From the Boundary Element Domain Decomposition Methods to Local Trefftz Finite Element Methods on Polyhedral Meshes

Dylan Copeland[1], Ulrich Langer[2], and David Pusch[3]

[1] Institute of Computational Mathematics, Johannes Kepler University Linz, dylan.copeland@jku.at
[2] Institute of Computational Mathematics, Johannes Kepler University Linz, ulanger@numa.uni-linz.ac.at
[3] RICAM, Austrian Academy of Sciences, david.pusch@oeaw.ac.at

Summary. We derive and analyze new boundary element (BE) based finite element discretizations of potential-type, Helmholtz and Maxwell equations on arbitrary polygonal and polyhedral meshes. The starting point of this discretization technique is the symmetric BE Domain Decomposition Method (DDM), where the subdomains are the finite elements. This can be interpreted as a local Trefftz method that uses PDE-harmonic basis functions. This discretization technique leads to large-scale sparse linear systems of algebraic equations which can efficiently be solved by Algebraic Multigrid (AMG) methods or AMG preconditioned conjugate gradient methods in the case of the potential equation and by Krylov subspace iterative methods in general.

1 Introduction

We introduce new finite element methods based on the symmetric boundary element domain decomposition method presented in [5], which can be applied with general polygonal or polyhedral meshes. That is, each element of the mesh may be any polygon or polyhedron, since we treat the elements as subdomains. There are many important practical applications where one wants to discretize PDEs on such kinds of meshes without further decomposition of the polyhedra, see e.g. [6] and [1]. Boundary integral operators are utilized to obtain a method which solves for traces of the solution on the element boundaries, from which the solution may be obtained via a representation formula. Simple, low-order boundary element spaces are used to approximate traces on the element surfaces, yielding a finite element method with PDE-harmonic basis functions.

Since boundary integral operators are used only locally, piecewise constant coefficients are admissible, and the coupling of boundary element functions is local. Consequently, sparse linear systems are obtained, which can be solved by Krylov iterative methods. For the potential equation, the resulting system is symmetric and

positive definite, and algebraic multigrid (see [9]) is a very effective preconditioner in the conjugate gradient solver.

2 The Potential Equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz boundary $\Gamma = \partial\Omega$, and let $d \in \{2,3\}$ be the dimension of the computational domain Ω. In this section, we assume for simplicity that $d = 2$. As a model problem, we consider the Dirichlet boundary value problem (BVP) for the potential equation

$$-\mathrm{div}(a(x)\nabla u(x)) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma. \tag{1}$$

We assume that the coefficient a is piecewise constant, $f \in L_2(\Omega)$, and $g \in H^{1/2}(\Gamma)$. Further, we suppose that there is a non–overlapping decomposition of our domain Ω into e_h shape-regular polygonal elements Ω_i such that

$$\overline{\Omega} = \bigcup_{i=1}^{e_h} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \quad \Gamma_i = \partial\Omega_i, \quad \overline{\Gamma}_{ij} = \overline{\Gamma}_i \cap \overline{\Gamma}_j \tag{2}$$

and that $a(x) = a_i > 0$ for $x \in \Omega_i$, $i = 1,\dots,e_h$. The domain Ω is assumed to be scaled in such a way that $\mathrm{diam}(\Omega_i) = O(h) \leq h_0 < 1/2$ for all $i = 1,\dots,e_h$. Under the assumptions made above, there obviously exists a unique weak solution $u \in H^1(\Omega)$ of the BVP (1).

Using the local Dirichlet-to-Neumann map

$$a_i \partial u/\partial \nu_i = a_i S_i u_{|\Gamma_i} - N_i f \quad \text{on } \Gamma_i, \tag{3}$$

we observe that the variational formulation of (1) is equivalent to the associated variational formulation on the skeleton $\Gamma_S = \Gamma_{S,h} = \cup_{i=1}^{e_h} \Gamma_i$ (see, e.g., [7]): find $u \in H^{1/2}(\Gamma_S)$ with $u = g$ on Γ such that

$$\sum_{i=1}^{e_h} \int_{\Gamma_i} a_i (S_i u_i)(x)\, v_i(x) ds_x = \sum_{i=1}^{e_h} \int_{\Gamma_i} (N_i f(x)) v_i(x) ds_x \tag{4}$$

for all $v \in H_0^{1/2}(\Gamma_S)$, where $u_i = u_{|\Gamma_i}$ and $v_i = v_{|\Gamma_i}$ denote the traces of u and v on Γ_i, respectively. The Steklov–Poincaré operator S_i and the Newton potential operator N_i have different representations (see again [7]). Here we are using the symmetric representation

$$S_i = D_i + (\frac{1}{2} I + K_i')V_i^{-1}(\frac{1}{2} I + K_i) : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i) \tag{5}$$

of the local Steklov–Poincaré operator S_i via the local single layer potential integral operator $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, the local double layer potential operator

