Non-overlapping Domain Decomposition for the Richards Equation via Superposition Operators

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Summary. Simulations of saturated-unsaturated groundwater flow in heterogeneous soil can be carried out by considering non-overlapping domain decomposition problems for the Richards equation in subdomains with homogeneous soil. By the application of different Kirchhoff transformations in the different subdomains local convex minimization problems can be obtained which are coupled via superposition operators on the interface between the subdomains. The purpose of this article is to provide a rigorous mathematical foundation for this reformulation in a weak sense. In particular, this involves an analysis of the Kirchhoff transformation as a superposition operator on Sobolev and trace spaces.

1 Introduction

The Richards equation, which describes saturated-unsaturated fluid flow in a homogeneous porous medium, reads

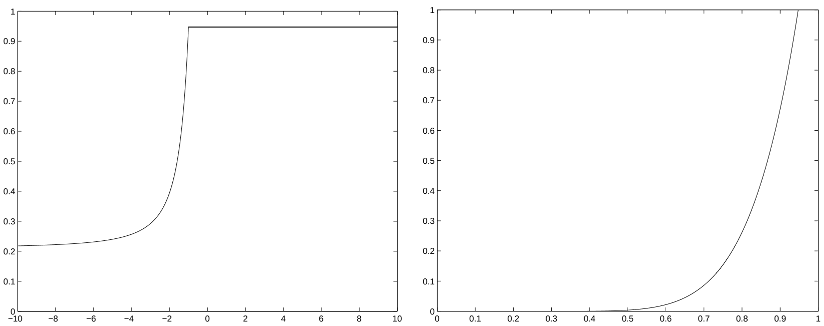
$$n\theta(p)_t - \operatorname{div}(K_h kr(\theta(p))(\nabla p - z)) = 0\,. \tag{1}$$

The unknown water or capillary pressure p, given as the height of a corresponding water column, is a function on $\Omega \times (0,T)$ for a time $T > 0$ and a domain $\Omega \subset \mathbb{R}^d$ ($d = 1,2,3$) inhibited by the porous medium. The function $n : \Omega \to (0,1)$ is the porosity of the soil, $K_h : \Omega \to \mathbb{R}^+$ is the hydraulic conductivity and z is the coordinate in the direction of gravity.

The saturation $\theta : \mathbb{R} \to [\theta_m, \theta_M]$ with $\theta_m, \theta_M \in [0,1]$ is an increasing function of p with $\theta(p) = \theta_M$ (the case of full saturation and ellipticity of (1)) if p is large enough. The relative permeability $kr : [\theta_m, \theta_M] \to [0,1]$ is an increasing function of θ with $kr(\theta_m) = 0$ (degeneracy in (1)) and $kr(\theta_M) = 1$. In this way the Richards equation contains the generalized law of Darcy

$$\mathbf{v} = -K_h kr(\theta(p))(\nabla p - z)\,,$$

for the water flux $\mathbf{v}$. Typical shapes of the nonlinearities θ and kr are depicted in Figs. (a) and (b). However, these functions depend on the soil type so that we



(a) $p \mapsto \theta(p)$ (b) $\theta \mapsto kr(\theta)$

have different nonlinearities θ_i, kr_i on different non-overlapping subdomains Ω_i, $i = 1, \ldots, N \in \mathbb{N}$, constituting a decomposition of Ω.

In the following, we assume $n = K_h = 1$ and $N = 2$ for simplicity. See Figure 1 for a decomposition of Ω into Ω_1 and Ω_2 where $\mathbf{n}$ denotes the outer normal of Ω_1. Moreover, we assume that (1) is discretized implicitly in time but with an explicit

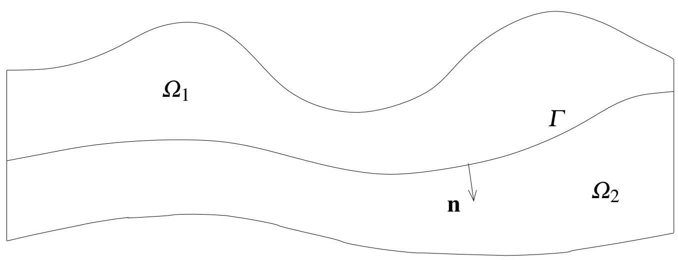


Fig. 1. 2D-domain Ω decomposed into two subdomains.

