Discontinuous Galerkin and nonconforming in time optimized Schwarz waveform relaxation

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1 Introduction

In many fields of applications such as reactive transport or ocean-atmosphere coupling, models with very different spatial and time scales have to be coupled. Optimized Schwarz Waveform Relaxation methods (OSWR), applied to linear advection-reaction-diffusion problems in Bennequin et al. [2009], Martin [2005], provide efficient solvers for this purpose. They have two main advantages: first, they are global in time and thus permit non conforming space-time discretization in different subdomains, and second, few iterations are needed to compute an accurate solution, due to optimized transmission conditions. It has been proposed in Halpern and Japhet [2008] to use a discontinuous Galerkin method in time as a subdomain solver. Rigorous analysis can be made for any degree of accuracy and local time-stepping, and finally time steps can be adaptively controlled by a posteriori error analysis, see Thomee [1997], Johnson et al. [1985], Makridakis and Nochetto [2006].

We present here the 2D analysis of the method. The time interval is split into time windows, and in each time window, a few iterations of an OSWR algorithm are computed, using second order optimized transmission conditions. The subdomain solver is the discontinuous Galerkin method in time, and classical finite elements in space. Coupling between subdomains is done by a simple and optimal projection algorithm without any additional grid (see Gander et al. [2003], Gander et al. [2005]). The mathematical analysis is carried out on the problem semi-discrete in time. The nonconforming DG-OSWR domain decomposition method is proved to be well-posed and convergent for a decomposition into strips, and the error analysis is performed in the case of Robin transmission conditions. We present numerical results in two dimen-

sions which extends the domain of validity of the approach to the fully discrete problem.

We consider the advection-reaction-diffusion equation in $\mathbb{R}^2$, written for variational purpose in the form

$$\partial_t u + \frac{1}{2}\nabla\cdot(bu) + \frac{1}{2}b\cdot\nabla u - \nabla\cdot(\nu\nabla u) + cu = f. \tag{1}$$

The initial condition is u_0. The advection and diffusion coefficients $b = (b^1, b^2)$ and ν, as well as the reaction coefficient c, are piecewise constant, *i.e.* constant in the subdomains $\Omega_i,\ i \in \{1, ..., I\}$. The subdomains are strips $\Omega_i = (\alpha_i, \alpha_{i+1}) \times \mathbb{R}$, with $\alpha_1 = -\infty$ and $\alpha_{I+1} = +\infty$. More general geometries as well as piecewise smooth coefficients will be studied in Halpern et al. [2009]. We suppose that $\nu > 0$ and $c > 0$.

2 Local problem and time discontinuous Galerkin

The optimized Schwarz waveform relaxation algorithm, as described in Bennequin et al. [2009], introduces a sequence of initial boundary value problems in $\Omega = (\alpha, \beta) \times \mathbb{R}$ of the following type:

$$\begin{aligned} &\partial_t u + \tfrac{1}{2}\nabla\cdot(bu) + \tfrac{1}{2}b\cdot\nabla u - \nabla\cdot(\nu\nabla u) + cu = f \text{ in } \Omega\times(0,T),\\ &(\nu\,\partial_n - \frac{b\cdot n}{2})u + \mathcal{S}u = g \text{ on } \Gamma\times(0,T), \end{aligned} \tag{2}$$

where n is the unit outward normal to Γ, and $\mathcal{S}$ is the boundary operator defined on $\Gamma = \{\alpha, \beta\} \times \mathbb{R}$ by $\mathcal{S}u = p\,u + q\,(\partial_t u + r\partial_y u - s\partial_{yy} u)$. Here p, q, r and s are real parameters, constrained to $p > 0, q \geq 0$, $s > 0$. If $q = 0$, the boundary condition reduces to a Robin boundary condition. We define the bilinear forms m and a by $m(u, v) = (u, v)_{L^2(\Omega)} + q(u, v)_{L^2(\Gamma)}$, and

$$\begin{aligned} a(u,v) := \int_\Omega \big(\frac{1}{2}((b\cdot\nabla u)v - (b\cdot\nabla v)u) + \nu\nabla u\cdot\nabla v\ + cuv\big)dx \\ + \int_\Gamma \big(qs\partial_y\, u\partial_y v + q\,r\partial_y uv + puv\big)\,dy. \end{aligned} \tag{3}$$

