Domain Decomposition Preconditioning for Discontinuous Galerkin Approximations of Convection-Diffusion Problems

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1 Introduction

In the classical Schwarz framework for conforming approximations of nonsymmetric and indefinite problems [5, 6] the finite element space is optimally decomposed into the sum of a finite number of uniformly overlapped, two-level subspaces. In each iteration step, a coarse mesh problem and a number of smaller linear systems, which correspond to the restriction of the original problem to subregions, are solved instead of the large original system of equations. Based on this decomposition, domain decomposition methods of three basic type—additive, multiplicative and hybrid Schwarz methods—have been studied in the literature (cf. [4, 5, 6]). In [1, 2] it was shown that for discontinuous Galerkin (DG) approximations of purely elliptic problems optimal nonoverlapping Schwarz methods (which have no analogue in the conforming case) can be constructed. Moreover, it was proved that they exhibit spectral bounds analogous to the one obtained with conforming finite element approximations in the case of "small" overlap, making Schwarz methods particularly well-suited for DG preconditioning. Motivated by the above considerations, we study a class of nonoverlapping Schwarz preconditioners for DG approximations of convection-diffusion equations. The *generalized minimal residual* (GMRES) Krylov space-based iterative solver is accelerated with the proposed preconditioners. We discuss the issue of convergence of the resulting preconditioned iterative method, and demonstrate through numerical computations that the classical Schwarz convergence theory cannot be applied to explain theoretically the converge observed numerically.

2 Statement of the Problem and its DG Approximation

Given a bounded polyhedral domain $\Omega \subseteq \mathbb{R}^d$, $d = 2,3$, $f \in L^2(\Omega)$, and $g \in H^{1/2}(\partial\Omega)$, we consider the following elliptic convection-diffusion problem with constant coefficients:

$$-\varepsilon\Delta u+\beta\cdot\nabla u=f \ \text{ in } \Omega, \quad u=g \ \text{ on } \Gamma\equiv\partial\Omega, \tag{1}$$

where $\varepsilon>0$ is the diffusion coefficient and $\beta\in\mathbb{R}^d$ is the velocity field.

We consider, for simplicity, shape-regular *quasi-uniform* partitions $\mathcal{T}_h$ of Ω with granularity $h>0$, where each $K\in\mathcal{T}_h$ is the affine image of a fixed master element $\mathcal{K}$, *i.e.*, $K=F_K(\mathcal{K})$, where $\mathcal{K}$ is either the open unit d-simplex or the open unit d-hypercube in $\mathbb{R}^d$, $d=2,3$. We denote by $\mathcal{F}_h$ the set of all faces of $\mathcal{T}_h$, and for $F\in\mathcal{F}_h$ we set $h_F=\mathrm{diam}(F)$. The symbol $\mathcal{F}_h^B$ will denote the set of all faces that lie on the boundary, Γ. For a given approximation order $\ell\geq 1$, we define the discontinuous Galerkin finite element space $V_h=\{v\in L^2(\Omega): v|_K\circ F_K\in\mathcal{M}^\ell(\mathcal{K}) \quad \forall K\in\mathcal{T}_h\}$, where $\mathcal{M}^\ell(\mathcal{K})$ is either the space of polynomials of degree at most ℓ on $\mathcal{K}$, if $\mathcal{K}$ is the reference d-simplex, or the space of polynomials of degree at most ℓ in each variable on $\mathcal{K}$, if $\mathcal{K}$ is the reference d-hypercube.

We denote by ∇_h the elementwise application of the operator ∇, and, for $v\in V_h$ and $K\in\mathcal{T}_h$, v^+ (respectively, v^-) denotes the interior (respectively, exterior) trace of v defined on ∂K (respectively, $\partial K\setminus\Gamma$). Given $K\in\mathcal{T}_h$, the inflow and outflow parts of ∂K are defined

$$\partial_- K := \{x\in\partial K : \beta(x)\cdot\mathbf{n}_K(x)<0\}, \quad \partial_+ K := \{x\in\partial K : \beta(x)\cdot\mathbf{n}_K(x)\geq 0\},$$

respectively, where $\mathbf{n}_K$ denotes the unit outward normal vector to ∂K.

