := w_e^+(x)v_h^-(x) + w_e^-(x)v_h^+(x),$$

where the weights necessarily satisfy $w_e^-(x) + w_e^+(x) = 1$. We say that these averages are conjugate, because they satisfy the following identity,

$$[\![v_h w_h]\!] = \{v_h\}_w[\![w_h]\!] + \{w_h\}^w[\![v_h]\!],\ \forall v_h, w_h \in V_h. \tag{2}$$

The role of $\{\cdot\}_w$ and $\{\cdot\}^w$ can also be interchanged, but for symmetry this choice does not affect the final setting of the method. Finally, there is no need to extend the definitions of jumps and averages on the boundary $\partial\Omega$, because the contributions of Γ_e and $\Gamma_{\partial\Omega}$ will be always treated separately.

To set up a numerical approximation scheme for problem (1), we assume for simplicity that ϵ is piecewise constant on T_h. We define $\sigma_h(v_h) := -\epsilon\nabla v_h + \beta v_h$ or simply σ_h if the flux is applied to the primal unknown u_h, and we consider the Galerkin discretization method in V_h, which originates from the following expression,

$$\int_\Omega f v_h = \int_\Omega \Big(\nabla\cdot\sigma_h v_h + \mu u_h v_h\Big) = \sum_{K\in T_h}\int_K \Big(\nabla\cdot\sigma_h v_h + \mu u_h v_h\Big)$$
$$= \sum_{K\in T_h}\Big[\int_K \Big(-\sigma_h\cdot\nabla v_h + \mu u_h v_h\Big) + \int_{\partial K}\sigma_h\cdot n_{\partial K} v_h\Big],\ \forall v_h\in V_h. \quad (3)$$

Then, considering the identity,

$$\sum_{K\in T_h}\int_{\partial K}\sigma_h\cdot n_{\partial K} v_h = \sum_{e\in\Gamma_e}\int_e [\![\sigma_h v_h]\!]\cdot n_e + \sum_{e\in\Gamma_{\partial\Omega}}\int_e (\sigma_h v_h)\cdot n,$$

and replacing it into (3), owing to (2) we obtain,

$$\sum_{e\in\Gamma_e}\int_e \Big(\{\sigma_h\}_w\cdot n_e[\![v_h]\!] + [\![\sigma_h]\!]\cdot n_e\{v_h\}^w\Big) + \sum_{e\in\Gamma_{\partial\Omega}}\int_e \sigma_h\cdot n v_h$$
$$+ \sum_{K\in T_h}\int_K \Big(-\sigma_h\cdot\nabla v_h + \mu u_h v_h\Big) = \int_\Omega f v_h,\ \forall v_h\in V_h. \quad (4)$$

We need now to apply suitable conditions on each inter-element interface and on the boundary of the domain. To this aim, we define $\gamma_e(\epsilon,\beta) := \frac{1}{2}\big(|\beta\cdot n_e| - \varphi_e(\epsilon)\beta\cdot n_e\big) + \chi_e(\epsilon)h_e^{-1}$, where $\chi_e(\epsilon)\geq 0$ such that $\chi_e(0)=0$ and $|\varphi_e(\epsilon)|\leq 1$ will be defined later, and we set $[\![\sigma_h]\!]\cdot n_e = 0$, $\gamma_e(\epsilon,\beta)[\![u_h]\!]=0$ on any $e\in\Gamma_e$. We also set $\gamma_{\partial\Omega}(\epsilon,\beta) := \frac{1}{2}\big(|\beta\cdot n| - \beta\cdot n\big) + \chi_{\partial\Omega}(\epsilon)h_e^{-1}$. Introducing the boundary and local interface conditions into (4) we obtain,

$$\sum_{K\in T_h}\int_K \Big(-\sigma_h\cdot\nabla v_h + \mu u_h v_h\Big) + \sum_{e\in\Gamma_e}\int_e \{\sigma_h\}_w\cdot n_e[\![v_h]\!] + \sum_{e\in\Gamma_{\partial\Omega}}\int_e \sigma_h\cdot n v_h$$
$$+ \sum_{e\in\Gamma_e}\int_e \gamma_e(\epsilon,\beta)[\![u_h]\!][\![v_h]\!] + \sum_{e\in\Gamma_{\partial\Omega}}\int_e \gamma_{\partial\Omega}(\epsilon,\beta)u_h v_h = \int_\Omega f v_h, \forall v_h\in V_h. \quad (5)$$