$K_i : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, its adjoint $K_i' : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$, and the local hypersingular boundary integral operator $D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$, see, e.g., [11] for the definition and properties of these boundary integral operators. The operator N_i is defined by the equation

$$N_i = V_i^{-1}\widetilde{N}_{i,0} : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i), \tag{6}$$

where the Newton potential operator $\widetilde{N}_{i,0}$ is given by the relation

$$(\widetilde{N}_{i,0}f)(x) = \int_{\Omega_i} U^*(x-y)f(y)dy, \;\; x \in \Gamma_i. \tag{7}$$

Here $U^*(x) = -(1/2\pi)\log|x|$ and $U^*(x) = 1/(4\pi|x|)$ denotes the fundamental solution of the negative Laplace operator $-\Delta$ for $d = 2$ and $d = 3$, respectively.

For simplicity (higher-order versions can be constructed in the same way), we use continuous piecewise linear boundary element functions for approximating the potential u on the skeleton Γ_S and piecewise constant boundary element functions for approximating the normal derivatives $t_i = \partial u/\partial \nu_i$ on the boundary Γ_i of the polygonal element Ω_i. This yields the element stiffness matrices

$$\mathbf{S}_{i,h} = a_i\mathbf{D}_{i,h} + a_i\big(0.5\,\mathbf{I}_{i,h}^\top + \mathbf{K}_{i,h}^\top\big)\big(\mathbf{V}_{i,h}\big)^{-1}\big(0.5\,\mathbf{I}_{i,h} + \mathbf{K}_{i,h}\big) \tag{8}$$

and the element load vectors

$$\mathbf{f}_{i,h} = \mathbf{I}_{i,h}^\top\big(\mathbf{V}_{i,h}\big)^{-1}\mathbf{f}_{i,h}^N, \tag{9}$$

where the matrices $\mathbf{V}_{i,h}$, $\mathbf{K}_{i,h}$, $\mathbf{D}_{i,h}$ and $\mathbf{I}_{i,h}$ arise from the BE Galerkin approximation to the single layer potential operator V_i, double layer potential operator K_i, hypersingular integral operator D_i and the identity operator I_i living on Γ_i, respectively. $\mathbf{I}_{i,h}$ is nothing but the mass matrix. The vector $\mathbf{f}_{i,h}^N$ is defined by the Newton potential identity

$$(\mathbf{f}_{i,h}^N, \mathbf{t}_{i,h}) = \int_{\Gamma_i}\int_{\Omega_i} U^*(x-y)f(y)dy\,t_{h,i}(x)ds_x \tag{10}$$

for all vectors $\mathbf{t}_{i,h}$ corresponding to the piecewise constant functions $t_{h,i}$ on Γ_i. Now, we obtain the BE-based FE system

$$\mathbf{S}_h\mathbf{u}_h = \mathbf{f}_h \tag{11}$$

by assembling the stiffness matrix $\mathbf{S}_h$ and the load vector $\mathbf{f}_h$ from the element stiffness matrices (8) and the element load vectors (9), respectively, and by incorporating the Dirichlet boundary condition as usual.

The solution of (11) provides an approximation to the Dirichlet trace of the solution to (1) on the boundary $\partial\Omega_i$ of all elements Ω_i, $i = 1,\ldots,e_h$. Applying the Dirichlet-to-Neumann map locally (i.e. element-wise), we may obtain an approximate solution $\tilde{u}_h$to u in each element Ω_i via the representation formula (see, e.g., [7] or [11]).

Following [5] we immediately obtain the discretization error estimate

$$\|u - u_h\|_h \leq c(u)h^{3/2} \tag{12}$$

in the mesh-dependent norm $\|v\|_h^2 := \sum_{i=1}^{e_h} \|v|_{\Gamma_i}\|^2_{H^{1/2}(\Gamma_i)}$ for a sufficiently (piecewise) smooth solution u, where u_h is the continuous piecewise linear function on the skeleton $\Gamma_{S,h}$ corresponding to the Dirichlet nodal values and to the nodal values from the solution vector $\mathbf{u}_h$ of (11). The discretization error estimate (12) yields the usual $O(h)$ estimate of the discretization error $u - \tilde{u}_h$ in the $H^1(\Omega)$-norm.