For a parameter $\alpha\geq\alpha_{\min}>0$ (at our disposal), and adopting the standard notation $\{\{\cdot\}\}$ for the face-average and $[\![\cdot]\!]$ for the jump operator [3], we define the bilinear form $B_h(\cdot,\cdot): V_h\times V_h\to\mathbb{R}$ as

$$\begin{aligned}B_h(u,v) &= \int_\Omega \varepsilon\nabla_h u\cdot\nabla_h v\,\mathrm{d}x - \sum_{F\in\mathcal{F}_h}\int_F \{\{\varepsilon\nabla_h u\}\}\cdot[\![v]\!]\,\mathrm{d}s \\ &\quad - \sum_{F\in\mathcal{F}_h}\int_F [\![u]\!]\cdot\{\{\varepsilon\nabla_h v\}\}\,\mathrm{d}s + \sum_{F\in\mathcal{F}_h}\int_F \alpha\,\varepsilon\,h_F^{-1}\,[\![u]\!]\cdot[\![v]\!] - \int_\Omega u\,\beta\cdot\nabla_h v\,\mathrm{d}x \\ &\qquad + \sum_{K\in\mathcal{T}_h}\int_{\partial_+ K}(\beta\cdot\mathbf{n}_K)\,u^+v^+\,\mathrm{d}s + \sum_{K\in\mathcal{T}_h}\int_{\partial_- K\setminus\Gamma}(\beta\cdot\mathbf{n}_K)\,u^-v^+\,\mathrm{d}s.\end{aligned}$$

Then, the DG approximation of problem (1) reads as follows:

$$\text{Find } u_h\in V_h \text{ such that } B_h(u_h,v)=F_h(v)\ \forall v\in V_h, \tag{2}$$

where the functional $F_h(\cdot): V_h\to\mathbb{R}$ is given by

$$\begin{aligned}F_h(v) &:= \int_\Omega f\,v\,\mathrm{d}x + \sum_{F\in\mathcal{F}_h^B}\int_F \varepsilon\,g\,\nabla v^+\cdot\mathbf{n}_K\,\mathrm{d}s \\ &\quad + \sum_{F\in\mathcal{F}_h^B}\int_F \varepsilon\,\alpha\,h_F^{-1}g\,v^+\,\mathrm{d}s + \sum_{K\in\mathcal{T}_h}\int_{\partial_- K\cap\Gamma}(\beta\cdot\mathbf{n}_K)\,g v^+\,\mathrm{d}s.\end{aligned}$$

Given a basis of V_h, any function $v \in V_h$ is uniquely determined by a set of degrees of freedom. Here and in the following, we use boldface notation to denote elements of the spaces of degrees of freedom (vectors in $\mathbb{R}^{\mathrm{n}}$, and matrices in $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$). If $\mathbf{B}$ is the *stiffness matrix* associated with the bilinear form $B_h(\cdot,\cdot)$ and the given basis, problem (2) can be rewritten as the system of linear equations $\mathbf{Bu} = \mathbf{F}$. In order to solve this system of linear equations efficiently by a Krylov space-based iterative solver (such as, for example, the GMRES method), suitable preconditioners have to be employed to accelerate the iterative scheme.

3 Nonoverlapping Schwarz Methods

We consider three levels of *nested* partitions of the domain Ω satisfying the previous assumptions: a subdomain partition $\mathcal{T}_N$ consisting of N nonoverlapping subdomains Ω_i, a coarse partition $\mathcal{T}_H$ (with mesh size H) and a fine partition $\mathcal{T}_h$ (with mesh size h). Next we introduce the key ingredients of the definition of the Schwarz preconditioners.

Local Solvers. For $i = 1,\ldots,N$, we define the local DG spaces by

$$V_h^i := \{v \in V_h : v|_K = 0 \;\; \forall K \in \mathcal{T}_h,\; K \subset \Omega \setminus \Omega_i\}.$$

We note that a function in V_h^i is discontinuous and, as opposed to the case of conforming approximations, does not in general vanish on $\partial\Omega_i$. The classical extension (injection) operator from V_h^i to V_h is denoted by $R_i^{\mathrm{T}} : V_h^i \longrightarrow V_h$, $i = 1,\ldots,N$. We define the local solvers $\mathcal{B}_i : V_h^i \times V_h^i \longrightarrow \mathbb{R}$ as

$$\mathcal{B}_i(u_i,v_i) := B_h(R_i^{\mathrm{T}} u_i, R_i^{\mathrm{T}} v_i) \;\; \forall u_i, v_i \in V_h^i, \quad i = 1,\ldots,N.$$

Remark 1. Approximate local solvers, such as the ones proposed in [1, 2], could also be considered for the definition of the local components of the preconditioner.