By the Green's formula, we obtain a variational formulation of (2):

$$\frac{d}{dt}\,m(u,v) + a(u,v) = (f,v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma)}, \quad \forall v \in V, \tag{4}$$

with $V = H^1(\Omega)$ if $q = 0$ and $V = H^1_1(\Omega)$ defined below, if $q > 0$. The problem is well-posed: if $q = 0$, if f is in $L^2(0, T, L^2(\Omega))$, u_0 is in $H^1(\Omega)$, and g is in $L^2(0, T, H^{1/2}(\Gamma))$, then the subdomain problem (2) has a unique solution u in $L^2(0, T, H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$. If $q > 0$, we introduce the

spaces $H^s_s(\Omega) = \{v \in H^s(\Omega), v_{|\Gamma} \in H^s(\Gamma)\}$ which are defined for $s > 1/2$. If f is in $L^2(0,T,L^2(\Omega))$, u_0 is in $H^1_1(\Omega)$, and g is in $L^2((0,T)\times\Gamma)$, then the subdomain problem (2) has a unique solution u in $L^2(0,T,H^2_2(\Omega))$ with $\partial_t u \in L^2(0,T;L^2(\Omega)\cap L^2(\Gamma))$, see Szeftel [2004], Bennequin et al. [2009].

We now introduce the time-discontinuous Galerkin method, as described and analysed in Johnson et al. [1985]. We are given a decomposition $\mathcal{T}$ of the time interval $(0,T)$, $I_n = (t_n, t_{n+1}]$, for $0 \le n \le N$, the mesh size is $k_n = t_{n+1} - t_n$. For $\mathcal{B}$ a Banach space and I an interval of $\mathbb{R}$, define for any integer $d \ge 0$

$$\mathbf{P}_d(\mathcal{B},\mathcal{T}) = \{\varphi : (0,T) \to \mathcal{B},\ \varphi_{|I_n} = \sum_{i=0}^{d} \varphi_i t^i,\ \varphi_i \in \mathcal{B},\ 0 \le n \le N\}.$$

Let $\mathcal{B} = H^1_1(\Omega)$ if $q > 0$, $\mathcal{B} = H^1(\Omega)$ if $q = 0$. We approximate u by a function $U \in \mathbf{P}_d(\mathcal{B},\mathcal{T})$ such that $U(0,\cdot) = u_0$ and for all V in $\mathbf{P}_d(\mathcal{B},\mathcal{T})$,

$$\int_{I_n} (\, m(\dot{U},V) + a(U,V))\, dt + m(U(t_n^+) - U(t_n^-), V(t_n^+)) = \int_{I_n} L(V)\, dt, \quad (5)$$

with $L(V) = (f,V)_{L^2(\Omega)} + (g,V)_{L^2(\Gamma)}$. Due to the discontinuous nature of the test and trial spaces, the method is an implicit time stepping scheme, and $U \in \mathbf{P}_d(\mathcal{B},\mathcal{T})$ is obtained recursively on each subinterval, which makes the method very flexible.

Theorem 1. *If $p > 0$, $q \ge 0$, $s > 0$, equation (5) defines a unique solution.*

The result relies on the fact that the bilinear form a is positive definite. This is most easily seen by using a basis of Legendre polynomials.

We will make use of the following remark due to Makridakis and Nochetto [2006]. Introduce the Gauss-Radau points, $(0 < \tau_1 < \ldots < \tau_{d+1} = 1)$, defined such that the quadrature formula $\int_0^1 f(t)dt \approx \sum_{q=1}^{d+1} w_q f(\tau_q)$ is exact in $\mathbf{P}_{2d}$, and the interpolation operator $\mathcal{I}_n$ on $[t_n, t_{n+1}]$ at points $(t_n, t_n + \tau_1 k_n, \ldots, t_n + \tau_{d+1} k_n)$. For any $\chi \in \mathbf{P}_d$, $\mathcal{I}_n\chi \in \mathbf{P}_{d+1}$, is such that $\mathcal{I}_n\chi(t_n) = \chi(t_n^-)$, $\mathcal{I}_n\chi(t_{n+1}) = \chi(t_{n+1}^-)$, and therefore for any ψ in $\mathbf{P}_d$, we have