The left hand side of equation (5) can be split in two parts. The former corresponds to the symmetric terms and it reads as follows,

$$a_h^s(u_h,v_h) := \sum_{K\in T_h}\int_K \Big[\epsilon\nabla u_h\cdot\nabla v_h + \big(\mu + \tfrac{1}{2}\nabla\cdot\beta\big)u_h v_h$$
$$+ \sum_{e\in\Gamma_e}\int_e \Big[-\{\epsilon\nabla u_h\}_w\cdot n_e[\![v_h]\!] - \{\epsilon\nabla v_h\}_w\cdot n_e[\![u_h]\!] + \big(\tfrac{1}{2}|\beta\cdot n_e| + \chi_e(\epsilon)h_e^{-1}\big)[\![u_h]\!][\![v_h]\!]\Big]$$
$$+ \sum_{e\subset\Gamma_{\partial\Omega}}\int_e \Big[-\epsilon\nabla u_h\cdot n v_h - \epsilon\nabla v_h\cdot n u_h + \big(\tfrac{1}{2}|\beta\cdot n| + \chi_{\partial\Omega}(\epsilon)h_e^{-1}\big)u_h v_h\Big],$$

where we have added the new terms $\{\epsilon\nabla v_h\}_w \cdot n_e[\![u_h]\!]$ on Γ_e and $\epsilon\nabla v_h \cdot n u_h$ on $\Gamma_{\partial\Omega}$ to preserve symmetry. The remaining part of the bilinear form is,

$$a_h^r(u_h,v_h) := -\sum_{K\in T_h}\int_K \Big[(\beta u_h)\cdot\nabla v_h + \tfrac{1}{2}(\nabla\cdot\beta)u_h v_h\Big]$$
$$+\sum_{e\in\Gamma_e}\int_e \Big[\{\beta u_h\}_w\cdot n_e[\![v_h]\!] - \tfrac{1}{2}\varphi_e(\epsilon)\beta\cdot n_e[\![u_h]\!][\![v_h]\!]\Big] + \sum_{e\in\Gamma_{\partial\Omega}}\int_e \tfrac{1}{2}\beta\cdot n u_h v_h.$$

Then, setting $a_h(u_h,v_h) := a_h^s(u_h,v_h) + a_h^r(u_h,v_h)$ and $F(v_h) := \int_\Omega f v_h$ our prototype of method reads as follows: find $u_h \in V_h$ such that,

$$a_h(u_h,v_h) = F(v_h),\ \forall v_h \in V_h. \tag{6}$$

Before proceeding, we choose the weights w_e^-, w_e^+ on each edge such that $w_e^-\epsilon^- = w_e^+\epsilon^+$, and accordingly we define, $\omega_e(\epsilon) := \frac{1}{2}\{\epsilon\}_w = w_e^-\epsilon^- = w_e^+\epsilon^+$. Together with $w_e^+ + w_e^- = 1$ this leads to the expressions,

$$\begin{aligned} w_e^- &= \frac{\epsilon^+}{\epsilon^- + \epsilon^+},\quad w_e^+ = \frac{\epsilon^-}{\epsilon^- + \epsilon^+},\quad \text{if } \epsilon^- + \epsilon^+ > 0,\\ \text{or } w_e^- &= w_e^+ = \tfrac{1}{2},\quad \text{if } \epsilon^- = \epsilon^+ = 0. \end{aligned}\tag{7}$$

Replacing (7) into $\{\epsilon\}_w$, we observe that it is equivalent to the harmonic mean of the coefficient ϵ across the edges. In what follows, we will see how this is related to the behavior of the method. For any admissible value of $\chi_e(\epsilon)$ and $\varphi_e(\epsilon)$ we observe that method (6) is by construction consistent with respect to the weak formulation of problem (1). Now, the definition of $\chi_e(\epsilon)$ and $\varphi_e(\epsilon)$ has to be made precise in order to enforce that $a_h(\cdot,\cdot)$ is coercive in the following norm,