Coarse Solver. For a given approximation order $0 \le p \le \ell$ we introduce the coarse space $V_H \equiv V_h^0 := \{v_0 \in L^2(\Omega) : v_0|_K \circ F_K \in \mathcal{M}^\ell(\mathcal{K}) \; \forall K \in \mathcal{T}_H\}$, and we define the *coarse solver* $\mathcal{B}_0 : V_h^0 \times V_h^0 \longrightarrow \mathbb{R}$ by

$$\mathcal{B}_0(u_0,v_0) := B_h(R_0^{\mathrm{T}} u_0, R_0^{\mathrm{T}} v_0) \quad \forall u_0, v_0 \in V_h^0,$$

where $R_0^{\mathrm{T}} : V_h^0 \longrightarrow V_h$ is the classical injection operator from V_h^0 to V_h.

For $0 \le i \le N$, let the projection operators $T_i : V_h \longrightarrow V_h^i \subset V_h$ be given by

$$\mathcal{B}_i(T_i u, v_i) := B_h(u, v_i) \quad \forall v_i \in V_h^i.$$

The additive and multiplicative Schwarz operator are defined by

$$T_{\mathrm{ad}} := \sum_{i=0}^{N} T_i, \quad T_{\mathrm{mu}} := I - (I - T_N)(I - T_{N-1}) \cdots (I - T_0),$$

respectively (cf. [5, 6]). The multiplicative Schwarz method is less amenable to parallelization than the additive method because the presence of the coarse solver T_0, which cannot be handled in parallel with the other local subproblem solvers, leads to a bottleneck for the whole algorithm. Motivated by the above observations, we also consider a *hybrid* operator in which the global operator T_0 is incorporated additively relative to the rest of the local solvers (see [4]):

$$T_{\mathrm{hy}} := T_0 + I - (I - T_N)(I - T_{N-1}) \cdots (I - T_1).$$

The Schwarz operators can be written as products of suitable preconditioners, namely $\mathbf{M}_{\mathrm{ad}}$, $\mathbf{M}_{\mathrm{mu}}$ or $\mathbf{M}_{\mathrm{hy}}$, and $\mathbf{B}$. Then, the Schwarz method consists of solving, by a suitable Krylov space-based iterative solver, the preconditioned system of equations $\mathbf{MBu} = \mathbf{MF}$, where $\mathbf{M}$ is either $\mathbf{M}_{\mathrm{ad}}$, $\mathbf{M}_{\mathrm{mu}}$ or $\mathbf{M}_{\mathrm{hy}}$.

4 The Issue of Convergence

The abstract analysis of Schwarz methods for conforming approximations to nonsymmetric elliptic problems, originally carried out by Cai and Widlund in [6], relies upon the GMRES convergence bounds of Eisenstat *et al.* [7]. According to [7], the GMRES method applied to the preconditioned system of equations does not stagnate (*i.e.*, the iterative method makes some progress in reducing the residual at each iteration step) provided that the symmetric part of T (where T is one of the Schwarz operators introduced in Sec. 3) is positive definite, and T is uniformly bounded. That is,

$$c_p(T) := \inf_{\substack{v \in V_h \\ v \neq 0}} \frac{\mathcal{S}_h(v, Tv)}{\mathcal{S}_h(v, v)} > 0, \quad C_p(T) := \sup_{\substack{v \in V_h \\ v \neq 0}} \frac{\|Tv\|_h}{\|v\|_h} \leq C, \tag{3}$$

where $\|\cdot\|_h$ is a suitable (mesh-dependent) norm on V_h in which the bilinear form $B_h(\cdot,\cdot)$ is continuous and coercive, and where $\mathcal{S}_h(\cdot,\cdot)$ denotes the symmetric part of $B_h(\cdot,\cdot)$. While the second condition can usually be shown to hold without difficulties, the first condition cannot, in general, be guaranteed. Indeed, as we demonstrate by numerical computations, $c_p(T)$ may be negative even in generic, non-pathological, cases. In Table 1 we show the computed values of $c_p(T_{\mathrm{ad}})$ and $c_p(T_{\mathrm{mu}})$ obtained with two choices of the global Péclet number $\mathtt{Pe} := \|\beta\|_\infty |\Omega| / \varepsilon$ (that relates the rate of convection of a flow to its rate of diffusion) for the first test case considered in Section 5. Even though GMRES applied to the preconditioned systems does not stagnate and, in fact, converges in only a few iterations (cf. Section 5), $c_p(T) < 0$ once the spacing of the fine grid is sufficiently small.

