A Robin Domain Decomposition Algorithm for Contact Problems: Convergence Results

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Summary. In this paper, we propose and study a Robin domain decomposition algorithm to approximate a frictionless unilateral problem between two elastic bodies. Indeed this algorithm combines, in the contact zone, the Dirichlet and Neumann boundaries conditions (Robin boundary condition). The primary feature of this algorithm is the resolution on each sub-domain of variational inequality.

1 Introduction

The numerical treatement of nonclassical contact problems leads to very large (due to the large ratio of degrees of freedom concerned by contact conditions) and ill-conditioned systems. Domain decomposition methods are good alternatives to overcome these difficulties (see [2, 3, 13, 14]).

The aim of this paper is to give an idea of the proof for iterative schemes based on domain decomposition techniques for a nonlinear problem modeling the frictionless contact of linear elastic bodies. They were introduced in [11] and can be considered as a generalization to variational inequality of the method described in [7, 15]. In [2, 3, 13, 14], the initial problem is transformed into a unilateral contact problem in the one body and a prescribed displacement problem in the other one. We propose, in this paper, another domain decomposition method in which we solve an unilateral contact problem in each subdomain.

2 Weak Formulation of the Continuous Problem

Let us consider two bodies occupying, in the reference configuration, bounded domains Ω^α, $\alpha = 1,2$, of the space $\mathbb{R}^2$ with sufficiently smooth boundaries. The boundary $\Gamma^\alpha = \partial\Omega^\alpha$ consists of three measurable, mutually disjoint parts Γ_u^α, Γ_ℓ^α, Γ_c^α so that $\Gamma^\alpha = \overline{\Gamma_u^\alpha} \cup \overline{\Gamma_\ell^\alpha} \cup \overline{\Gamma_c^\alpha}$. The body $\overline{\Omega}^\alpha$ is fixed on the set Γ_u^α. It is subject to surface traction forces $\Phi^\alpha \in (L^2(\Gamma_\ell^\alpha))^2$ and the body forces are denoted by $f^\alpha \in (L^2(\Omega^\alpha))^2$.

On the contact interface determined by Γ_c^1 and Γ_c^2, we consider the contact condition that is characterized by the non-penetration of the bodies and the transmission of forces. To describe the non-penetration of the bodies, we shall use a pre-defined bijective mapping $\chi : \Gamma_c^1 \longrightarrow \Gamma_c^2$, which assigns to each $x \in \Gamma_c^1$ some nearby point $\chi(x) \in \Gamma_c^2$. Let $v^1(x)$ and $v^2(\chi(x))$ denote the displacement vectors at x and $\chi(x)$, respectively. Assuming the small displacements, the *non-penetration condition* reads as follows:

$$v_\nu^1(x) - v_\nu^2(x) = [v_\nu] \leq g(x) \quad \text{with} \quad v_\nu^1(x) \equiv v^1(x)\cdot\nu(x),\; v_\nu^2(x) \equiv v^2(\chi(x))\cdot\nu(x),$$

where $g(x) = (\chi(x) - x)\cdot\nu(x)$ is the initial gap and $\nu(x)$ is the critical direction defined by $\nu(x) = (\chi(x)-x)/\|\chi(x)-x\|$ or, if $\chi(x) = x$, by the outer unit normal vector to Γ_c^1. We seek the displacement field $u = (u^1, u^2)$ (where the notation u^α stands for $u|_{\Omega^\alpha}$) and the stress tensor field $\sigma = (\sigma(u^1), \sigma(u^2))$ satisfying the following equations and conditions (1)–(2) for $\alpha = 1,2$:

$$\begin{cases} div\,\sigma(u^\alpha) + f^\alpha = 0 & \text{in } \Omega^\alpha, \\ \sigma(u^\alpha)n^\alpha - \Phi_\ell^\alpha = 0 & \text{on } \Gamma_\ell^\alpha, \\ u^\alpha = 0 & \text{on } \Gamma_u^\alpha, \\ \sigma_\nu \leq 0, \sigma_T = 0, [u_\nu] \leq 0, & \text{on } \Gamma_c^\alpha, \\ \sigma_\nu \cdot [u_\nu] = 0 & \text{on } \Gamma_c^\alpha. \end{cases} \tag{1}$$

