A Maximum Principle for L^2-Trace Norms with an Application to Optimized Schwarz Methods

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Summary. Harmonic functions attain their pointwise maximum on the boundary of the domain. In this article, we analyze the relationship between various norms of nearly harmonic functions and we show that the trace norm is maximized on the boundary of the domain. One application is that the Optimized Schwarz Method with two subdomains converges for all Robin parameters $\alpha > 0$.

1 Introduction

Given a domain $\Omega \subset \mathbb{R}^2$, consider the model problem

$$-\nabla \cdot (a\nabla u) + cu = f \;\text{ in } \Omega \quad \text{and} \quad u = 0 \;\text{ on } \partial\Omega, \tag{1}$$

where $a : \Omega \to M_{2\times 2}$ is a symmetric and coercive 2×2 matrix valued function of $x \in \Omega$ and c is a non-negative function of $x \in \Omega$. If we have a domain decomposition $\Omega = \Omega_1 \cup \Omega_2$ and given functions v_0, w_0 on Ω_1, Ω_2, respectively, typical domain decomposition algorithms iteratively solve problems of the type $(-\nabla \cdot (a\nabla) + c)v_k = f$ in Ω_1 and $(-\nabla \cdot (a\nabla) + c)w_k = f$ in Ω_2, $k \geq 1$, with some boundary conditions. In the classical Schwarz algorithm, the local problems use Dirichlet data. Optimized Schwarz Methods replace the Dirichlet subproblems by Robin subproblems; see [5] for a detailed discussion and bibliography. The analysis of the convergence of Optimized Schwarz Methods turned out to be more complicated than that of classical Schwarz methods; see [7, 8, 9].

Schwarz's idea [11] to prove the convergence was to use the maximum principle. This is based on the observation that the *error* iterates $u_k^{(i)} - u$ are (a,c)-harmonic on each subdomain, for every $k \geq 1$; i.e., they solve the PDE $-\Delta \cdot (a\Delta u_k^{(i)}) + cu_k^{(i)} = 0$. In a recent paper [10], we have shown that trace norms can be used in a similar way to show the convergence of Optimized Schwarz Methods, which use Robin boundary conditions on the interfaces between the subdomains.

Let Ω be a domain and $\Gamma_1 \subset \Omega$, $\Gamma_2 \subset \partial\Omega$ be curves. Our goal is to give some conditions under which there is a positive $\omega < 1$ such that the inequality

$$\int_{\Gamma_1} v^2 \le \omega \int_{\Gamma_2} v^2, \tag{2}$$

is satisfied for every (nearly) (a,c)-harmonic function v on Ω satisfying suitable boundary conditions. This result is more general than the one we presented in [10], where it is assumed that v is exactly (a,c)-harmonic.

The structure of this article is as follows. In Section 2, we introduce the notion of (ε,a,c)-harmonicity, and ε-relative uniformity. With these definitions, we are able to prove our main result, which is that the maximum trace L^2 norm is attained on the boundary. In Section 3, we discuss some applications to Schwarz methods.

2 The Maximum Principle for L^2-Trace Norms

To describe our main result, we must first discuss certain Sobolev estimates; we refer the reader to [1, 2, 7], and references therein for details.

2.1 Preliminaries on the Domain and the Interfaces

Let ρ be a nonnegative function on Ω. Let $H^1(\Omega,\rho)$ be the space of functions v of finite weighted Sobolev norm

$$\|v\|^2_{H^1(\Omega,\rho)} = \int_\Omega (|\nabla v|^2 + |v|^2)\rho.$$

If $\rho(x)$ goes to zero linearly as x approaches the boundary $\partial\Omega$, then the trace map $u \to u|_{\partial\Omega}$ is discontinuous, i.e., there is no trace space [6].