treatment of the gravitational (convective) term so that with a suitable function f on Ω we arrive at spatial problems of the form

$$\theta_i(p_i) - \operatorname{div}(kr_i(\theta_i(p_i))\nabla p_i) = f \quad \text{on } \Omega_i, \quad i = 1, 2. \tag{2}$$

Appropriate interface conditions on $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$, which are motivated hydrologically, are the continuity of the pressure and the normal water flux $\mathbf{v} \cdot \mathbf{n}$ across Γ. After our implicit–explicit time discretization, this leads to

$$p_1 = p_2 \quad \text{on } \Gamma, \tag{3}$$

$$kr_1(\theta_1(p_1))\nabla p_1 \cdot \mathbf{n} = kr_2(\theta_2(p_2))\nabla p_2 \cdot \mathbf{n} \quad \text{on } \Gamma. \tag{4}$$

In case of $\theta_1 = \theta_2$ and $kr_1 = kr_2$, these interface conditions can be mathematically derived in a weak sense (and in a very general setting) as a multi-domain formulation for the corresponding global problem, see [2, pp. 131–139].

A powerful tool for the treatment of the Richards equation is Kirchhoff's transformation. It leads to spatial convex minimization problems after time discretization (see [2] for details). Here, we need to apply two different Kirchhoff transformations in the two subdomains. More concretely, we define

$$u_i(x) := \kappa_i(p_i(x)) = \int_0^{p_i(x)} kr_i(\theta_i(q))\,dq \quad \text{a.e. on } \Omega_i\,, \quad i = 1,2\,. \tag{5}$$

Consequently, we obtain

$$kr_i(\theta_i(p_i))\nabla p_i = \nabla u_i\,, \quad i = 1,2\,, \tag{6}$$

by the chain rule so that with the saturation

$$M_i(u_i) = \theta_i(\kappa_i^{-1}(u_i))\,, \quad i = 1,2\,, \tag{7}$$

with respect to the new variables the equations (2) are transformed into

$$M_i(u_i) - \Delta u_i = f \quad \text{on } \Omega_i\,, \quad i = 1,2\,. \tag{8}$$

Moreover, the Kirchhoff–transformed interface conditions read

$$\kappa_1^{-1}(u_1) = \kappa_2^{-1}(u_2) \quad \text{on } \Gamma\,, \tag{9}$$

$$\nabla u_1 \cdot \mathbf{n} = \nabla u_2 \cdot \mathbf{n} \quad \text{on } \Gamma\,. \tag{10}$$

Accordingly, boundary conditions on $\partial\Omega$ for (1) and (2) are transformed.

Applying Kirchhoff's transformation is straightforward in the strong formulations above. However, regarding the weak forms, the proof for the equivalence of the physical and the transformed versions is more sophisticated. For example, we need the chain rule (6) in a weak sense in $H^1(\Omega_i)$. Furthermore, $\kappa_i^{-1}(u_i)$, $i = 1,2$, in (9) has to be understood as an element of some trace space. In order to clarify these issues, which already occur in case of a single domain, one has to study the Kirchhoff transformation as a superposition operator in Sobolev and trace spaces. This is the purpose of this paper.

Concretely, we present weak forms of the domain decomposition problems for the time-discretized Richards equation and its transformed version in Section 2. Then we carry out some analysis for the Kirchhoff transformation as a superposition operator in Section 3. Finally, the obtained results are exploited to prove the equivalence of the weak formulations in Section 4.

2 Weak Forms of the Domain Decomposition Problems

In this section we give variational formulations of the domain decomposition problems (2)–(4) and (8)–(10) with homogeneous Dirichlet boundary conditions (compare [3]). We start with some notation and assumptions.