$$|||v_h|||^2 := \|\epsilon^{\frac{1}{2}}\nabla v_h\|^2_{0,T_h} + \|\mu_0^{\frac{1}{2}} v_h\|^2_{0,T_h}$$
$$+ \|(\tfrac{1}{2}|\beta\cdot n_e| + \tfrac{1}{2}\{\epsilon\}_w h_e^{-1})^{\frac{1}{2}}[\![v_h]\!]\|^2_{0,\Gamma_e} + \|(\tfrac{1}{2}|\beta\cdot n| + \epsilon h_e^{-1})^{\frac{1}{2}} v_h\|^2_{0,\Gamma_{\partial\Omega}},$$

where we have introduced the notation $\|v_h\|^2_{0,T_h} := \sum_{K\in T_h}\|v_h\|^2_{0,K}$, and $\|v_h\|^2_{0,\Gamma_e} := \sum_{e\in\Gamma_e}\|v_h\|^2_{0,e}$, being $\|\cdot\|_{0,K}$ and $\|\cdot\|_{0,e}$ the L^2-norms on K and e respectively. Then, we consider the bilinear form $a_h^r(\cdot,\cdot)$ that can be manipulated as follows,

$$\begin{aligned} a_h^r(u_h,u_h) &= \sum_{e\in\Gamma_e}\int_e \Big[\beta\cdot n_e\{u_h\}_w[\![u_h]\!] - \beta\cdot n_e\{u_h\}[\![u_h]\!] - \tfrac{1}{2}\beta\cdot n_e\varphi_e(\epsilon)[\![u_h]\!]^2\Big]\\ &= \tfrac{1}{2}\sum_{e\in\Gamma_e}\int_e (2w_e^- - \varphi_e(\epsilon) - 1)\beta\cdot n_e[\![u_h]\!]^2 = 0, \end{aligned}\tag{8}$$

provided that we set $\varphi_e(\epsilon) := (2w_e^- - 1)$ or equivalently, owing to (7),

$$\varphi_e(\epsilon) = \frac{2\epsilon^+}{\epsilon^+ + \epsilon^-} - 1 = -\frac{[\![\epsilon]\!]}{2\{\epsilon\}},\quad \text{if } \{\epsilon\} > 0, \tag{9}$$

and $\varphi_e(\epsilon) = 0$ if $\{\epsilon\} = 0$. Definition (9) satisfies $|\varphi_e(\epsilon)| \le 1$ and the expression $\frac{1}{2}\big(|\beta\cdot n_e| - \varphi_e(\epsilon)\beta\cdot n_e\big)$ represents a natural generalization of the standard upwind scheme. As a consequence of (8), the coercivity only depends on the properties of $a_h^s(\cdot,\cdot)$. First of all, it is straightforward to verify that,

$$\begin{aligned}
&\sum_{K\in T_h}\int_K\left[\epsilon(\nabla u_h)^2+\left(\mu+\tfrac{1}{2}\nabla\cdot\beta\right)u_h^2\right]\\
&+\sum_{e\in\Gamma_e}\int_e\left(\tfrac{1}{2}|\beta\cdot n_e|+\chi_e(\epsilon)h_e^{-1}\right)[\![u_h]\!]^2+\sum_{e\in\Gamma_{\partial\Omega}}\int_e\left(\tfrac{1}{2}|\beta\cdot n|+\chi_{\partial\Omega}(\epsilon)h_e^{-1}\right)u_h^2\\
&\geq\|\epsilon^{\frac{1}{2}}\nabla u_h\|_{0,T_h}^2+\|\mu_0^{\frac{1}{2}}u_h\|_{0,T_h}^2+\|\left(\tfrac{1}{2}|\beta\cdot n_e|+\chi_e(\epsilon)h_e^{-1}\right)^{\frac{1}{2}}[\![u_h]\!]\|_{0,\Gamma_e}^2\\
&+\|\left(\tfrac{1}{2}|\beta\cdot n|+\chi_{\partial\Omega}(\epsilon)h_e^{-1}\right)^{\frac{1}{2}}u_h\|_{0,\Gamma_{\partial\Omega}}^2. \qquad (10)
\end{aligned}$$