Let Ω be parametrized by a function $\Phi(x,y)$. The domain of Φ is $Z = \{(x,y)|x_0 \le x \le x_2 \text{ and } p(x) \le y \le q(x)\}$, where $x_0 < x_2$ are real numbers. For simplicity, all domains in this paper are Lipschitz.[1] We assume that p and q are C^1 and that Φ is C^2. Because of the parametrization, Ω is furthermore piecewise C^1 and connected.[2] For each fixed x, we define Γ_x to be the curve parametrized by $y \to \Phi(x,y)$. In the context of domain decompositions, Ω is one of the two overlapping subdomains. Choosing x_1 between x_0 and x_2, Γ_{x_1} and Γ_{x_2} are the interfaces defined by the boundary of the overlap. We define $U_x = \cup_{x_0 \le \xi \le x} \Gamma_\xi$, the part of Ω "to the left" of Γ_x; see Fig. 1. If v is in $H^1(\Omega)$, we define

$$e(x) = e(x,v) = \int_{\Gamma_x} v^2. \tag{3}$$

Thus, our goal is to find a positive $\omega < 1$, and conditions on Ω, Φ, v so that $e(x_1) < \omega e(x_2)$, i.e., that (2) holds.

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[1] We do not assume that Ω is convex.

[2] We could relax the connectedness hypothesis by using one chart per component.

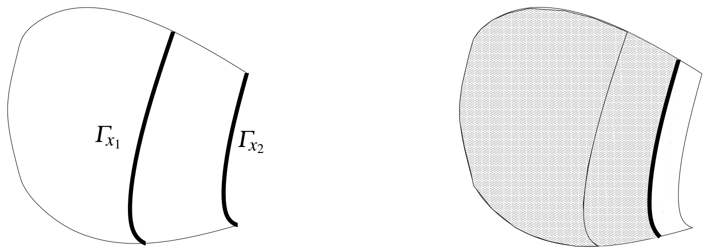

Fig. 1. Left: the domain Ω and the two interfaces Γ_{x_1} and Γ_{x_2} (in bold). Right: U_x (shaded) and Γ_x (bold).

2.2 (ε, a, c)-Harmonicity

A function v in $H^1(\Omega)$ is said to be (a,c)-*harmonic* if $(-\nabla \cdot a\nabla + c)v = 0$. Such functions obey the maximum principle: the L^∞ norm of an (a,c)-harmonic nonconstant function is attained on the boundary, and not on the interior. We want to find a notion of near-(a,c)-harmonicity, which we will call (ε,a,c)-harmonicity, such that a related maximum principle holds. To that end, let v be in $H^1(\Omega)$ with $v = 0$ on $\partial\Omega \setminus \Gamma_{x_2}$ and let ν be the outer normal to U_x. Consider the quantity

$$\begin{aligned} S &= \int_{x_1}^{x_2} \int_{p(x)}^{q(x)} v(a\nu) \cdot \nabla v |\Phi_y| \, dy \, dx \\ &= \int_{x_1}^{x_2} \int_{U_x} -v(-\nabla \cdot a\nabla + c)v + (\nabla v)^T a(\nabla v) + cv^2 \, ds \, dx \quad (4) \\ &= \int_\Omega [-v(-\nabla \cdot a\nabla + c)v + (\nabla v)^T a(\nabla v) + cv^2]\rho, \quad (5) \end{aligned}$$

where we have used Green's integration by parts to obtain (4), and Fubini's Theorem to obtain (5). The function $\rho(\mathbf{x})$ is therefore the Lebesgue measure of the set $\{\xi \in (x_1,x_2) : \mathbf{x} \in U_\xi\}$, and hence $\rho(\mathbf{x}) = O(dist(\mathbf{x},\Gamma_{x_2}))$. We want to be able to compare S with

$$\|v\|_L^2 = \int_\Omega [(\nabla v)^T a(\nabla v) + cv^2]\rho, \quad (6)$$

which is a norm that is equivalent to $\|v\|_{H^1(\Omega,\rho)}$.[3] To that end, for $\varepsilon \geq 0$, we say that $v \in H^1(\Omega)$ is (ε,a,c)-*harmonic* if

$$-\varepsilon\|v\|_L^2 \leq \int_\Omega v(-\nabla \cdot a\nabla + c)v\rho \leq \varepsilon\|v\|_L^2. \quad (7)$$