In our first numerical experiments we solve the Laplace equation in $\Omega = (0,1) \times (0,1)$ with prescribed Dirichlet conditions on the boundary $\Gamma = \partial\Omega$. The Dirichlet datum g is given as the trace of the function $g(x) = \log\|x - x^*\|$ on Γ, where the singularity $x^* = (1.1, 1.1)^\top$ is located outside the computational domain Ω.

Figure 1 shows a close-up view of a polygonal mesh that was generated with the help of a software tool from the group of Olaf Steinbach at the TU Graz. Table 1

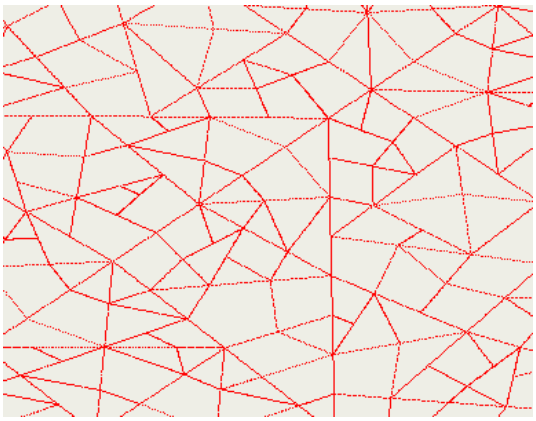

Fig. 1. Close-up view of a polygonal mesh

provides numerical results for 3 meshes (with N_h nodes) which were separately generated by the software tool mentioned above. The coarsest, the intermediate, and the finest mesh contain polygonal elements with a maximum of 13, 7 and 8 nodes, respectively. The systems of algebraic equations were solved by the Preconditioned Conjugate Gradient (PCG) method. The preconditioner is defined by a standard Algebraic MultiGrid (AMG) method implemented in the AMG package PEBBLES developed by [9]. More precisely, the AMG preconditioning step consists of one symmetric V-cycle with one pre-smoothing step and one post-smoothing step. The AMG level denotes the number of levels used in the algebraic multigrid process. The auxiliary coarse grid matrices are constructed by Galerkin projection. We observe that the times for constructing the stiffness matrices and for setting up the AMG also depend on the number of nodes of the polygons. The termination condition for the PCG iterations is defined as the reduction of the initial error by the factor $\varepsilon_{it} = 10^{-12}$

with respect to the $S_h C_h^{-1} S_h$-energy norm. We remark that the $S_h C_h^{-1} S_h$-energy norm is close to the S_h-energy norm if C_h is a good preconditioner for S_h. The nearly constant iteration numbers demonstrate the excellent preconditioning properties of the AMG preconditioner. The last two columns provide discretization errors in the $L_2(\Gamma_S)$- and $L_2(\Omega)$-norms. Note that the $L_2(\Gamma_S)$-norm is a mesh-dependent norm.

	AMG			PCG			
N_h	Level	S_h	Setup	Cycle	Iter.	$\|u-u_h\|_{0,\Gamma_S}$	$\|u-u_h\|_{0,\Omega}$
44249	4	4.3	17.2	0.35	15	1.29 E-4	4.88 E-6
71735	4	4.3	9.1	0.42	17	1.42 E-4	3.85 E-6
247250	5	17.0	82.0	1.80	17	6.73 E-5	1.51 E-6

Table 1. Numerical results for the polygonal mesh (CPU time in seconds).

3 The Helmholtz Equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a polyhedral Lipschitz boundary $\Gamma = \partial\Omega$. As a model problem, we consider the interior Dirichlet BVP for the Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma. \tag{13}$$

We assume that the wavenumber $\kappa > 0$ is piecewise constant and not an interior eigenvalue, and $g \in H^{1/2}(\Gamma)$. The case of a non-zero source function can be treated via the Newton potential operator as in the previous section, but we omit this for simplicity.

Given a domain decomposition satisfying (2), the BE-based FE method for the Helmholtz equation is formally identical to the method presented in the previous section for the potential equation (with $a = 1$ and $f = 0$). That is, the variational method is simply equation (4), with the Steklov–Poincaré operator S_i still given by (5) but with different operators D_i, K_i, and V_i. The appropriate boundary integral operators are given, e.g., in [8], [10], or [11], and the representation formula holds with the Helmholtz fundamental solution $U^*(x) = e^{i\kappa|x|}/(4\pi|x|)$.