The symbol *div* denotes the divergence operator of a tensor function and is defined as

$$div\,\sigma = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)_i.$$

The summation convention of repeated indices is adopted. The elastic constitutive law, is given by Hooke's law for homogeneous and isotropic solids:

$$\sigma(u^\alpha) = A^\alpha(x)\varepsilon(u^\alpha), \tag{2}$$

where $A^\alpha(x) = (a_{ijkh}^\alpha(x))_{1\leq i,j,k,h\leq 2} \in (L^\infty(\Omega^\alpha))^{16}$ is a fourth-order tensor satisfying the usual symmetry and ellipticity conditions in elasticity. The linearized strain tensor $\varepsilon(u^\alpha)$ is given by

$$\varepsilon(u^\alpha) = \frac{1}{2}(\nabla u^\alpha + (\nabla u^\alpha)^T).$$

We will use the usual notations for the stress vector on the contact zone Γ_c^α:

$$\sigma_\nu^\alpha = \sigma_{ij}(u^\alpha)\nu_i^\alpha \nu_j^\alpha, \quad \sigma_T^\alpha = \sigma_{ij}(u^\alpha)\nu_j^\alpha - \sigma_\nu^\alpha \nu_i^\alpha.$$

In order to give the variational formulation corresponding to the problem (1)–(2), let us introduce the following spaces

$$V^\alpha = \{v^\alpha \in (H^1(\Omega^\alpha))^2,\; v = 0 \text{ on } \Gamma_u^\alpha\}, \quad V = V^1 \times V^2.$$

Now, we denote by K the following non-empty closed convex subset of V:

$$K = \{v = (v^1, v^2) \in V,\ [v_\nu] \le 0 \text{ on } \Gamma_c^1\}.$$

The variational formulation of problem (1)–(2) is

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \ge L(v-u) \quad \forall v \in K, \end{cases} \tag{3}$$

where

$$\begin{aligned} a(u,v) &= a^1(u,v) + a^2(u,v), \\ a^\alpha(u,v) &= \int_{\Omega^\alpha} A^\alpha(x)\varepsilon(u^\alpha)\cdot\varepsilon(v^\alpha)dx, \end{aligned} \tag{4}$$

and

$$L(v) = \sum_{\alpha=1}^{2} \int_{\Omega^\alpha} f^\alpha \cdot v^\alpha\, dx + \int_{\Gamma_\ell^\alpha} \Phi^\alpha \cdot v^\alpha\, d\sigma.$$

There exists a unique solution u to problem (3) (see [5, 6, 12]).

3 The Domain Decomposition Algorithm

In order to split the problem (3) into two subproblems coupled through the contact interface, we first introduce the following spaces and mappings:

$$\begin{aligned} V_0^\alpha &= \{v^\alpha \in V^\alpha,\ v_\nu^\alpha = 0 \text{ on } \Gamma_c^\alpha\}, \quad \alpha = 1,2, \\ \mathcal{H}^{1/2}(\Gamma_c) &= \{\varphi \in (L^2(\Gamma_c))^2,\ \exists v \in V^\alpha,\ \gamma v_{|\Gamma_c} = \varphi\}, \\ H^{1/2}(\Gamma_c) &= \{\varphi \in L^2(\Gamma_c),\ \exists v \in H^1(\Omega^\alpha),\ \gamma v_{|\Gamma_c} = \varphi\}, \end{aligned}$$

where γ is the usual trace operator.

By $P^\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow V^\alpha$, we denote the extension operator from Γ_c^α in Ω^α defined by : $P^\alpha\varphi = v^\alpha, \varphi \in H^{1/2}(\Gamma_c^\alpha)$, where $v^\alpha \in V^\alpha$ satisfies

$$\begin{cases} a^\alpha(v^\alpha, w^\alpha) = 0 & \forall w^\alpha \in V_0^\alpha; \\ v_\nu^\alpha = \varphi & \text{on } \Gamma_c^\alpha. \end{cases}$$

Remark 1. For the sake of simplicity, we shall write $P^1(v_\nu^2)$ and $P^2(v_\nu^1)$ instead of $P^1(v^2 \circ \chi.\nu)$ and $P^2(v^1 \circ \chi^{-1}.\nu)$, respectively.