Note that an (a,c)-harmonic function is $(0,a,c)$-harmonic. If v is (ε,a,c)-harmonic, and if $v = 0$ on $\partial\Omega \setminus \Gamma_{x_2}$, then we have

[3] The equivalence of norms is a variant of the standard argument that the bilinear form of the elliptic operator is equivalent to the H^1 norm, but with the added weight ρ; see [2] for details. In the case $c = 0$, a variant of the Friedrichs' inequality is used.

$$(1-\varepsilon)\|v\|_L^2 \leq S \leq (1+\varepsilon)\|v\|_L^2.$$

We define

$$s(x) = \frac{\int_{\Gamma_x} \Phi_x^T a\nu}{\int_{\Gamma_x} (a\nu)^2}, \tag{8}$$

that is, for any $x \in (x_1, x_2)$, $s(x)a(\Phi(x,\cdot))\nu(\Phi(x,\cdot))$ is the orthogonal projection[4] of $\Phi_x(x,\cdot)$ onto the span of $a\nu$ in $L^2(\Gamma_x)$. Let g be a C^0 vector field. Then for $v \in H^1(\Omega)$, $D_g v = g \cdot \nabla v \in L^2(\Omega)$. We will use the field $g = \Phi_x - sa\nu$.

2.3 ε-Relative Uniformity

We now turn to the notion of *relative uniformity*. We first want to impose a condition so that Φ_x is not too tangent to Γ_x in the sense that there are constants $C_1 > 0$ and $C_2 < \infty$, such that $C_1 \leq s(x) \leq C_2$ for all x.

If a is the identity and Φ is conformal, then $\Phi_x \cdot \Phi_y = 0$, i.e., ν is parallel to Φ_y and $s(x)$ is strictly positive and bounded, cf. (8). If Φ is not conformal, but still $0 < C_1 \leq s \leq C_2 < \infty$, we say that Φ is "nearly conformal". Using (6), if v is (ε, a, c)-harmonic, for a fixed $\varepsilon > 0$, and $v = 0$ on $\partial\Omega \setminus \Gamma_{x_2}$, then there are constants $C_a = C_a(\varepsilon)$ and $C'_a = C'_a(\varepsilon)$ such that

$$C_a(\varepsilon)\|v\|_L^2 \leq \int_{x_1}^{x_2} s \int_{p(x)}^{q(x)} 2v(a\nu)\cdot\nabla v|\Phi_y|\,dy\,dx \leq C'_a(\varepsilon)\|v\|_L^2. \tag{9}$$

Specifically, one may use $C_a(\varepsilon) = 2(C_1 - \varepsilon C_2)$ and $C'_a(\varepsilon) = 2(C_2 + \varepsilon C_2)$. Further assume that there are constants $C_\nu < \infty$ and $C_0 < \infty$ such that

$$\left| \int_{x_1}^{x_2} \int_{p(x)}^{q(x)} 2v(\Phi_x - sa\nu)\cdot\nabla v|\Phi_y|\,dy\,dx \right| \leq C_\nu \|v\|_L^2, \tag{10}$$

$$\left| \int_{x_1}^{x_2} \int_{p(x)}^{q(x)} v^2 \frac{\Phi_y \cdot \Phi_{xy}}{|\Phi_y|}\,dy\,dx \right| \leq C_0 \|v\|_L^2. \tag{11}$$

If a is the identity and Φ is conformal, then $C_\nu = C_0 = 0$. Our allowances for $C_\nu, C_0 > 0$ means that we can use a Φ which is "nearly conformal".

Furthermore, if there is a diffeomorphism that turns a into the identity and Φ into a conformal map, we also have $C_\nu = C_0 = 0$. Thus, if the interfaces Γ_{x_1} and Γ_{x_2} are "nearly parallel" in the metric induced by a, these constants will be small. In our Definition 1 we want C_ν and C_0 to be small in the sense that their sum is smaller than $C_a(\varepsilon)$.