To treat the remaining terms of $a_h^s(u_h,u_h)$, as usual for DG methods, we make use of the following trace/inverse inequality,

$$h_e\|\nabla v_h\cdot n_e\|_{0,e}^2\leq C_I\|\nabla v_h\|_{0,K}^2,\ \forall K\in T_h \text{ and } \forall e\in\partial K,$$

where $C_I>0$ does not depend on h_e. Then, we obtain the following bounds,

$$\begin{aligned}
&2\sum_{e\in\Gamma_e}\int_e\{\epsilon\nabla u_h\}_w\cdot n_e[\![u_h]\!]+2\sum_{e\in\Gamma_{\partial\Omega}}\int_e\epsilon\nabla u_h\cdot nu_h\\
=&2\sum_{e\in\Gamma_e}\int_e\omega_e(\nabla u_h^-+\nabla u_h^+)\cdot n_e[\![u_h]\!]+2\sum_{e\in\Gamma_{\partial\Omega}}\int_e\epsilon\nabla u_h\cdot nu_h\\
\leq&\sum_{e\in\Gamma_e}\left[\alpha h_e\left(\|(\epsilon^-)^{\frac{1}{2}}\nabla u_h^-\cdot n_e\|_{0,e}^2+\|(\epsilon^+)^{\frac{1}{2}}\nabla u_h^+\cdot n_e\|_{0,e}^2\right)+\frac{1}{\alpha h_e}\|\omega_e^{\frac{1}{2}}[\![u_h]\!]\|_{0,e}^2\right]\\
&+\sum_{e\in\Gamma_{\partial\Omega}}\left[\alpha h_e\|\epsilon^{\frac{1}{2}}\nabla u_h\cdot n\|_{0,e}^2+\frac{1}{\alpha h_e}\|\epsilon^{\frac{1}{2}}u_h\|_{0,e}^2\right]\\
\leq&6\alpha C_I\|\epsilon^{\frac{1}{2}}\nabla u_h\|_{0,T_h}^2+\frac{1}{\alpha}\|(\tfrac{1}{2}\{\epsilon\}_w)^{\frac{1}{2}}h_e^{-\frac{1}{2}}[\![u_h]\!]\|_{0,\Gamma_e}^2+\frac{1}{\alpha}\|\epsilon^{\frac{1}{2}}h_e^{-\frac{1}{2}}u_h\|_{0,\Gamma_{\partial\Omega}}^2. \qquad (11)
\end{aligned}$$

The coercivity of $a_h(\cdot,\cdot)$ in the norm $|||\cdot|||$ directly follows from the combination of (8), (10) and (11) provided α is such that $6\alpha C_I<1$ and,

$$\chi_e(\epsilon):=\tfrac{1}{2}\zeta\{\epsilon\}_w,\quad \chi_{\partial\Omega}(\epsilon):=\zeta\epsilon, \qquad (12)$$

where ζ is a suitable constant such that $\zeta>\frac{1}{\alpha}$. Due to (9) and (12) the method (6) is completely determined.

By virtue of the second Strang lemma and owing to the continuity (not addressed here), the consistency and the coercivity of the bilinear form $a_h(\cdot,\cdot)$, it is possible to prove optimal a-priori error estimates in the norm $|||\cdot|||$ for problem (6). This analysis has been fully addressed in [3] in the case of a similar method applied to anisotropic diffusivity.

3 Numerical Results and Conclusions

In order to pursue a quantitative comparison between our scheme and the standard interior penalty method, we aim to build up a test problem featuring discontinuous coefficients which allows us to analytically compute the exact solution. To this aim, we consider the following test case, already proposed in [2]. We split the domain Ω into two subregions, $\Omega_1=(x_0,x_{\frac{1}{2}})\times(y_0,y_1)$, $\Omega_2=(x_{\frac{1}{2}},x_1)\times(y_0,y_1)$ and we choose for simplicity $x_0 = 0$, $x_{\frac{1}{2}} = 1$, $x_1 = 2$ while $y_0 = 0$, $y_1 = \frac{1}{2}$. The viscosity $\epsilon(x, y)$ is a discontinuous function across the interface $x = x_{\frac{1}{2}}$, for any $y \in (y_0, y_1)$. Precisely, we consider a constant $\epsilon(x, y)$ in each subregion with several values for ϵ_1 in Ω_1 and a fixed $\epsilon_2 = 1.0$ in Ω_2. In the case $\beta = [1, 0]$, $\mu = 0$, $f = 0$ and the boundary conditions $u_1(x_0, y) = 1$, $u_2(x_1, y) = 0$ for simplicity, the exact solution of the problem on each subregion Ω_1, Ω_2 can be expressed as an exponential function with respect to x independently from y. The global solution $u(x, y)$ is then provided by choosing the value at the interface, $u(x_{\frac{1}{2}}, y)$, in order to ensure the continuity of both $u(x, y)$ and the normal fluxes with respect to the interface, namely $-\epsilon(x, y)\partial_x u(x, y)$. For the corresponding explicit expressions of $u\Big(x_{\frac{1}{2}}, y\Big)$ and $u_1(x, y)$, $u_2(x, y)$, we remand to [2].