Let $S_\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow H^{-1/2}(\Gamma_c^\alpha)$ be the following Steklov-Poincaré operator (see [1]), for any $\varphi \in H^{1/2}(\Gamma_c^\alpha)$

$$S_\alpha\varphi = (\sigma(u^\alpha)\nu^\alpha)\nu^\alpha = \sigma_\nu(u^\alpha) \quad \text{on } \Gamma_c^\alpha, \tag{5}$$

where u^α solves the problem

$$\begin{cases} div(\sigma(u^\alpha)) = 0 & \text{in } \Omega^\alpha, \\ \sigma(u^\alpha)\nu^\alpha = 0 & \text{on } \Gamma_\ell^\alpha, \\ u^\alpha = 0 & \text{on } \Gamma_u^\alpha, \\ \sigma_T(u^\alpha) = 0 & \text{on } \Gamma_c^\alpha, \\ u^\alpha \nu^\alpha = \varphi & \text{on } \Gamma_c^\alpha. \end{cases} \tag{6}$$

Finally, with any $\varphi \in H^{1/2}(\Gamma_c^\alpha)$, we associate the closed convex set

$$V_-^\alpha(\varphi) = \{v^\alpha \in V^\alpha / v^\alpha \nu^\alpha \leq \varphi \text{ on } \Gamma_c^\alpha\}.$$

The two-body contact problem (3) is approximated by an iterative procedure involving a contact problem for each body Ω^α with a rigid foundation described by:

Given $g_0^\alpha \in H^{1/2}(\Gamma_c)$, $\alpha = 1,2$. For $m \geq 1$, we build the sequence of functions $(u_m^1)_{m\geq 0} \in V^1$ and $(\mathbf{u}_m^2)_{m\geq 0} \in V^2$ by solving the following problems:

$$\textit{Step } 1\text{:} \begin{cases} \text{Find } u_m^\alpha \in V_-^\alpha(g_{m-1}^\alpha), \\ a^\alpha\big(u_m^\alpha, w + P^\alpha(g_{m-1}^\alpha) - u_m^\alpha\big) \geq L^\alpha\big(w + P^\alpha(g_{m-1}^\alpha) - u_m^\alpha\big) \quad \forall w \in V_-^\alpha(0). \end{cases} \tag{7}$$

$$\textit{Step } 2\text{:} \begin{cases} \text{Find } w_m^1 \in V^1, \\ a^1(w_m^1, v) = -a^2(u_m^2, P^2(v_\nu)) + L^2(P^2(v_\nu)) - a^1(u_m^1, v) + L^1(v) \quad \forall v \in V^1. \\ \text{Find } w_m^2 \in V^2, \\ a^2(w_m^2, v) = a^1(u_m^1, P^1(v_\nu)) - L^1(P^1(v_\nu)) + a^2(u_m^2, v) - L^2(v) \qquad \forall v \in V^2. \end{cases} \tag{8}$$

$$\textit{Step } 3\text{:} \begin{cases} g_m^1 = (1-\theta)g_{m-1}^1 + \theta(w_m^2 \nu^2 - u_m^2 \nu^2) & \text{on } \mathcal{G}_c^1, \\ g_m^2 = (1-\theta)g_{m-1}^2 + \theta(w_m^1 \nu^1 - u_m^1 \nu^1) & \text{on } \mathcal{G}_c^2. \end{cases} \tag{9}$$

Theorem 1. *The fixed point of the algorithm* (7)–(9) *is the unique solution of the problem* (3).

Proof. We refer to [10] for the proof of this theorem.

4 Convergence

The convergence of iterative schemes (7)–(9) is proven by the application of Banach's fixed point theorem to a suitable defined operator. In this following, we reformulate (7)–(9) with operators representation.

In order to decouple the influence of exterior forces and boundary data, we define U^α, $\alpha = 1,2$, as solutions of the problems:

$$\begin{cases} -div(\sigma(U^\alpha)) = f^\alpha & \text{in } \Omega^\alpha, \\ \sigma(U^\alpha)\nu^\alpha = \Phi_\ell^\alpha & \text{on } \Gamma_\ell^\alpha, \\ U^\alpha = 0 & \text{on } \Gamma_u^\alpha, \\ \sigma(U^\alpha)\nu^\alpha = 0 & \text{on } \Gamma_c^\alpha. \end{cases} \tag{10}$$

