A Multilevel Domain Decomposition Solver Suited to Nonsmooth Mechanical Problems

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1 Introduction

A particular class of mechanical systems concerns diffuse non smooth problems for which unilateral conditions may occur within the whole studied domain. For instance, when contact and friction occur as interactions between a large number of bodies, such as for granular media, or with tensegrity structures, [14], when cable slackening may occur on the whole structure.

When such large scale structures are studied, their numerical simulation may take advantage of using domain decomposition (DD) solvers. We do not consider here an outer loop for dealing with non linearities and non smoothness that lead to a series of linear problems, each of them being solved with a classical DD solver as a black box, but we focus on algorithms that allow us to tackle the non smoothness issue at the subdomain level, with a single iterative loop. Such approaches have already be designed for mechanical assemblings with a limited number of unilateral conditions, [5, 6, 7], or for multicontact situations, [1, 2, 3, 4, 12, 15].

We consider in this article approaches suited to multicontact cases, that focus on the non smooth interactions by solving them locally on the one hand, and by solving the global equations on the other hand, iteratively. Among these, one may consider the LArge Time INcrement (LATIN) approaches, [2, 12, 15], that embed a multiscale aspect to derive an scalable DD method, and the Non Smooth Contact Dynamics (NSCD) approaches, genuinely designed for granular media, [10, 13].

Herein, we proposed an approach based on the Gauss-Seidel (or more precisely Jacobi for parallelization purposes) interpretation, [11], of the NSCD method, embedding the same multiscale description used in the LATIN method, and we design the solver in the case of a tensegrity grid steady-state simulation.

2 A Multiscale Description

With a given discretized structure (for instance, with finite element for an elastic problem on a continuum domain, of directly on the equilibrium equations of a truss-

like elastic problem), there are at least two choices for a partitioning into substructures. On the one hand, one can split the nodes into different sets, leading to an interface between two substructures composed by elements linking the nodes of the two corresponding sets. On the other hand, one can split the elements into different sets, leading to an interface defined as shared nodes. The interface behavior is therefore either the behavior of the shared elements (that may be nonlinear), or the behavior of the shared nodes. In this last case, unless if unilateral conditions occur at the interface, [5, 7, 12], the behavior is linear, [3, 15].

This last case is chosen herein. Once the substructuring is performed and the interfaces between each pair of connected substructures are defined, the multiscale description is performed at the interface level. For the quasi-static or steady-state problems we are concerned herein, two dual fields are involved on each interface: the trace of the substructure displacement, and the forces acting on the interface from the neighboring substructures. In the case of discrete structures, the interface is a set of nodes, Figure 1, and the displacement field V on the interface is split into a macro part (denoted with superscript M) and a micro part (denoted with a superscript m): $V = V^M + V^m$. The macro part is chosen as the average translation, rotation and extension in the average plane of the interface, [15]. Therefore, it can be defined with a small number of parameters (9 values per interface in 3D case) stored in a small vector w: $V^M = Pw$, where P stores the basis vectors of macro space. The dual field, i.e. the forces on interface, is split in a similar way: $F = F^M + F^m$. The macro part is also defined with the same small number of parameters, here, the dual macro quantities f, and we chose the basis macro functions such as $F^M = Pf$ and $P^T P$ is the identity matrix.

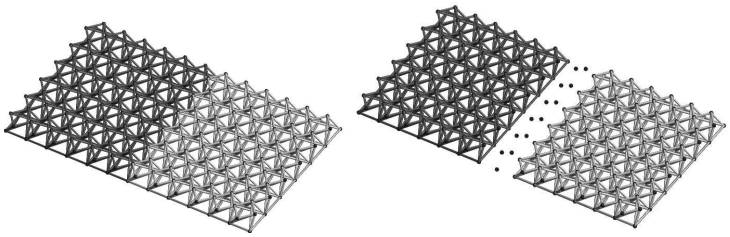

Fig. 1. Element oriented partitioning (left) and perfect discrete interface between the substructures (right).

3 Preliminary: Linear Elastic Case

Substructure Behavior. If we consider a single substructure E, the actions of its neighboring interfaces are the forces $-F_{E\Gamma}$ and the displacement on the boundary $V_{E\Gamma}$. The subscript $E\Gamma$ is used to denote the assembly of the local interfaces of the substructure E. The corresponding balance equation is

$$F_E^d - F_E - C_{E\Gamma}^T F_{E\Gamma} = 0, \tag{1}$$

where $C_{E\Gamma}$ is a boolean mapping matrix that selects the trace on the local interfaces, F_E^d are the prescribed external forces, and F_E are the internal forces. For an elastic media, the internal forces can be expressed with the nodal displacement V_E via the stiffness matrix: $F_E = K_E V_E$.