Remark 2. Closer inspection reveals that, in the case of elliptic convection-dominated diffusion equations, the theory in [6] is far from satisfactory since, on the one hand, it relies upon the GMRES bounds from [7] that only provide *sufficient* conditions for non-stagnation of GMRES and, on the other hand, it requires the skew-symmetric part of the operator to be "small" relative to the symmetric part (typically a low-order compact perturbation). Clearly, such a requirement cannot be satisfied in the

Table 1. Estimate of $c_p(T)$: $\ell = p = 1$, $N = 16$, Cartesian grids.

(a) $c_p(T_{\mathrm{ad}})$: $\varepsilon = 10^{-1}$, $\beta = (1,1)^{\mathrm{T}}$

$H \downarrow h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	0.077	-0.008	-0.047	-0.067
$H_0/2$	-	0.101	0.037	0.005
$H_0/4$	-	-	0.117	0.050
$H_0/8$	-	-	-	0.119

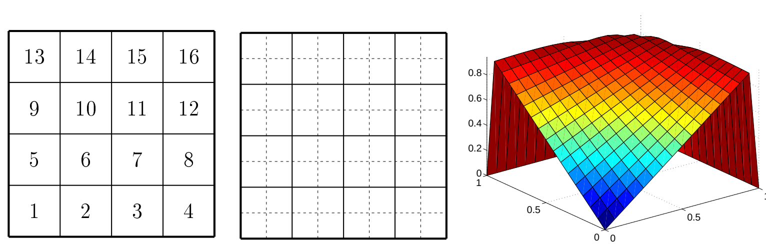
(b) $c_p(T_{\mathrm{mu}})$: $\varepsilon = 10^{-3}$, $\beta = (1,1)^{\mathrm{T}}$

$H \downarrow h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	0.225	-0.553	-1.484	-2.795
$H_0/2$	-	0.114	-0.628	-1.554
$H_0/4$	-	-	0.114	-0.570
$H_0/8$	-	-	-	0.077

convection-dominated case. Similar conclusions have been drawn in [1, 2] in the case of nonoverlapping preconditioners for nonsymmetric DG approximations of the Laplace operator (where the skew-symmetric part of the operator happens to be of the same order as the symmetric part).

Remark 3. The comments above also apply in to the case of *generous overlapping* partitions (cf. [8]) under suitable additional assumptions on the size of the coarse mesh, *i.e.*, $H < H_0$. Closer inspection reveals that H_0 strongly depends on the size of the global Péclet number, making the analysis inapplicable in the convection-dominated case.

5 Numerical Experiments



(a) (b) (c)

Fig. 1. (a) Subdomain ordering for $N = 16$; (b) initial coarse (solid line) and fine (dashed line) meshes; (c) the exact solution (4) for $\varepsilon = 10^{-2}$ (right).

We investigate the performance of our preconditioners while varying h, H and the Péclet number. We use a uniform subdomain partition of $\Omega = (0,1)^2$ consisting of 16 squares ordered as in Fig. 1(a). The initial coarse and fine refinements are depicted in Fig. 1(b). We denote by H_0 and h_0 the corresponding initial coarse and fine mesh sizes, respectively, and we consider $n = 1,2,3$ successive uniform refinements of the initial grids. The linear systems of equations have been solved by GMRES with a (relative) tolerance set equal to 10^{-6} allowing a maximum of 100 (respectively, 600) iterations for the preconditioned (respectively, unpreconditioned) systems.

We set $\beta = (1,1)^{\mathrm{T}}$ and adjust the source term f and the boundary condition so that the exact solution is given by

$$u(x,y) = x + y - xy + \frac{1}{1-\mathrm{e}^{-1/\varepsilon}}\left[\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-(1-x)(1-y)/\varepsilon}\right]. \tag{4}$$

We note that for $0 < \varepsilon \ll 1$, *i.e.*, for $\mathtt{Pe} \gg 1$, the solution exhibits boundary layers along $x = 1$ and $y = 1$ (cf. Fig. 1(c) for $\varepsilon = 10^{-2}$).

Table 2. GMRES iteration counts: $\varepsilon = 1$.