Moreover, we introduce the operator $Q^\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow H^{1/2}(\Gamma_c^\alpha)$ defined by $Q^\alpha g_{m-1}^\alpha = \tilde{u}_{\nu,m}^\alpha$, $\forall g_{m-1}^\alpha \in H^{1/2}(\Gamma_c^\alpha)$, where $\tilde{u}_m^\alpha$ is the solution of

$$\begin{cases} -div(\sigma(\tilde{u}_m^\alpha)) = 0 & \text{in } \Omega^\alpha, \\ \sigma(\tilde{u}_m^\alpha)\nu^\alpha = 0 & \text{on } \Gamma_\ell^\alpha, \\ \tilde{u}_m^\alpha = 0 & \text{on } \Gamma_u^\alpha, \\ \sigma_T(\tilde{u}_m^\alpha) = 0, \sigma_\nu(\tilde{u}_m^\alpha) \le 0 & \text{on } \Gamma_c^\alpha, \\ \tilde{u}_m^\alpha \nu^\alpha \le g_{m-1}^\alpha & \text{on } \Gamma_c^\alpha, \\ \sigma_\nu(\tilde{u}_m^\alpha)(\tilde{u}_m^\alpha \nu^\alpha - g_{m-1}^\alpha) = 0 & \text{on } \Gamma_c^\alpha. \end{cases} \tag{11}$$

Then the solution of the problem (7) can be expressed by

$$u_m^\alpha = U^\alpha + P^\alpha(Q^\alpha g_{m-1}^\alpha). \tag{12}$$

Using the Steklov-Poincaré operator, the Step 2 of (7)–(9) can be written as follows:

$$\begin{cases} w_{\nu,m}^1 = S_1^{-1}(\sigma_\nu(u_m^2) - \sigma_\nu(u_m^1)), \\ w_{\nu,m}^2 = S_2^{-1}(\sigma_\nu(u_m^1) - \sigma_\nu(u_m^2)). \end{cases} \tag{13}$$

Then, we have

$$\begin{cases} w_{\nu,m}^1 = a - (Q^1 g_{m-1}^1 + S_1^{-1} S_2 g_{m-1}^2), \\ w_{\nu,m}^2 = b - (Q^2 g_{m-1}^2 + S_2^{-1} S_1 g_{m-1}^1), \end{cases} \tag{14}$$

where $a = -S_1^{-1} S_2 U_\nu^2 - U_\nu^1$ and $b = -S_2^{-1} S_1 U_\nu^1 - U_\nu^2$.

From (14), we obtain a new expression of (9)

$$\begin{cases} g_m^1 = (1-\theta) g_{m-1}^1 - \theta(2Q^2 g_{m-1}^2 + S_2^{-1} S_1 Q^1 g_{m-1}^1) + \theta b_1, \\ g_m^2 = (1-\theta) g_{m-1}^2 - \theta(2Q^1 g_{m-1}^1 + S_1^{-1} S_2 Q^2 g_{m-1}^2) + \theta a_1, \end{cases} \tag{15}$$

with $a_1 = -S_1^{-1} S_2 U_\nu^2 - 2U_\nu^1$ and $b_1 = -S_2^{-1} S_1 U_\nu^1 - 2U_\nu^2$.

Let us introduce, the operator T defined by

$$\begin{aligned} T : (H^{1/2}(\Gamma_c^\alpha))^2 &\longrightarrow (H^{1/2}(\Gamma_c^\alpha))^2 \\ \mathbf{g} \longmapsto T(\mathbf{g}) = \begin{pmatrix} w_\nu^2 - u_\nu^2 \\ w_\nu^1 - u_\nu^1 \end{pmatrix} &= \begin{pmatrix} -2Q^2 g^2 - S_2^{-1} S_1 Q^1 g^1 + b_1 \\ -2Q^1 g^1 - S_1^{-1} S_2 Q^2 g^2 + a_1 \end{pmatrix} \end{aligned} \tag{16}$$

and T_θ

$$\begin{aligned} T_\theta : (H^{1/2}(\Gamma_c^\alpha))^2 &\longrightarrow (H^{1/2}(\Gamma_c^\alpha))^2 \\ \mathbf{g} &\longmapsto T_\theta(\mathbf{g}) = (1-\theta)\mathbf{g} + \theta T(\mathbf{g}). \end{aligned} \tag{17}$$

Using the definition of the operators T and T_θ, (15) can be expressed by

$$\mathbf{g}_m = T_\theta(\mathbf{g}_{m-1}) = (1-\theta)\mathbf{g}_{m-1} + \theta T(\mathbf{g}_{m-1}). \tag{18}$$