We require $kr_i \in L^\infty(\mathbb{R})$ with $kr_i \geq \alpha$ for some $\alpha > 0$ and $i = 1,2$. (For the general case $\alpha = 0$ as in Figs. 1(a) and 1(b), the results are weaker; see [2, Sec. 1.5.4]). Let θ_i, $i = 1,2$, be bounded Borel–measurable functions on $\mathbb{R}$ and $f \in L^2(\Omega)$. Furthermore, in a decomposition as above, let Ω and Ω_i, $i = 1,2$, be bounded Lipschitz domains in $\mathbb{R}^d$ and Γ a Lipschitz $(d-1)$-dimensional manifold. Now we introduce the spaces

$$V_i := \{v_i \in H^1(\Omega_i) | \, v_{i|\partial\Omega\cap\partial\Omega_i} = 0\}, \quad V_i^0 := H_0^1(\Omega_i), \quad \Lambda := \{v_{|\Gamma} : v \in H_0^1(\Omega)\},$$

and for $w_i, v_i \in V_i$, the forms

$$a_i(w_i, v_i) := (\nabla w_i, \nabla v_i)_{\Omega_i}\,, \qquad b_i(w_i, v_i) := (kr_i(\theta_i(w_i))\nabla w_i, \nabla v_i)_{\Omega_i}\,,$$

where $(\cdot,\cdot)_{\Omega_i}$ stands for the L^2–scalar product on Ω_i. The norm in $H^1(\Omega)$ will be denoted by $\|\cdot\|_1$. Recall that the trace space Λ is either $H_{00}^{1/2}(\Gamma)$ in case of $\Gamma \cap \partial\Omega \neq \emptyset$ (as in Figure 1) or $H^{1/2}(\Gamma)$ otherwise [8, p. 7]. The restriction $w_{i|\Gamma}$ of a function $w_i \in V_i$ on the interface Γ has to be understood as the application of the corresponding trace operator on w_i.

Finally, let R_i, $i = 1,2$, be any continuous extension operator from Λ to V_i. Then the variational formulation of problem (2)–(4) with homogeneous Dirichlet boundary conditions reads as follows:

Find $p_i \in V_i$, $i = 1,2$, such that

$$(\theta_i(p_i), v_i)_{\Omega_i} + b_i(p_i, v_i) = (f, v_i)_{\Omega_i} \quad \forall v_i \in V_i^0, \;\; i = 1,2\,, \tag{11}$$

$$p_{1|\Gamma} = p_{2|\Gamma} \qquad \text{in } \Lambda\,, \tag{12}$$

$$\begin{aligned}(\theta_1(p_1), R_1\mu)_{\Omega_1} + b_1(p_1, R_1\mu) - (f, R_1\mu)_{\Omega_1} = \\ -(\theta_2(p_2), R_2\mu)_{\Omega_2} - b_2(p_2, R_2\mu) + (f, R_2\mu)_{\Omega_2} \quad \forall \mu \in \Lambda\,.\end{aligned} \tag{13}$$

Analogously, the weak formulation of the transformed problem (8)–(10) with homogeneous Dirichlet boundary conditions reads:

Find $u_i \in V_i$, $i = 1,2$, such that

$$(M_i(u_i), v_i)_{\Omega_i} + a_i(u_i, v_i) = (f, v_i)_{\Omega_i} \qquad \forall v_i \in V_i^0, \;\; i = 1,2\,, \tag{14}$$

$$\kappa_1^{-1}(u_{1|\Gamma}) = \kappa_2^{-1}(u_{2|\Gamma}) \quad \text{in } \Lambda\,, \tag{15}$$

$$\begin{aligned}(M_1(u_1), R_1\mu)_{\Omega_1} + a_1(u_1, R_1\mu) - (f, R_1\mu)_{\Omega_1} = \\ -(M_2(u_2), R_2\mu)_{\Omega_2} - a_2(u_2, R_2\mu) + (f, R_2\mu)_{\Omega_2} \quad \forall \mu \in \Lambda\,.\end{aligned} \tag{16}$$

The rest of this paper is devoted to prove the equivalence of the variational formulations (11)–(13) and (14)–(16).

3 Kirchhoff Transformation as a Superposition Operator

The difficulties encountered to prove the equivalence of the weak forms in physical and in transformed variables already occur for a single domain. Therefore, we omit the indices $i \in \{1,2\}$ in this section in which we want to address these difficulties. We start with an important definition [1].

