Optimized Schwarz Methods

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Summary. The strategy of domain decomposition methods is to decompose the computational domain into smaller subdomains. Each subdomain is assigned to one processor. The equations are solved on each subdomain. In order to enforce the matching of the local solutions, interface conditions have to be written on the boundary between subdomains. These conditions are imposed iteratively. The convergence rate is very sensitive to these interface conditions. The Schwarz method is based on the use of Dirichlet boundary conditions. It can be slow and requires overlapping decompositions. In order to improve the convergence and to be able to use non-overlapping decompositions, it has been proposed to use more general boundary conditions. It is even possible to optimize them with respect to the efficiency of the method. Theoretical and numerical results are given along with open problems.

1 Introduction: Original Schwarz Method (1870)

The first domain decomposition method was developed at the end of the 19th century by the mathematician H. A. Schwarz. His goal was to study the Laplace operator. At that time, the main tool for this purpose was Fourier analysis and more generally the use of special functions. Geometries of the domain were essentially restricted to simple configurations: rectangles and disks, see Fig. 1. His idea was to study the case of a domain that is the union of simple domains. For example, let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$. We want to solve

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Schwarz proposed the following algorithm (Alternating Schwarz Method):

Let (u_1^n, u_2^n) be an approximation to $(u_{|\Omega_1}, u_{|\Omega_2})$ at step n of the algorithm, (u_1^{n+1}, u_2^{n+1}) is defined by

$$\begin{aligned} -\Delta u_1^{n+1} &= f \quad \text{in } \Omega_1 \\ u_1^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{n+1} &= u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. \end{aligned} \qquad \begin{aligned} -\Delta u_2^{n+1} &= f \quad \text{in } \Omega_2 \\ u_2^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\ u_2^{n+1} &= u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{aligned}$$

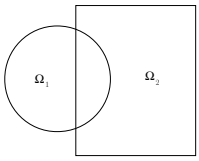

Fig. 1. Overlapping domain decomposition

The problem in domain Ω_1 has to be solved before the problem in domain Ω_2. This algorithm is sequential. Schwarz proved linear convergence of (u_1^n, u_2^n) to $(u_{|\Omega_1}, u_{|\Omega_2})$ as n tends to infinity.

A slight modification of the algorithm is

$$\begin{array}{llll} -\Delta u_1^{n+1} = f & \text{in } \Omega_1 & -\Delta u_2^{n+1} = f & \text{in } \Omega_2 \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega & u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ u_1^{n+1} = u_2^n & \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. & u_2^{n+1} = u_1^n & \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{array} \quad (2)$$

Problems in domains Ω_1 and Ω_2 may be solved concurrently. The algorithm is parallel and is adapted to parallel computers.

The discrete version of (2) is the **RAS** algorithm, see [7, 8].

1.1 Towards Faster Methods: Two Families of Methods

The benefit of the above Schwarz algorithms is the saving in memory requirements. Indeed, if the problems are solved by direct methods, the cost of the storage is nonlinear with respect to the number of unknowns. By dividing the original problem into smaller pieces the amount of storage can be significantly reduced. As far as CPU is concerned, the original Schwarz algorithms work fine for some problems but may be very slow for others. Roughly speaking for time dependent problems with relatively small time steps, the methods will perform well (e.g. transient compressible flow computations). But for steady state problems (e.g. Helmholtz or harmonic Maxwell's equations), it can be very slow. Another weakness is the need of overlapping subdomains. Indeed, only the continuity of the solution is imposed and nothing is imposed on the matching of the fluxes. When there is no overlap convergence is thus impossible.

The slowness of the method and the need for overlapping subdomains are linked. Indeed, it can be proved that the convergence rate of the Schwarz method is a continuous function of the size of the overlap denoted δ. For small overlaps the convergence rate is close to one. Actually it can be proved that for small overlaps the convergence rate varies as $1 - C^t\delta$.

In order to remedy the drawbacks of the original Schwarz method, two families of methods have been developed. They both work in the non-overlapping case and consist of introducing the normal derivative of the solution, but in two very different ways:

- write a substructured formulation of the domain decomposition problem where the matching of the solution and of its normal derivative along the interface are imposed explicitly.
- Modify the original Schwarz method by replacing the Dirichlet interface conditions on $\partial\Omega_i \setminus \partial\Omega$, $i = 1,2$, by Robin interface conditions ($\partial_{n_i} + \alpha$, where n is the outward normal to subdomain Ω_i), see [17].

The first approach corresponds to "Neumann-Neumann or FETI Methods". The second approach is developped in what follows.

More generally, a complete overview of various domain decomposition methods may be found in a few books [4, 22, 30, 32] or in the proceedings of various conferences on domain decomposition methods, see e.g. [1, 3, 16] and references therein.

2 Modified Schwarz Method

The Restrictive Additive Schwarz Method presents the drawback of needing overlapping subdomains in order to converge. In this chapter, we consider several improvements:

- replacement of the Dirichlet interface conditions by mixed interface conditions which yield convergence for non overlapping domain decompositions, see section 2.1;
- optimization of the interface conditions for faster convergence, see section 2.2;
- replacement of the fixed point iterative strategy of (2) by Krylov type methods, see [4].