Theorem 2. *The operator T is a Lipschitz operator.*

Theorem 3. *There exists $\theta_0 \in]0,1[$ such that for θ in $]0,\theta_0[$, the operator T_θ is a contraction in a suitable norm equivalent to the $H^{1/2}(\Gamma_c^\alpha)$-norm.*

Remark 2. To prove Theorems 2 and 3, the properties of the operators S_α, Q^α and P^α are very important. Indeed $S_\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow H^{-1/2}(\Gamma_c^\alpha)$ is bounded, bijective, self-adjoint and coercive. The operator $Q^\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow H^{1/2}(\Gamma_c^\alpha)$ is a Lipschitz operator. $S_\alpha Q^\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow H^{-1/2}(\Gamma_c^\alpha)$ is Lipschitz and monotone. The extension operator $P^\alpha : H^{1/2}(\Gamma_c^\alpha) \longrightarrow P^\alpha(H^{1/2}(\Gamma_c^\alpha))$ is continuous and bijective (see [9]).

5 Numerical Experiments

In this section, we describe some numerical results obtained with algorithm (7)–(9) for various values of the parameter θ and various problem sizes. Our implementation uses a recently developed algorithm of quadratic programming with proportioning and gradient projections [4]. The numerical implementations are performed in Scilab 2.7 on a Pentium 4, 1.80 GHz workstation with 256 MB RAM. We set $tol = 10^{-8}$ and we stop the iterations, if their number is greater than eight hundred. For all experiments to be described below, the stopping criterion of algorithm (7)–(9) is

$$\frac{\|g_m^1 - g_{m-1}^1\|}{\|g_m^1\|} + \frac{\|g_m^2 - g_{m-1}^2\|}{\|g_m^2\|} \le tol,$$

where $\|\cdot\|$ denotes the Euclidean norm. The precision in the inner iterations are adaptively adjusted by the precision achieved in the outer loop.

Let us consider the plane elastic bodies

$$\Omega^1 = (0,3)\times(1,2) \quad \text{and} \quad \Omega^2 = (0,3)\times(0,1)$$

made of an isotropic, homogeneous material characterized by Young's modulus $E_\alpha = 2.1\ 10^{11}$ and Poisson's ratio $\nu_\alpha = 0.277$. The decomposition of Γ^1 and Γ^2 read as:

$$\begin{aligned}
&\Gamma_u^1 = \{0\}\times(1,2), \quad \Gamma_c^1 = (0,3)\times\{1\}, \quad \Gamma_\ell^1 = \Gamma^1 \setminus \overline{\Gamma_u^1 \cup \Gamma_c^1},\\
&\Gamma_u^2 = \{0\}\times(0,1), \quad \Gamma_c^2 = (0,3)\times\{1\}, \quad \Gamma_\ell^2 = \Gamma^2 \setminus \overline{\Gamma_u^2 \cup \Gamma_c^2}.
\end{aligned}$$

The volume forces vanish for both bodies. The non-vanishing surface traction $\ell^1 = (\ell_1^1, \ell_2^1)$ and $\ell^2 = (\ell_1^2, \ell_2^2)$ on Γ_ℓ^1 and on Γ_ℓ^2, respectively:

$$\begin{aligned}
&\ell_1^1(s,2) = 0, \quad \ell_2^1(s,2) = -100, \quad s \in (0,3),\\
&\ell_1^1(3,s) = 0, \quad \ell_2^1(3,s) = 0, \quad s \in (1,2),\\
&\ell_1^2(s,0) = 0, \quad \ell_2^2(s,0) = 0, \quad s \in (0,3),\\
&\ell_1^2(3,s) = 0, \quad \ell_2^2(3,s) = 0, \quad s \in (0,1).
\end{aligned}$$

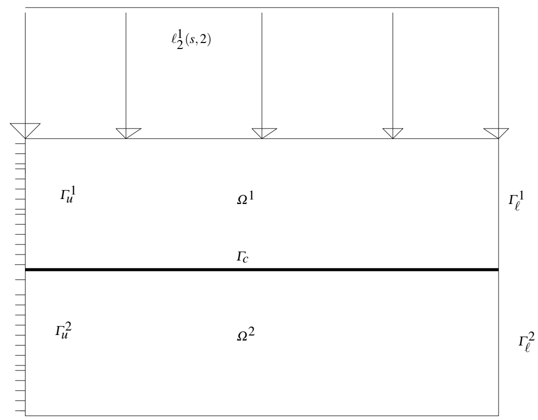

Fig. 1. Setting of the problem.