4 The Maxwell Equations

Under the same assumptions on Ω and κ, we consider the interior Dirichlet BVP for the time-harmonic Maxwell equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2 \mathbf{u} = \mathbf{0} \qquad \text{in } \Omega, \quad \gamma_t \mathbf{u} := \mathbf{u} \times \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma, \tag{14}$$

where $\mathbf{n}$ is the outward unit normal vector. Developing a method of the form (4) for (14) involves quite technical trace spaces and boundary integral operators, so we only outline the main results here.

[2] defined the operator div_Γ and the appropriate function space $\mathbf{X} := \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ for the tangential trace γ_t on Lipschitz polyhedral domains, for which $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{X}$ is linear and continuous. In [3], potential operators $\mathbf{\Psi}_E, \mathbf{\Psi}_M : \mathbf{X} \to \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega)$ are defined such that the representation formula

$$\mathbf{u} = \mathbf{\Psi}_M(\gamma_t \mathbf{u}) + \mathbf{\Psi}_E(\gamma_N \mathbf{u}) \tag{15}$$

holds, where $\gamma_N \mathbf{u} := \kappa^{-1} \gamma_t \mathbf{curl}\,\mathbf{u}$ is the Neumann trace.

Defining the boundary integral operators $C, M : \mathbf{X} \to \mathbf{X}$ by

$$C := \{\gamma_t\}_\Gamma \circ \mathbf{\Psi}_E = \{\gamma_N\}_\Gamma \circ \mathbf{\Psi}_M,$$
$$M := \{\gamma_t\}_\Gamma \circ \mathbf{\Psi}_M = \{\gamma_N\}_\Gamma \circ \mathbf{\Psi}_E,$$

where $\{\}_\Gamma$ denotes the average across Γ, the Dirichlet-to-Neumann map can be expressed as

$$S : \mathbf{X} \to \mathbf{X}, \qquad S := C + (\frac{1}{2} I + M) C^{-1} (\frac{1}{2} I - M).$$

The inverse $C^{-1} : \mathbf{X} \to \mathbf{X}$ is given by [3, Corollary 5.5], and this representation of S is symmetric with respect to the bilinear form $< \mathbf{v}, \mathbf{w} >_{\tau,\Gamma} := \int_\Gamma (\mathbf{w} \times \mathbf{n}) \cdot \mathbf{v}$.

Since the trace operator γ_t is oriented with respect to the normal vector, in order to define a trace operator on the mesh skeleton we arbitrarily choose a global normal vector field $\mathbf{n}_S$ on the skeleton Γ_S. Then $\gamma_t^S := (\mathbf{n}_S \cdot \mathbf{n}_i)\gamma_{t,i}$, on each Γ_i^h, uniquely defines a tangential trace on Γ_S^h. Now the space $\gamma_t^S(\mathbf{H}(\mathbf{curl}, \Omega))$ is denoted $\mathbf{X}_S := \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, with the mesh-dependent norm $\|\mathbf{v}\|_{\mathbf{X}_s}^2 := \sum_{i=1}^{e_h} \|(\mathbf{n}_S \cdot \mathbf{n}_i)\mathbf{v}|_{\Gamma_i^h}\|_{\mathbf{X}_i}^2$.

The Maxwell skeleton variational formulation is to find $\mathbf{v} \in \mathbf{X}_S$ satisfying $\mathbf{v} = \mathbf{g}$ on Γ and

$$\sum_{i=1}^{e_h} < S_i \mathbf{v}_i, \mathbf{w}_i >_{\tau,\Gamma_i^h} = 0, \qquad \text{for all } \mathbf{w} \in \mathbf{X}_S \text{ such that } \mathbf{w}_{|\Gamma} = 0.$$

The space $\mathbf{X}_S$ can be approximated by the lowest-order Raviart-Thomas space defined on a triangular mesh of Γ_S. Galerkin discretization of the operators C_i, M_i, and I_i then yields matrices $\mathbf{C}_{i,h}$, $\mathbf{M}_{i,h}$, and $\mathbf{I}_{i,h}$, respectively, whereby we define the approximation

$$\mathbf{S}_{i,h} := \mathbf{C}_{i,h} + (\frac{1}{2}\mathbf{I}_{i,h} + \mathbf{M}_{i,h})\mathbf{C}_{i,h}^{-1}(\frac{1}{2}\mathbf{I}_{i,h} - \mathbf{M}_{i,h})$$

to S_i. Assembling the local element matrices $\mathbf{S}_{i,h}$ and incorporating the boundary conditions (where $\mathbf{u}_h \times \mathbf{n}$ approximates $\mathbf{g}$) results in a system of linear algebraic equations similar to the system (11).