Definition 1. *Let p be a real-valued function defined on a subset $S \subset \mathbb{R}^d$, possibly almost everywhere w.r.t. an appropriate measure. Furthermore, let $\kappa : \mathbb{R} \to \mathbb{R}$ be a real function. The* superposition operator *(or* Nemytskij operator*) $\kappa_S : p \mapsto \kappa(p)$ is defined by pointwise application*

$$(\kappa_S(p))(x) := \kappa(p(x)),$$

of κ to p (for x almost everywhere) on S. Let X be a normed space consisting of a subset of all measurable functions on S. If the superposition operator satisfies $\kappa_S(p) \in X$ for all $p \in X$, we say that it acts on *the space X. In this case we write $\kappa_X : X \to X$ for the restriction of κ_S on the space X and call κ_X superposition operator on X (induced by κ).*

Here, S will be either Ω or a submanifold Σ of $\partial\Omega$. If not otherwise stated, we assume the conditions listed at the beginning of Section 2 and the Kirchhoff transformation κ given as in (5). We begin by stating the weak chain rule which goes back to J. Serrin (see [5]). Recall that $\kappa' = kr \circ \theta \in L^\infty(\mathbb{R})$ holds for any Lipschitz continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ due to the fundamental theorem of calculus.

Theorem 1. *If $\kappa : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous then the weak chain rule*

$$\kappa'(p)\nabla p = \nabla(\kappa(p)) \quad \text{a.e. on } \Omega\,,$$

holds for any $p \in W^{1,1}_{\mathrm{loc}}(\Omega)$ provided $\kappa'(p(x))\nabla p(x)$ is interpreted as 0 whenever $\nabla p(x) = 0$.

We remark that the last condition is an essential part of the theorem since $\kappa'(p(x))$ does not have to be defined for any $x \in \Omega$. Indeed, for $kr \in L^\infty(\mathbb{R})$ the composition $kr \circ \theta(p)$ alone does not make sense for $p \in W^{1,1}_{\mathrm{loc}}(\Omega)$ since it depends on the choice of the representative in the equivalence class kr.

The next lemma is not hard to prove (see [2, Sec. 1.5.4]), however, we must apply the weak chain rule twice in order to obtain (iii).

Lemma 1. *The Kirchhoff transformation κ has the following properties.*

(i) *$\kappa : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and has a Lipschitz continuous inverse.*
(ii) *$\kappa : \mathbb{R} \to \mathbb{R}$ and $\kappa^{-1} : \mathbb{R} \to \mathbb{R}$ induce Lipschitz continuous superposition operators acting on $L^2(\Omega)$ and on $L^2(\Sigma)$ for any submanifold $\Sigma \subset \partial\Omega$.*
(iii) *$\kappa : \mathbb{R} \to \mathbb{R}$ induces an invertible superposition operator on $H^1(\Omega)$ with*

$$\alpha^{-1}\|p\|_1 \le \|\kappa(p)\|_1 \le \|kr \circ \theta\|_\infty \|p\|_1 \quad \forall p \in H^1(\Omega)\,.$$

By imposing further conditions on the function $kr \circ \theta$, e.g. its boundedness and uniform continuity, the continuity of the superposition operator $\kappa_{H^1(\Omega)}$ can be proved by elementary means (compare [2, Prop. 1.5.14]) — if one assumes $kr \circ \theta$ to be Lipschitz continuous, one even obtains local Lipschitz continuity of $\kappa_{H^1(\Omega)}$ in one space dimension.

The following remarkable characterization of superposition operators acting on $H^1(\Omega)$, however, is a quite profound result, see Marcus and Mizel [6, 7].

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $\kappa : \mathbb{R} \to \mathbb{R}$ a Borel function. The superposition operator κ_Ω acts on $H^1(\Omega)$ if and only if it is continuous on $H^1(\Omega)$ or, equivalently, if and only if κ is Lipschitz continuous for $d > 1$ or locally Lipschitz in the case $d = 1$, respectively.*

The following proposition contains an important commutativity result. Strangely enough, in order to derive this algebraic property, it seems necessary to assume the continuity of $\kappa_{H^1(\Omega)}$. In the proof we also apply the well-known trace theorem for trace operators $tr_\Sigma : H^1(\Omega) \to H^{1/2}(\Sigma)$ (compare e.g. [4, pp. 1.61, 1.65]).