To prescribe the force equilibrium on each interface, the local forces $F_{E\Gamma}$ can be derived from a single field on the global interface (the gathering of all local interfaces) F_Γ with a signed boolean matrix B_E as for dual approaches, [8, 9]: $F_{E\Gamma} = B_E F_\Gamma$. The dual quantity, the trace of displacement on the local interfaces is: $V_{E\Gamma} = C_{E\Gamma} V_E$.

Interface Behavior. Once the equilibrium of forces on the interfaces is automatically satisfied, their perfect behaviors lead to a displacement continuity:

$$\sum_E B_E^T V_{E\Gamma} = 0 \tag{2}$$

Solution Algorithm. The balance equation (1) for the substructure E can be recast as:

$$K_E V_E + C_{E\Gamma}^T B_E F_\Gamma = F_E^d \tag{3}$$

The first step to design the proposed approach is to condense the information on the interfaces. For sake of simplicity, we consider here that the stiffness matrix K_E is invertible. If this is not the case, for floating substructures for instance, the same procedure can be derived, provided that a suited generalized inverse is used. The gluing condition (2), using (3) to eliminate the internal degrees of freedom, reads:

$$X F_\Gamma = \tilde{F}^d \tag{4}$$

with $\tilde{F}^d = \sum_E B_E^T C_{E\Gamma} K_E^{-1} F_E^d$ and $X = \sum_E B_E^T C_{E\Gamma} K_E^{-1} {C_{E\Gamma}}^T B_E$

In order to solve this problem, we propose a stationary iterative method based on the splitting of the global operator X into $X = X^D - (X^D - X)$. Different choices can be selected to split X (Jacobi, Gauss-Seidel...), that lead to different algorithms. In each case, the iterate number $i+1$ consists in solving: $X^D F_\Gamma^{i+1} = \tilde{F}^d - (X - X^D) F_\Gamma^i$, or when developping $\tilde{F}^d$:

$$X^D (F_\Gamma^{i+1} - {F_\Gamma}^i) = \sum_E B_E^T C_{E\Gamma} V_E^i \tag{5}$$

with $K_E V_E^i = F_E^d - C_{E\Gamma}^T B_E F_\Gamma^i$.

Splitting Choice. X is a dense operator coupling all the degrees of freedom on the global interface. X^D is similar to a preconditioner, or a search direction. Choosing for instance a 'lumped' approximation on the local interfaces on each subdomain leads to $(X^D)^{-1} = \sum_E B_E^T (C_{E\Gamma} K_E C_{E\Gamma}^T) B_E$. An even simpler version uses a constant scalar stiffness d as: $(X^D)^{-1} = \sum_E B_E^T d I_{E\Gamma} B_E$, where $I_{E\Gamma}$ is the identity matrix on the boundary degrees of freedom of the subdomain E. In such a case, due to the fact that the global interface is merely the gathering of all the local interfaces, $\sum_E B_E^T B_E =$

$2I_{E\Gamma}$ and $(X^D)^{-1} = 2d\,I_\Gamma$. Applying $(X^D)^{-1}$ to a vector simply leads to explicit and local computations on each interface independently.

Multiscale Approach. Up to this point, no multilevel feature is involved in the previous algorithm. To do so, the micro-macro description of Section 2 should be used. We propose here to enforce the continuity of the macro displacement and the equilibrium of the macro forces at each iteration, while it is satisfied previously only when the solution has converged. Therefore, for the interfaces connected to each substructure E, the macro generalized forces are supposed to be extracted from a unique macro vector defined on the global interface: $f_{E\Gamma} = c_E f_\Gamma$, where c_E is a signed boolean matrix selecting the entries in f_Γ. The dual quantity is the gap of macro displacements on interfaces $w_\Gamma = \sum_E c_E^T w_{E\Gamma}$. The macro displacement continuity on interfaces gives $w_\Gamma = 0$, which reads:

$$\sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} V_E = 0 \tag{6}$$

This constraint is therefore to be prescribed at each iteration for the displacement field V_E^i in (5), for which f_Γ is the associated Lagrange multiplier. Therefore, the displacement V_E^i is now obtained by solving:

$$\begin{cases} K_E V_E = F_E^d - C_{E\Gamma}^T B_E F_\Gamma - C_{E\Gamma}^T P_{E\Gamma} c_E f_\Gamma \\ \sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} V_E = 0 \end{cases} \tag{7}$$

The local condensation on f_Γ for each substructure, and the assembly in (6) leads to the macroscopic (coarse) problem:

$$L_\Gamma f_\Gamma = \sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} K_E^{-1} (F_E^d - C_{E\Gamma}^T B_E F_\Gamma) \tag{8}$$

with $L_\Gamma = \sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} K_E^{-1} C_{E\Gamma}^T P_{E\Gamma} c_E$ which is explicitly assembled to maintain the globality of the coarse problem. The size of L_Γ corresponds to the number of macro degrees of freedom involved in the coarse problem.