Fig. 2 illustrates the convergence of the algorithm (7)–(9) for different values of the relaxation parameter θ and various problem sizes with n the number of d.o.f. in $\Omega^1 \cup \Omega^2$ and m the number of d.o.f. on Γ_c^α. The results obtained show that the convergence of algorithm (7)–(9) does not depend on the mesh size h. Moreover, this algorithm (7)–9 converges for all $\theta \in]0,1[$.

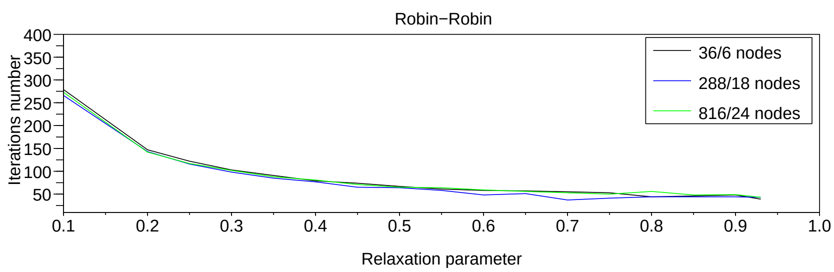

Fig. 2. Convergence rate of the algorithm.

References

[1] Agoshkov, V.I.: *Poincaré-Steklov's operators and domain decomposition methods in finite-dimensional spaces*. First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987), 73–112, SIAM, Philadelphia, PA, 1988.

[2] Bayada, G., Sabil, J., Sassi, T.: *Algorithme de Neumann-Dirichlet pour des problèmes de contact unilatéral: résultat de convergence*. (French)

[Neumann-Dirichlet algorithm for unilateral contact problems: convergence results] C. R. Math. Acad. Sci. Paris 335 (2002), no. 4, 381–386.

[3] Bayada, G., Sabil, J., Sassi, T.: *A Neumann-Neumann domain decomposition algorithm for the Signorini problem*. Appl. Math. Lett. 17 (2004), no. 10, 1153–1159.

[4] Dostal, Z., Schöberl, J.: *Minimizing quadratic functions over non-negative cone with the rate of convergence and finite termination*. Optim. Appl., 30, (2000), no. 1, 23–44.

[5] Duvaut, G., Lions, J.-L.: *Les inéquations en mécanique et en physique*, Dunod, Paris, (1972).

[6] Glowinski, R., Lions, J.-L., Trémolière, R.: *Numerical Analysis of variational Inequalities*, North-Holland, 1981.

[7] Guo, W., Hou, L.S.: *Generalizations and acceleration s of Lions' nonoverlapping domain decomposition method for linear elliptic PDE*. SIAM J. Numer. Anal. 41 (2003), no. 6, 2056–2080

[8] Haslinger, J., Hlavacek, I., Nečas, J.: *Numerical methods for unilateral problems in solid mechanics*. Handbook of numerical analysis, Vol. IV, 313–485, Handb. Numer. Anal., IV, North-Holland, Amsterdam, 1996.

[9] Ipopa, M.A.: *Algorithmes de d'composition de domaine pour les problèmes de contact: Convergence et simulations numériques*. Thesis, Université de Caen, 2008.

[10] Ipopa, M.A., Sassi, T.: *A Robin algorithm for unilateral contact problems*. C.R. Math Acad. Sci. Paris, Ser. I, 346 (2008), 357-362.

[11] Sassi, T., Ipopa, M.A., Roux, F.-X.: *Generalization of Lion's nonoverlapping domain decomposition method for contact problems*. Lect. Notes Comput. Sci. Eng., Vol 60, pp 623-630, 2008.

[12] Kikuchi, N., Oden, J.T.: *Contact problems in elasticity: a study of variational inequalities and finite element methods*, SIAM, Philadelphia, (1988)

[13] Koko, J.: *An Optimization-Based Domain Decomposition Method for a Two-Body Contact Problem*. Num. Funct. Anal. Optim., Vol. 24 , no. 5-6, 587–605, (2003).

[14] Krause, R.H., Wohlmuth, B.I.: *Nonconforming domain decomposition techniques for linear elasticity*. East-West J. Numer. Math. 8 (2000), no. 3, 177–206.

[15] Lions, P.-L.: *On the Schwarz alternating method. III. A variant for nonoverlapping subdomains*. Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989), 202–223, SIAM, Philadelphia, PA, 1990.