In Table 2 we report the results of some numerical experiments in solving the linear system (11) in the Maxwell case by the GMRES iterative solver without a preconditioner. A preconditioner remains to be derived, so the iteration counts grow in these computations. The error relative to the exact solution $\nabla \times (U^*(x - (1.5, 0, 0))x)$ in the mesh-dependent norm of $\mathbf{L}_2(\Gamma_S)$ is of order $O(h^{1/2})$. Since the area of the mesh skeleton Γ_S grows in proportion to h^{-1}, one may consider the $\mathbf{L}_2(\Gamma_S)$-error of the trace

to be of order $O(h)$. Also, the $\mathbf{L}_2(\Omega)$-norm of the error is of order $O(h)$, comparable to the standard finite element method with the lowest-order Nedelec elements. We refer the reader to [4] for more numerical results.

h	Edges	Iter.	$\|u-u_h\|_{0,\Gamma_S}$	$\|u\|_{0,\Gamma_S}$	$\|u-u_h\|_{0,\Omega}$
1/8	2156	243	5.09 E-2	0.490	9.19 E-3
1/16	16024	556	3.70 E-2	0.677	4.81 E-3
1/32	123440	1219	2.73 E-2	0.948	2.52 E-3
1/64	968800	2987	2.13 E-2	1.33	1.39 E-3

Table 2. Tetrahedral mesh of the unit cube $\Omega = (0,1)^3$, $\kappa = 1$.

5 Conclusions

Our technique can obviously be generalized to potential equations with piecewise smooth coefficients $a(.)$ and, therefore, to nonlinear potential equations arising, e.g., in electromagnetics. On each element Ω_i, $a(.)$ can be approximated by its value at the center of gravity of Ω_i. Moreover, we can easily construct polygonal and polyhedral elements with special geometric features like small holes and inclusions. In particular, periodic structures allow a fast generation of the finite element equations or a fast matrix-vector multiplication. The generalization to problems for which the fundamental solution is locally known (for frozen coefficients) is obviously feasible. The methods presented here can be applied to acoustic and electromagnetic scattering problems by coupling with BEM in the unbounded exterior domain. We mention that one and the same technique is used for generating the finite and the boundary element equations. The latter issues is addressed in detail in a paper by [4].

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References

[1] Brezzi, F., Lipnikov, K., Shashkov, M.: Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(3):1872–1896, 2005.

[2] Buffa, A., Ciarlet, Jr., P.: On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra. *Math. Methods Appl. Sci.*, 24:9–30, 2001.

[3] Buffa, A., Hiptmair, R., von Petersdorff, T., Schwab, C.: Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95:459–485, 2003.

[4] Copeland, D.: Boundary-element-based finite element methods for Helmholtz and Maxwell equations on general polyhedral meshes. *Int. J. Appl. Math. Comput. Sci.*, 5(1):60–73, 2009.

[5] Hsiao, G.C., Wendland, W.L.: Domain decomposition in boundary element methods. In R. Glowinski, Y.A. Kuznetsov, G. Meurant, J. Périaux and O.B. Widlund, eds., *Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Moscow, May 21-25, 1990*, pages 41–49, Philadelphia, 1991. SIAM.

[6] Kuznetsov, Y., Lipnikov, K., Shashkov, M.:
Mimetic finite difference method on polygonal meshes for diffusion-type problems. *Comput. Geosci.*, 8(4):301–324, December 2004.

[7] Langer, U., Steinbach, O.: Coupled finite and boundary element domain decomposition methods. In M. Schanz and O. Steinbach, eds., *Boundary Element Analysis: Mathematical Aspects and Application*, vol. 29 of *Lecture Notes in Applied and Computational Mechanics*, pages 29–59, Berlin, 2007. Springer.

[8] McLean, W.: *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[9] Reitzinger, S.: *Algebraic Multigrid Methods for Large Scale Finite Element Equations*. Reihe C - Technik und Naturwissenschaften. Universitätsverlag Rudolf Trauner, Linz, 2001.

[10] Sauter, S., Schwab, C.: *Randelementemethoden: Analyse, Numerik und Implementierung schneller Algorithmen*. Teubner, Stuttgart, Leipzig, Wiesbaden, 2004.

[11] Steinbach, O.: *Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements*. Springer, New York, 2008.