Definition 1. *Let Ω be a domain, fix $\varepsilon > 0$ and consider the elliptic problem (1). Let Φ be a parametrization of Ω as above. Let $s(x)$ be as in (8), and let the positive constants $C_a(\varepsilon)$, C_ν, and C_0 be such that (9), (10), and (11) hold. We say that the parametrization is* ε-relatively uniform *if the inequality $C_a(\varepsilon) - C_\nu - C_0 > 0$ holds, for every v which is (ε, a, c)-harmonic with $v = 0$ on $\partial\Omega \setminus \Gamma_{x_2}$.*

[4] There is no reason to prefer this particular choice of $s(x)$, but we hope that it makes $\Phi_x - sa\nu$ small in a useful way.

2.4 Maximum Principle for L^2-Trace Norms

We present our main Theorem which depends on the elliptic operator (as parametrized by a and c), on the domain Ω and its parametrization Φ, as well as on $\varepsilon > 0$.

Theorem 1. (Maximum principle for a trace norm) *Fix $\varepsilon > 0$ and let Φ be an ε-relatively uniform parametrization of Ω. Then, there exists a positive $\omega < 1$ such that, for every (ε, a, c)-harmonic $v \in H^1(\Omega)$ with $v = 0$ on $\partial\Omega \setminus \Gamma_{x_2}$, then the estimate $e(x_1) \leq \omega e(x_2)$ is satisfied.*

Proof. This proof proceeds in two steps. First, we justify differentiating under the integral sign, then we use Green's Theorem and various trace estimates to show that $e' \geq 0$. Let K be an upper bound for $|p(x)|$ and $|q(x)|$. Let $w \in H^1(Z)$ with $w(x, p(x)) = w(x, q(x)) = 0$ for $x \in (x_1, x_2)$. Let

$$\varphi(x) = \int_{p(x)}^{q(x)} w^2(x,y)\,dy. \tag{12}$$

Let $\varphi \in C_c^\infty(x_1, x_2)$, i.e., an infinitely differentiable, compactly supported test function on the interval (x_1, x_2), and consider the number

$$\eta = \eta(w) = \int_{x_1}^{x_2} \varphi(x)\varphi'(x)\,dx = \int_{x_1}^{x_2} \int_{p(x)}^{q(x)} w^2(x,y)\varphi'(x)\,dy\,dx.$$

We can extend w, first by zero to the strip $S = \{(x,y) | x_1 < x < x_2\}$ (since $p, q \in C^1$), then using a continuous extension operator to $H^1(\mathbb{R}^2)$ to obtain a new function $\tilde{w}$; see, e.g., [1]. We have $\tilde{w}|_Z = w$ and $\tilde{w}|_{S \setminus Z} = 0$. Likewise, by trivial extension of φ, we can consider $\tilde{\varphi}(x,y) = \pi(y)\varphi(x) \in C_c^\infty(\mathbb{R}^2)$ where $\pi(y)$ a smooth function which is uniformly one on $[-K, K]$, and zero outside of $[-K-1, K+1]$. Then we have

$$\begin{aligned}
\eta = \int_{\mathbb{R}^2} \tilde{w}^2(x,y) D_x \tilde{\varphi}(x,y)\,dx\,dy &= -\int_{\mathbb{R}^2} D_x(\tilde{w}^2(x,y))\tilde{\varphi}(x,y)\,dx\,dy \\
&= -\int_Z 2w(x,y)w_x(x,y)\varphi(x)\,dx\,dy \\
&= -\int_{x_1}^{x_2} \int_{p(x)}^{q(x)} 2w(x,y)w_x(x,y)\,dy\varphi(x)\,dx.
\end{aligned}$$