With a given approximation of the solution $(F_\Gamma^i, V_E^i, f_\Gamma^i)$, one iteration provides the update $(F_\Gamma^{i+1}, V_E^{i+1}, f_\Gamma^{i+1})$:

- During the 'local stage', F_Γ^{i+1} is computed using (5), locally on each interface:

$$F_\Gamma^{i+1} = (X^D)^{-1} \sum_E B_E^T C_{E\Gamma} V_E^i + {F_\Gamma}^i \tag{9}$$

- During the 'coarse step', the macro problem is solved to get f_Γ^{i+1}

$$L_\Gamma f_\Gamma^{i+1} = \sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} K_E^{-1} (F_E^d - C_{E\Gamma}^T B_E F_\Gamma^{i+1}) \tag{10}$$

- During the 'global step' per subdomain E independently, V_E^{i+1} is updated by solving:

$$K_E V_E^{i+1} = F_E^d - C_{E\Gamma}^T B_E F_\Gamma^{i+1} - C_{E\Gamma}^T P_{E\Gamma} c_E f_\Gamma^{i+1} \tag{11}$$

4 Nonsmooth Case: a Tensegrity Grid

Tensegrity Structures. Tensegrity systems are reticulated spatial structures constituted with rectilinear elements such as 'cables' or 'bars', [14]. Bars are subjected to compression loading, while cables are subjected to traction loading. Joining elements are perfect articulations called 'nodes'. These systems allow for selfstressed states, i.e. stress states that satisfy the equilibrium without external loading. These stress states are mandatory to ensure the overall structure rigidity. The reference problem is herein related to the static behavior of such a structure, with a small perturbation assumption.

Model of a Tensegrity Structure. In the case of a tensegrity structure, the equilibrium (1) remains identical, but the internal forces arise from the internal tension (or compression) r_E in the elements (or the links) that constitute the structure: the cables (with a superscript c) or the bars (with a superscript b)

$$F_E = H_E r_E = H_E^c r_E^c + H_E^b r_E^b \tag{12}$$

where H_E is a mapping from the link set to the node set.

Additionally, the trace of nodal displacement $V_{E\Gamma} = C_{E\Gamma} V_E$ remains identical, but we add the length variation of the links, e_E as the dual quantity of the strength r_E:

$$e_E^b = {H_E^b}^T V_E \quad \text{and} \quad e_E^c = {H_E^c}^T V_E \tag{13}$$

The interface behavior (2) holds again, but the constitutive relations of the links are a linear elastic behavior for the bars,

$$r_E^b + {r_E^b}^0 = k_b (e_E^b + {e_E^b}^0), \tag{14}$$

and a nonsmooth complementary condition for the cables, [15],

$$0 \leq \tau_E^c \perp \lambda_E^c \geq 0 \tag{15}$$

where $\tau_E^c = r_E^c + {r_E^c}^0$ and $\lambda_E^c = -e_E^c + k_c^{-1} r_E^c = -(e_E^c + {e_E^c}^0) + k_c^{-1}(r_E^c + {r_E^c}^0)$. The superscript 0 denotes the prestress or prestrain that have to be initially prescribed for the structure to exhibit stiffness.

Multiscale Solver. With (12), (13), (14) and (15), the equilibrium reads:

$$K_E V_E + C_{E\Gamma}^T B_E F_\Gamma + H_E^c k_c \lambda_E^c = \tilde{F}_E^d, \tag{16}$$

with the stiffness matrix of the underlying truss $K_E = H_E^b k_b {H_E^b}^T + H_E^c k_c {H_E^c}^T$, and the given right hand side $\tilde{F}_E^d = F_E^d - H_E^b k_b {e_E^b}^0$.

As for the linear case, the first step consists in condensing the problem on the local interface quantities, but keeping the variable λ_E^c traducing the nonsmooth interaction as an unknown, using (16) and (2):

$$X F_\Gamma + \sum_E B_E^T C_{E\Gamma} K_E^{-1} H_E^c k_c \lambda_E^c = \sum_E B_E^T C_{E\Gamma} K_E^{-1} \tilde{F}_E^d, \tag{17}$$

with $X = \sum_E B_E^T C_{E\Gamma} K_E^{-1} {C_{E\Gamma}}^T B_E$.

Additionally, one must keep the local nonsmooth relationship traducing the behavior of the cables: condensing the equilibrium (1) on cable quantities, while keeping the constitutive equation (15) leads to the Linear Complementary Problem (LCP) with (λ_E, τ_E) as unknowns (and parametrized by F_Γ):

$$\begin{cases} W_E \lambda_E^c - \tau_E^c = -{H_E^c}^T K_E^{-1} \tilde{F}_E^d + {H_E^c}^T K_E^{-1} C_{E\Gamma}^T B_E F_\Gamma - {r_E^c}^0 \\ 0 \leq \tau_E^c \perp \lambda_E^c \geq 0 \end{cases} \tag{18}$$

where $W_E = k_c I_E - k_c {H_E^c}^T K_E^{-1} H_E^c k_c$ (I_E is the identity matrix), is the linear part of the relationship, [15].