Proposition 1. *For a submanifold $\Sigma \subset \partial\Omega$ and κ as in Theorem 2, we have the commutativity*

$$\kappa_\Sigma(tr_\Sigma v) = tr_\Sigma(\kappa_\Omega v) \quad \forall v \in H^1(\Omega)\,. \tag{17}$$

Proof. We prove that for any $v \in H^1(\Omega)$

$$\|tr_\Sigma(\kappa_\Omega v) - \kappa_\Sigma(tr_\Sigma v)\|_{L^2(\Omega)} \tag{18}$$

is arbitrarily small by considering a sequence $(v_n)_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ converging to v in $H^1(\Omega)$. In fact, since Theorem 2 provides the continuity of κ and the trace of a continuous function on Σ coincides with its restriction to Σ, the norm in (18) can be estimated by

$$\|tr_\Sigma(\kappa_\Omega v) - (\kappa_\Omega v_n)_{|\Sigma}\|_{L^2(\Omega)} + \|\kappa_\Sigma(v_{n|\Sigma}) - \kappa_\Sigma(tr_\Sigma v)\|_{L^2(\Omega)}\,. \tag{19}$$

The first term in (19) is at most

$$\|tr_\Sigma\|\,\|\kappa_\Omega v - \kappa_\Omega v_n\|_1\,,$$

due to the trace theorem, and this estimate goes to 0 for $n \to \infty$ by the continuity of $\kappa_{H^1(\Omega)}$. For $d > 1$ where $\kappa : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, the second term in (19) can be estimated by

$$L(\kappa_{L^2(\Sigma)})\,\|v_{n|\Sigma} - tr_\Sigma v\|_{L^2(\Sigma)} \le L(\kappa_{L^2(\Sigma)})\,\|tr_\Sigma\|\,\|v_n - v\|_1\,,$$

with Lemma 1 (ii) ($L(\kappa_{L^2(\Sigma)})$ denotes the Lipschitz constant of $\kappa_{L^2(\Sigma)}$) and the trace theorem and, therefore, tends to 0 for $n \to \infty$, too. In one space dimension, (17) is clear since both κ (Theorem 2) and v (Sobolev's embedding theorem) are continuous. $\square$

Note that Proposition 1 does not guarantee $\kappa_\Sigma(tr_\Sigma v) \in H_{00}^{1/2}(\Sigma)$ for $tr_\Sigma v \in H_{00}^{1/2}(\Sigma)$. However, we even have

Proposition 2. *For a submanifold $\Sigma \subset \partial\Omega$ the function κ as in Theorem 2 induces a continuous superposition operator on $H^{1/2}(\Sigma)$ and, if $\kappa(0) = (0)$, on $H_{00}^{1/2}(\Sigma)$, too.*

Proof. With the continuous extension operator $R_\Sigma : H^{1/2}(\Sigma) \to H^1(\Omega)$ given by the trace theorem and using Proposition 1, we can write

$$\kappa_\Sigma = \kappa_\Sigma \circ tr_\Sigma \circ R_\Sigma = tr_\Sigma \circ \kappa_{H^1(\Omega)} \circ R_\Sigma \,,$$

and the operator on the right hand side is a composition of continuous operators which obviously acts on $H^{1/2}(\Sigma)$.

Regarding the second case we recall (see [4, p. 1.60]) that $H_{00}^{1/2}(\Sigma)$ is the space of all functions $\mu \in H^{1/2}(\Sigma)$ allowing trivial extensions $\tilde{\mu} \in H^{1/2}(\partial\Omega)$ with the norm

$$\|\mu\|_{H_{00}^{1/2}(\Sigma)} = \|\tilde{\mu}\|_{H^{1/2}(\partial\Omega)} \,. \tag{20}$$