Hence, φ has a weak derivative and it is given by $\varphi'(x) = \int_{p(x)}^{q(x)} 2ww_x\,dy$. If we use $w(x,y) = v(\Phi(x,y))\sqrt{|\Phi_y|}$ in (12), we recover $e(x)$ as in (3). We thus obtain that $e(x)$ is weakly differentiable and that its weak derivative is

$$(D^w e)(x) = e'(x) = \int_{p(x)}^{q(x)} \left(2v\Phi_x \cdot \nabla v |\Phi_y| + v^2 \frac{\Phi_y \cdot \Phi_{xy}}{|\Phi_y|} \right) dy.$$

Therefore, by adding and subtracting the appropriate term and using (9), (10), and (11), we have

$$\begin{aligned}
e(x_2)-e(x_1) &= \int_{x_1}^{x_2} e'(x)\,dx \\
&= \int_{x_1}^{x_2}\int_{p(x)}^{q(x)} 2v\Phi_x\cdot\nabla v|\Phi_y|\,dy\,dx + \int_{x_1}^{x_2}\int_{p(x)}^{q(x)} v^2\frac{\Phi_y\cdot\Phi_{xy}}{|\Phi_y|}\,dy\,dx \\
&= \int_{x_1}^{x_2} s\int_{p(x)}^{q(x)} 2v(a\nu)\cdot\nabla v|\Phi_y|\,dy\,dx \qquad (13)\\
&\quad + \int_{x_1}^{x_2}\int_{p(x)}^{q(x)} 2v(\Phi_x - sa\nu)\cdot\nabla v|\Phi_y|\,dy\,dx \\
&\quad + \int_{x_1}^{x_2}\int_{p(x)}^{q(x)} v^2\frac{\Phi_y\cdot\Phi_{xy}}{|\Phi_y|}\,dy\,dx,
\end{aligned}$$

and thus

$$e(x_2)-e(x_1) \geq (C_a - C_\nu - C_0)\|v\|_L^2. \qquad (14)$$

Similarly, one obtains

$$\begin{aligned}
e(x_2)-e(x_1) &\leq (C_a' + C_\nu + C_0)\|v\|_L^2 \\
e(x_2) &\leq (C_a' + C_\nu + C_0 + C_T)\|v\|_L^2 \qquad (15)
\end{aligned}$$

where C_T is the constant of the trace inequality from $(H^1(\Omega), \|\cdot\|_L)$ to $L^2(\Gamma_{x_1})$.[5] Combining (14) and (15), one obtains the desired inequality

$$e(x_1) \leq \left(1 - \frac{C_a - C_\nu - C_0}{C_a' + C_\nu + C_0 + C_T}\right) e(x_2). \qquad \square$$

If we use the estimates $C_a = C_1 - \varepsilon C_2$ and $C_a' = C_2 + \varepsilon C_2$, we can make the dependence of ω on ε explicit:

$$\omega \leq 1 - \frac{C_1 - \varepsilon C_2 - C_\nu - C_0}{C_2 + \varepsilon C_2 + C_\nu + C_0 + C_T} < 1,$$

so long as the numerator $C_1 - \varepsilon C_2 - C_\nu - C_0 > 0$.

We mention that in [10] a version of our Theorem 1 for a block Gauss-Seidel algorithm is proved in the special case of a rectangular domain with $\Phi(x,y) = (x,y)$. We also mention that, while we proved Theorem 1 in the plane, it also holds in higher dimensions and on manifolds, under suitable hypotheses; see [10]. This is important because one cannot rely, e.g., on conformal maps in dimensions higher than 2 to prove results for general domains.

[5] This trace inequality follows from the following argument: since $\rho \gg 0$ on Γ_{x_1}, there is some neighborhood U of Γ_{x_1} such that $\rho(\mathbf{x}) > a > 0$ everywhere in U. Then, $\int_U (\nabla v)^2 + v^2 < \frac{1}{a}\int_\Omega ((\nabla v)^2 + v^2)\rho$, and we use the Trace Theorem [2] for $H^1(U)$.