As for the linear case, the left hand side X is split, and so is the (local per subdomain) operator W_E: $W_E = W_E^D - (W_E^D - W_E)$. The proposed algorithm iterates both on interface forces F_Γ and on the nonsmooth pair of variables (λ_E, τ_E). Knowing quantities with a superscript i, one iteration will give the update, with superscript $i+1$, such as:

$$\begin{cases} W_E^D \lambda_E^{i+1} - \tau_E^{i+1} = W_E^D \lambda_E^i - k_c \lambda_E^i - k_c {H_E^c}^T V_E^i - {r_E^c}^0 \\ 0 \leq \tau_E^{i+1} \perp \lambda_E^{i+1} \geq 0 \end{cases} \tag{19}$$

for the nonsmooth part, and (5), that remains unchanged, for the interface part, where V_E^i satisfies $K_E V_E^i = \tilde{F}_E^d - C_{E\Gamma}^T B_E F_\Gamma^i - H_E^c k_c \lambda_E^i$.

Splitting Choice. W_E is a global operator on each subdomain. Choosing a diagonal W_E^D leads to independant LCP on each cable; for instance, one can select $W_E^D = k_c I_E$. Other choices are obviously possible.

This constitutes the monolevel algorithm, whose multilevel version is obtained with a similar procedure as already done for the linear case. With a given approximation of the solution $(F_\Gamma^i, V_E^i, \lambda_E^i, \tau_E^i, f_\Gamma^i)$, one iteration provides the update:

- During 'local stage', F_Γ^{i+1} is computed locally on each interface:

$$F_\Gamma^{i+1} = (X^D)^{-1} \sum_E B_E^T C_{E\Gamma} V_E^i + F_\Gamma^i \tag{20}$$

  and $(\lambda_E^{i+1}, \tau_E^{i+1})$ is computed locally on each subdomain E by solving the nonsmooth but local problem:

$$\begin{cases} W_E^D \lambda_E^{i+1} - \tau_E^{i+1} = W_E^D \lambda_E^i - \tau_E^i \\ 0 \leq \tau_E^{i+1} \perp \lambda_E^{i+1} \geq 0 \end{cases} \tag{21}$$

- During the 'coarse step', the macro multiplier f_Γ^{i+1} is obtained by solving the coarse problem:

$$L_\Gamma f_\Gamma^{i+1} = \sum_E c_E^T P_{E\Gamma}^T C_{E\Gamma} K_E^{-1} (F_E^d - C_{E\Gamma}^T B_E F_\Gamma^{i+1} - H_E^c k_c \lambda_E^{i+1}) \tag{22}$$

- During the 'global step' per subdomain E independently, V_E^{i+1} is updated by solving:

$$K_E V_E^{i+1} = \tilde{F}_E^d - C_{E\Gamma}^T B_E F_\Gamma^{i+1} - C_{E\Gamma}^T P_{E\Gamma} c_E f_\Gamma^{i+1} - H_E^c k_c \lambda_E^{i+1} \tag{23}$$

At to this point, the algorithm is on its way to be tested and compared with previous algorithm based on a LArge Time INcrement approach, [15].

5 Conclusions

The proposed method constitutes a first attempt to combine a multilevel Domain Decomposition technique with a Non Smooth Gauss-Seidel (NSGS) type algorithm. The NSGS algorithm is classically associated with the Non Smooth Contact Dynamics approach and provides a robust solver for the simulation of dense granular media involving not only unilateral contact but also frictional contact or more general localized interactions. The multilevel DD technique ensures the scalability of the solver to deal with large-scale mechanical systems. Moreover the multiscale approach may allow replacing the fine description of some subdomains by their homogenized behavior under additional assumptions. The simulation cost of large-scale systems as granular media may be drastically decreased. For that the previous solver may be extended to dynamical problems without conceptual difficulty. Nevertheless the meaning of the homogenized coarse problem has to be investigated. From a computational viewpoint the chosen formulation, close to the FETI approach, suggests to replace the stationary iterative method by a conjugate gradient algorithm. The projected conjugate gradient method developed in [16] for granular media in sub domains would be usefully combined with the conjugate gradient algorithm of the standard FETI method for solving the interface problem in linear problems. Such a combination may provide a more efficient solver even if the non smoothness does not preserve the conjugating property from one iteration to the following one. A large range of numerical tests have to be performed to validate this strategy.

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