	Additive				Multiplicative				Hybrid			
$H\downarrow\ h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	20	30	40	54	8	13	17	24	11	15	20	27
$H_0/2$	-	19	27	37	-	7	10	13	-	11	15	20
$H_0/4$	-	-	20	28	-	-	6	8	-	-	12	17
$H_0/8$	-	-	-	19	-	-	-	5	-	-	-	12
#iter($\mathbf{B}$)	58	109	204	371	58	109	204	371	58	109	204	371

Table 3. GMRES iteration counts: $\varepsilon = 10^{-1}$.

	Additive				Multiplicative				Hybrid			
$H\downarrow\ h\rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	23	34	48	62	11	15	21	29	12	17	24	30
$H_0/2$	-	20	30	41	-	8	11	16	-	12	16	20
$H_0/4$	-	-	21	29	-	-	7	10	-	-	12	17
$H_0/8$	-	-	-	19	-	-	-	6	-	-	-	11
#iter($\mathbf{B}$)	59	110	209	396	59	110	209	396	59	110	209	396

We compare the GMRES iteration counts for the additive, multiplicative and hybrid Schwarz preconditioners for different values of the Péclet number, working for the sake of simplicity with approximations with $\ell = p = 1$. The computed iteration counts obtained for $\varepsilon = 1, 10^{-1}, 10^{-3}, 10^{-4}$ are shown in Tables 2–5, respectively. Clearly, the multiplicative and the hybrid Schwarz preconditioners perform far better than the additive preconditioner. The results in Tables 2–5 show that for small Péclet numbers the iteration counts seem to increase with the Péclet number; whereas, whenever the problem becomes convection-dominated, *i.e.*, for $\mathtt{Pe} \gg 1$, the iteration counts needed for achieving the fixed tolerance decrease with the increase of the Péclet number. Moreover, in the convection-dominated regime the performance of the additive nonoverlapping preconditioner is comparable with the one in [8] in

Table 4. GMRES iteration counts: $\varepsilon = 10^{-3}$.

	Additive				Multiplicative				Hybrid			
$H \downarrow\ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	15	21	26	33	6	8	10	14	8	10	12	16
$H_0/2$	-	17	24	32	-	5	8	13	-	8	11	15
$H_0/4$	-	-	18	27	-	-	6	10	-	-	9	14
$H_0/8$	-	-	-	20	-	-	-	5	-	-	-	10
#iter(**B**)	41	68	115	213	41	68	115	213	41	68	115	213

Table 5. GMRES iteration counts: $\varepsilon = 10^{-4}$.

	Additive				Multiplicative				Hybrid			
$H \downarrow\ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	14	16	17	18	3	4	4	6	6	6	7	8
$H_0/2$	-	14	16	18	-	3	4	5	-	6	6	7
$H_0/4$	-	-	14	17	-	-	3	4	-	-	6	7
$H_0/8$	-	-	-	14	-	-	-	4	-	-	-	6
#iter(**B**)	40	67	119	215	40	67	119	215	40	67	119	215

the overlapping case, making the nonoverlapping version competitive in practical applications.

Table 6. GMRES iteration counts: multiplicative and hybrid (between parenthesis) Schwarz preconditioners.

	$\varepsilon = 10^{-1}$				$\varepsilon = 10^{-4}$			
$H \downarrow\ h \rightarrow$	h_0	$h_0/2$	$h_0/4$	$h_0/8$	h_0	$h_0/2$	$h_0/4$	$h_0/8$
H_0	13 (15)	20 (21)	27 (29)	36 (39)	5 (8)	6 (9)	8 (10)	10 (12)
$H_0/2$	-	11 (15)	16 (19)	21 (26)	-	5 (7)	6 (10)	9 (12)
$H_0/4$	-	-	10 (13)	14 (19)	-	-	5 (9)	7 (11)
$H_0/8$	-	-	-	8 (11)	-	-	-	6 (10)
#iter(**B**)	79	152	290	551	39	65	113	203

Finally, we investigate the effect of the subdomain ordering on the performance of the Schwarz preconditioner. We set $\beta = (-1,-1)^{\mathrm{T}}$, and we choose as exact solution one that is analogous to the exact solution considered so far but now such that u exhibits boundary layers along $x = 0$ and $y = 0$ for $0 < \varepsilon \ll 1$, so that the subdomains turn out to be ordered "downwind" (cf. Fig. 1(a)). In Table 6 we report the GMRES iteration counts obtained with the multiplicative and hybrid (in parenthesis) Schwarz method using $\ell = p = 1$. As expected, the subdomain ordering does affect the performance of the preconditioner and "downwind" ordering of subdomains can lead to an increase in the number of GMRES iterations.

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