3 Applications to Schwarz Methods

This maximum principle can be used to prove convergence of the classical Schwarz iteration. If Σ is a domain in the plane and $\Sigma = \Sigma_1 \cup \Sigma_2$ is an overlapping domain decomposition, and given $u_0 \in H_0^1(\Sigma)$, the alternating Schwarz method is

$$\begin{aligned} (-\nabla \cdot a\nabla + c)u_{k+j/2} &= f && \text{in } \Sigma_j, \\ u_{k+j/2} &= 0 && \text{on } \partial\Sigma, \\ u_{k+j/2} &= u_{k+(j-1)/2} && \text{on } \partial\Sigma_j \cap \Sigma_{3-j}; \end{aligned} \tag{16}$$

with $k = 0,1,2,\dots$ and $j = 1,2$. Obviously, the iteration converges if the Dirichlet data (16) converge to zero in $L^2(\partial\Sigma_j)$. Let $v_{k+j/2}^{(\ell)} := u - u_{k+j/2}^{(\ell)}$ be the error terms. By setting $\Omega = \Sigma_1$ in Theorem 1, we see that $\|v_{k+1/2}\|_{L^2(\partial\Sigma_2)} < \sqrt{\omega}\|v_k\|_{L^2(\partial\Sigma_1)}$. Similarly, if Theorem 1 holds with $\Omega = \Sigma_2$, we obtain that $\|v_{k+1}\|_{L^2(\partial\Sigma_1)} < \sqrt{\omega}\|v_{k+1/2}\|_{L^2(\partial\Sigma_2)}$. Chaining these together, one obtains

$$\|v_{k+1}\|_{L^2(\partial\Sigma_1)} < \omega\|v_k\|_{L^2(\partial\Sigma_1)}$$

and so the classical Schwarz iteration converges, and the error is multiplied by ω at every full iteration.

It is commonplace to use inexact solvers for the local problems, e.g., the multigrid algorithm. Such methods generate *inner iterates*: for each j,k, one obtains a sequence $u_{k+j/2}^{(\ell)}$, $\ell = 1,2,\dots$ which converges to $u_{k+j/2}$ in the limit. However, the iteration is typically stopped before the residual is zero. The inequality (7) is a condition on the size of the residual. If the residual $f - (-\nabla \cdot a\nabla + c)u_{k+j/2}^{(\ell)}$ is small, then for the error term, we have that $(-\nabla \cdot a\nabla + c)v_{k+j/2}^{(\ell)}$ is small, and so $v_{k+j/2}^{(\ell)}$ is (ε, a, c)-harmonic for some small ε.[6] Hence, the Schwarz iteration is robust in the sense that it will tolerate inexact local solvers.

Less obviously, a consequence of Theorem 1 is that the Optimized Schwarz Method converges. To that end, we say that a domain decomposition for which Theorem 1 holds for $\Omega = \Sigma_1$ as well as $\Omega = \Sigma_2$ is said to be ε-relatively uniform.

Theorem 2. *Let $\varepsilon > 0$ and assume that the domain decomposition is ε-relatively uniform. Then the Optimized Schwarz Method for the general elliptic problem* (1) *converges geometrically for any Robin parameter $\alpha > 0$.*

We prove this theorem in [10], but without the benefit of the $\varepsilon > 0$ parameter. Although we have not proved it, we hope that the robustness of the ε-harmonicity condition can be used to show that the Optimized Schwarz Method can also use inexact local solvers.

[6] If exact solvers are used, then (7) is verified with $\varepsilon = 0$ and $v = v_k$. If an inexact solver is used, then (7) can be used as a stopping criterion for $v = v_{k+j/2}^{(\ell)}$, assuming that the inexact solver can reach (7). If the inexact solver stops and (7) is not satisfied, it is possible for the outer iteration to stagnate, never reaching an error of 0.

Acknowledgement. We thank a referee for helpful comments. This research was supported in part by the U.S. Department of Energy under grant DE-FG02-05ER25672.

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