2.1 Generalized Schwarz Methods

A major improvement of the Schwarz method comes from the use of other interface conditions. It has first been proposed by P.L. Lions to replace the Dirichlet interface conditions by Robin interface conditions, see [17]. Let α be a positive number; the modified algorithm is:

$$\begin{aligned} -\Delta u_1^{n+1} &= f \quad \text{in } \Omega_1, \\ u_1^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \left(\frac{\partial}{\partial n_1} + \alpha\right) u_1^{n+1} &= \left(-\frac{\partial}{\partial n_2} + \alpha\right) u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2} \end{aligned}$$

($\mathbf{n}_1$ and $\mathbf{n}_2$ are the outward normals on the boundary of the subdomains),

$$\begin{aligned} -\Delta u_2^{n+1} &= f \quad \text{in } \Omega_2, \\ u_2^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\ \left(\frac{\partial}{\partial n_2} + \alpha\right) u_2^{n+1} &= \left(-\frac{\partial}{\partial n_1} + \alpha\right) u_1^n \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{aligned}$$

The convergence proof given by P. L. Lions in the elliptic case was extended by B. Desprès [6] to the Helmholtz equation. A general presentation is given in [5]. It can also be extended to more general interface conditions with second order tangential derivatives in the interface conditions, see [19].

2.2 Optimal Interface Conditions

In the preceding section, we have seen that a general convergence result holds for interface conditions with Robin or second order tangential derivatives. Actually these conditions are not the most general. Rather than giving the general conditions in an *a priori* form, we shall derive them in this section so as to have the fastest convergence. We establish the existence of interface conditions which are optimal in terms of iteration counts. The corresponding interface conditions are pseudo-differential and are not practical. Nevertheless, this result is a guide for the choice of partial differential interface conditions. Moreover, this result establishes a link between the optimal interface conditions and artificial boundary conditions. This is also a help when dealing with the design of interface conditions since it gives the possibility to use the numerous papers and books published on the subject of artificial boundary conditions, see e.g. [8, 12].

We consider a general linear second order elliptic partial differential operator $\mathcal{L}$ and the problem:

$$\text{Find } u \text{ such that } \mathcal{L}(u) = f \text{ in a domain } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

The domain Ω is decomposed into two subdomains Ω_1 and Ω_2. We suppose that the problem is regular so that $u_i := u|_{\Omega_i}$, $i = 1,2$, is continuous and has continuous normal derivatives across the interface $\Gamma_i = \partial\Omega_i \cap \bar{\Omega}_j$, $i \neq j$. A modified Schwarz type method is considered.

$$\begin{aligned}
\mathcal{L}u_1^{n+1} &= f \quad \text{in } \Omega_1 \qquad u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega \\
\mu_1 \nabla u_1^{n+1}.\mathbf{n}_1 + \mathcal{B}_1 u_1^{n+1} &= -\mu_1 \nabla u_2^n.\mathbf{n}_2 + \mathcal{B}_1 u_2^n \quad \text{on } \Gamma_1 \\
\mathcal{L}u_2^{n+1} &= f \quad \text{in } \Omega_2 \qquad u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\
\mu_2 \nabla u_2^{n+1}.\mathbf{n}_2 + \mathcal{B}_2 u_2^{n+1} &= -\mu_2 \nabla u_1^n.\mathbf{n}_1 + \mathcal{B}_2 u_1^n \quad \text{on } \Gamma_2
\end{aligned} \tag{3}$$

where μ_1 and μ_2 are real-valued functions and $\mathcal{B}_1$ and $\mathcal{B}_2$ are operators acting on the interfaces Γ_1 and Γ_2. For instance, $\mu_1 = \mu_2 = 0$ and $\mathcal{B}_1 = \mathcal{B}_2 = \text{Id}$ correspond to the algorithm (2); $\mu_1 = \mu_2 = 1$ and $\mathcal{B}_i = \alpha \in \mathbf{R}$, $i = 1,2$, has been proposed in [17] by P.L. Lions.

The question is:

Are there other possibilities in order to have convergence in a minimal number of steps?