Now, let $\eta \in H_{00}^{1/2}(\Sigma)$ and $\tilde{\eta}$ be a trivial extension of η in $H^{1/2}(\partial\Omega)$. Then, since $\kappa(0) = 0$ and $\kappa_{\partial\Omega}$ acts on the space $H^{1/2}(\partial\Omega)$, we can conclude $\kappa_{\partial\Omega}(\tilde{\eta}) \in H^{1/2}(\partial\Omega)$ and $\kappa_{\partial\Omega}(\tilde{\eta})_{|\Sigma}$ is a trivial extension of $\kappa_\Sigma(\eta) \in H^{1/2}(\Sigma)$, i.e. by definition $\kappa_\Sigma(\eta) \in H_{00}^{1/2}(\Sigma)$. Moreover, if $\mu \in H_{00}^{1/2}(\Sigma)$ is treated as η, then $\kappa_{\partial\Omega}(\tilde{\eta}) - \kappa_{\partial\Omega}(\tilde{\mu}) \in H^{1/2}(\partial\Omega)$ is a trivial extension of $\kappa_\Sigma(\eta) - \kappa_\Sigma(\mu) \in H_{00}^{1/2}(\Sigma)$. Now, (20) and the continuity of $\kappa_{\partial\Omega}$ provide that, for any $\varepsilon > 0$, we have

$$\|\kappa_\Sigma(\eta) - \kappa_\Sigma(\mu)\|_{H_{00}^{1/2}(\Sigma)} = \|\kappa_{\partial\Omega}(\tilde{\eta}) - \kappa_{\partial\Omega}(\tilde{\mu})\|_{H^{1/2}(\partial\Omega)} \leq \varepsilon \,,$$

if $\|\tilde{\eta} - \tilde{\mu}\|_{H^{1/2}(\partial\Omega)} = \|\eta - \mu\|_{H_{00}^{1/2}(\Sigma)} \leq \delta$ holds with a suitable $\delta > 0$. $\square$

For completeness we remark that Proposition 2 also holds for the trace space $H_0^{1/2}(\Sigma)$, see [2, Prop. 1.5.17].

4 Equivalence of the Weak Formulations

We are now in a position to prove our main result.

Theorem 3. *With the assumptions on θ_i and kr_i, $i = 1,2$, the domain decomposition problem (11)–(13) is equivalent to its transformed version (14)–(16).*

Proof. The following statements are all valid for $i = 1,2$. First, Lemma 1 (iii) provides

$$p_i \in H^1(\Omega_i) \quad \Longleftrightarrow \quad u_i \in H^1(\Omega_i) \,.$$

Therefore, using (5), by Proposition 1 we can conclude

$$u_{i|\partial\Omega\cap\partial\Omega_i} = \kappa_i(p_i)_{|\partial\Omega\cap\partial\Omega_i} = \kappa_i(p_{i|\partial\Omega\cap\partial\Omega_i}) = \kappa_i(0) = 0 \,,$$

i.e. $u_i \in V_i$ if $p_i \in V_i$. In light of Lemma 1 (i), the converse is true, too.

Now, since θ_i are bounded Borel–measurable functions on $\mathbb{R}$ we have

$$\theta_i(p_i(x)) = M_i(u_i(x)) \quad \text{a.e. on } \Omega_i \,, \tag{21}$$

due to (7) for all $p_i \in V_i$ with $u_i = \kappa_i(p_i)$, and the functions given in (21) are Lebesgue–measurable L^∞–functions on Ω_i. Therefore, the L^2–scalar products, which correspond to each other in (11) and (14) as well as in (13) and (16), respectively, are equivalently reformulated.

Furthermore, the equivalent reformulation of the terms $b_i(\cdot,\cdot)$ in (11) and (13) into the terms $a_i(\cdot,\cdot)$ in (14) and (16), respectively, is provided by the identity

$$kr_i(\theta_i(p_i))\nabla p_i = \kappa_i'(p_i)\nabla p_i = \nabla u_i \quad \text{a.e. on } \Omega_i\,,$$

understood as functions in $(L^2(\Omega_i))^d$. This is a consequence of Theorem 1.

Finally, the equivalence (12)$\Leftrightarrow$(15) requires the commutativity

$$\kappa_i^{-1}(u_i)_{|\Gamma} = \kappa_i^{-1}(u_{i|\Gamma}) \quad \text{in } \Lambda\,,$$

which is obtained by Proposition 1 and 2. □

We close this investigation by noting that, in addition to Dirichlet and Neumann boundary conditions, which have been considered above, boundary conditions of "Signorini-type" can also be suitably Kirchhoff–transformed in a weak sense. However, as in the degenerate case $\alpha = 0$, one can no longer establish the full equivalence result, compare [2, Thm. 1.5.18].

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