In the following numerical simulations, our reference standard interior penalty method (IP) is obtained by replacing the weights $w_e^- = w_e^+ = \frac{1}{2}$ into (6). To compare the method proposed here (WIP) with IP we consider a uniform triangulation T_h with $h = 0.05$ and we apply piecewise linear elements. We perform a quantitative comparison based on the energy norm of the error $|||u - u_h|||$ and on the following indicator, $\Delta_{extr} := \max(|\max_\Omega(u_h) - \max_\Omega(u)|, |\min_\Omega(u_h) - \min_\Omega(u)|)$ which quantifies to which extent the numerical solution exceeds the extrema of the exact one. The results reported in table 1 and in figure 1 put into evidence that the WIP scheme performs better than the standard IP method, particularly in those cases where the solution is non smooth and at the same time the computational mesh is not completely adequate to capture the singularities. This happens in particular for the smallest value of ϵ_1, precisely $\epsilon_1 = 5\ 10^{-3}$, while in the other cases the two methods are equivalent. In the case $\epsilon_1 = 5\ 10^{-3}$ the weighted interior penalties turn out to be very effective, since they allow the scheme to approximate the very steep boundary layer at the interface $x = x_{\frac{1}{2}}$ with a jump. Conversely, the standard interior penalty scheme computes a solution that is almost continuous. As can be observed in figure 1, this behavior promotes the instability of the approximate solution in the neighborhood of the boundary layer, because the computational mesh is not adequate to smoothly approximate the very high gradients across the interface. The quantity Δ_{extr} shows that the the spurious oscillations generated in this case reach the 40% of the maximum of the exact solution. The different behavior of the two methods can also be interpreted observing that, disregarding the advective terms, in the case of the standard IP scheme the satisfaction of the inter-element continuity is proportional to $\{\epsilon\}$, as the neighboring elements of each edge were ideally connected by two adjacent springs of stiffness ϵ^- and ϵ^+. Conversely, in the WIP case the mortar between elements is proportional to $\{\epsilon\}_w$, which is the harmonic mean of the values ϵ^-, ϵ^+ and corresponds to the stiffness of two sequential springs of stiffness ϵ^- and ϵ^+ respectively. The latter case seems to be more natural for problems with discontinuous coefficients.

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Table 1. Quantitative comparison between WIP and a standard IP method.

	$\|\|\|u - u_h\|\|\|$			Δ_{extr}		
ϵ_1	$5\ 10^{-1}$	$5\ 10^{-2}$	$5\ 10^{-3}$	$5\ 10^{-1}$	$5\ 10^{-2}$	$5\ 10^{-3}$
WIP	8.151e-03	5.629e-02	1.858e-01	1.069e-04	1.016e-04	7.302e-02
IP	8.137e-03	5.779e-02	3.208e-01	1.069e-04	1.016e-04	4.412e-01

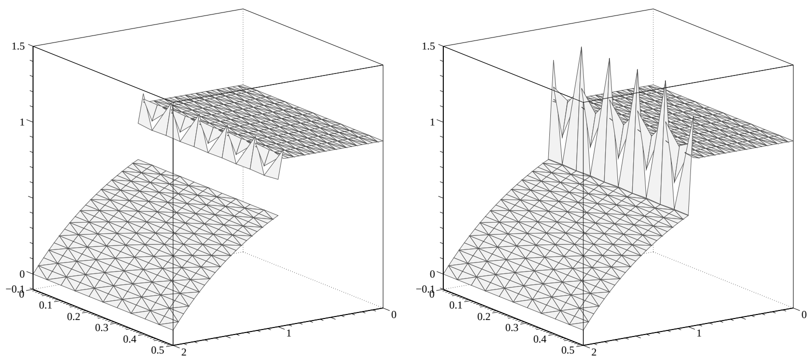

Fig. 1. The solutions computed by WIP (left) and IP (right) for $\epsilon_1 = 5\ 10^{-3}$.

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