In order to answer this question, we note that by linearity, the error e satisfies ($\mu_1 = \mu_2 = 1$)

$$\begin{aligned}
\mathcal{L}(e_1^{n+1}) = 0 \quad \text{in } \Omega_1 \qquad & e_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega \\
\nabla e_1^{n+1} \cdot \mathbf{n}_1 + \mathcal{B}_1(e_1^{n+1}) = -\nabla e_2^n \cdot \mathbf{n}_2 + \mathcal{B}_1(e_2^n) \quad & \text{on } \Gamma_1 \\
\mathcal{L}(e_2^{n+1}) = 0 \quad \text{in } \Omega_2 \qquad & e_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\
\nabla e_2^{n+1} \cdot \mathbf{n}_2 + \mathcal{B}_2(e_2^{n+1}) = -\nabla e_1^n \cdot \mathbf{n}_1 + \mathcal{B}_2(e_1^n) \quad & \text{on } \Gamma_2
\end{aligned}$$

The initial guess e_i^0 is arbitrary so that it is impossible to have convergence at step 1 of the algorithm. Convergence needs at least two iterations. Having $e_1^2 \equiv 0$ requires $-\nabla e_2^1.\mathbf{n}_2 + \mathcal{B}_1(e_2^1) \equiv 0$. The only meaningful information on e_2^1 is that $\mathcal{L}(e_2^1) = 0 \quad \text{in } \Omega_2$. In order to use this information, we introduce the DtN (Dirichlet to Neumann) map (a.k.a. Steklov-Poincaré): Let

$$\begin{aligned}
u_0 &: \Gamma_1 \to \mathbf{R} \\
\mathrm{DtN}_2(u_0) &:= \nabla v.\mathbf{n}_{2|\partial\Omega_1 \cap \bar{\Omega}_2},
\end{aligned} \tag{4}$$

where $\mathbf{n}_2$ is the outward normal to $\Omega_2 \setminus \bar{\Omega}_1$, and v satisfies the following boundary value problem:

$$\begin{aligned}
\mathcal{L}(v) = 0 \quad & \text{in } \Omega_2 \setminus \bar{\Omega}_1 \\
v = 0 \quad & \text{on } \partial\Omega_2 \cap \partial\Omega \\
v = u_0 \quad & \text{on } \partial\Omega_1 \cap \bar{\Omega}_2.
\end{aligned}$$

Let $\mathcal{B}_1 := \mathrm{DtN}_2$. This choice is optimal since we can check that $-\nabla e_2^1 \cdot \mathbf{n}_2 + \mathcal{B}_1(e_2^1) \equiv 0$. *The use of* $\mathcal{B}_i = \mathrm{DtN}_j$ $(i \neq j)$ *as interface conditions in* (3) *is optimal*: *we have* (*exact*) *convergence in two iterations*.

The two-domain case for an operator with constant coefficients has been first treated in [13]. The multidomain case for a variable coefficient operator with both positive results [20] and negative conjectures [21] has been considered as well.

Remark 1. The main feature of this result is to be very general since it does not depend on the exact form of the operator $\mathcal{L}$ and can also be extended to more general systems or to coupled systems of equations as well with a proper care of the well posedness of the algorithm.

As an application, we take $\Omega = \mathbf{R}^2$ and $\Omega_1 = (-\infty, 0) \times \mathbf{R}$. Using a Fourier technique, it is possible to give the explicit form of the DtN operator for a constant coefficient operator. If $\mathcal{L} = \eta - \Delta$, the DtN map is a pseudo-differential operator whose symbol is

$$B_{i,\mathrm{opt}}(k) = \sqrt{\eta + k^2},$$

i.e., $\mathcal{B}_{i,\mathrm{opt}}(u)(0,y) = \int_{\mathbf{R}} B_{i,\mathrm{opt}}(k)\widehat{u}(0,k)e^{iky}\,dk$. This symbol is not polynomial in the Fourier variable k so that the operators and the above example shows, exact absorbing conditions are in general pseudo-differential. They correspond to exact absorbing conditions. These conditions are used on the artificial boundary resulting from the truncation of a computational domain. On this boundary, boundary conditions have to be imposed. The solution on the truncated domain depends on the choice of this

artificial condition. We say that it is an exact absorbing boundary condition if the solution computed on the truncated domain is the restriction of the solution of the original problem. Surprisingly enough, the notions of exact absorbing conditions for domain truncation and that of optimal interface conditions in domain decomposition methods coincide.

As the above examples show, they are pseudodifferential. Therefore they are difficult to implement. Moreover, in the general case of variable coefficient operators and/or a curved boundary, the exact form of these operators is not known, although they can be approximated by partial differential operators which are easier to implement. The approximation of the DtN has been addressed by many authors since the seminal paper [8] by Engquist and Majda on this question.
It turns out that the approximations designed for domain truncation perform poorly when used in domain decomposition methods. There have been many research efforts in the last 15 years on how to tune approximate DtN maps so that they perform well in domain decomposition methods. The first works were based on Fourier techniques, see e.g. [3, 4, 5] and references therein. These approaches work fine for smooth coefficients operators. But when dealing with highly discontinuous coefficients, it is necessary to take a more algebraic approach, see [11, 18] in this direction. Results are promising but many issues are still open, see below.

3 Conclusion and Open Problems

Both approaches (Neumann-Neumann and optimized Schwarz methods) are robust thanks to Krylov methods. Neumann-Neumann, BDDC and FETI type methods are optimal but lack generality. Optimized Schwarz methods are general but are more difficult to tune. The main open problems are from a practical point of view

- the design of algebraic optimized interface conditions that are as efficient as the analytic ones
- the interplay between the optimized interface conditions and a coarse grid (see [15])

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