$$\int_{I_n} \frac{d}{dt}(\mathcal{I}_n\chi)\psi\, dt - \int_{I_n} \frac{d\chi}{dt}\,\psi\, dt = (\chi(t_n^+) - \chi(t_n^-))\psi(t_n^+). \quad (6)$$

As a consequence, we have a very useful inequality:

$$\int_{I_n} \frac{d}{dt}(\mathcal{I}_n\psi)\psi dt \ge \frac{1}{2}[\psi(t_{n+1}^-)^2 - \psi(t_n^-)^2]. \quad (7)$$

Equation (5) can be written in a strong form as

$$\begin{array}{l} \partial_t(\mathcal{I}\,U) + \frac{1}{2}\nabla\cdot(bU) + \frac{1}{2}b\cdot\nabla U - \nabla\cdot(\nu\nabla U) + c\,U = \mathcal{P}f \text{ in } \Omega\times(0,T), \\ \big(\nu\,\partial_n - \frac{b\cdot n}{2}\big)U + p\,U + q(\partial_t(\mathcal{I}U) + r\partial_y U - s\partial_{yy}U) = \mathcal{P}g \text{ on } \Gamma\times(0,T), \end{array} \quad (8)$$

where $\mathcal{P}$ is the L^2 projection in time on $\mathbf{P}_d(\mathcal{B},\mathcal{T})$ ($\mathcal{B}$ is defined by the underlying space), and $\mathcal{I}$ is the operator whose restriction to each subinterval is $\mathcal{I}_n$. We discuss now the iterative algorithm.

3 The optimized Schwarz waveform relaxation algorithm discretized in time with different subdomain grids

For each subdomain Ω_i, the indices of the neighbouring subdomains are $j \in \mathcal{N}_i$. Since b is constant in Ω_i, equal to b_i, $\frac{1}{2}\nabla\cdot(b_i u_i^k) + \frac{1}{2} b_i\cdot\nabla u_i^k = \nabla\cdot(b_i u_i^k)$. At the continuous level, the algorithm is

$$\partial_t u_i^k + \nabla\cdot(b_i u_i^k - \nu_i \nabla u_i^k) + c_i u_i^k = f \text{ in } \Omega_i\times(0,T), \tag{9}$$
$$(\nu_i\partial_{n_i} - \frac{b_i\cdot n_i}{2}) u_i^k + \mathcal{S}_{ij} u_i^k = (\nu_j\partial_{n_i} - \frac{b_j\cdot n_i}{2})\, u_j^{k-1} + \mathcal{S}_{ij}\, u_j^{k-1} \text{ on } \Gamma_{ij}, j\in\mathcal{N}_i,$$

with $\nu = \nu_i$ in Ω_i, $\mathcal{S}_{ij}u = p_{ij}\,u + q_{ij}\,(\partial_t u + r_{ij}\partial_y u - s_{ij}\partial_{yy}u)$.

Theorem 2. *For any value of $p_{ij} > 0$, $q_{ij} = q \geq 0$, $r_{ij} = r$ and $s_{ij} = s > 0$, the algorithm* (9) *converges in each subdomain to the solution u of problem* (1).

The proof of this theorem will be given in Halpern et al. [2009], for general geometries and variable coefficients. It relies on elaborate energy estimates, the use of Trace Theorems and the Gronwall Lemma.

Our purpose here is to describe the discrete formulation in detail. The time partition in subdomain Ω_i is $\mathcal{T}_i$, with N_i+1 intervals I_n^i, and mesh size k_n^i. In view of formulation (8), we define interpolation operators $\mathcal{I}^i$ and projection operators $\mathcal{P}^i$ in each subdomain, and we solve

$$\partial_t(\mathcal{I}^i U_i^k) + \nabla\cdot(b_i U_i^k - \nu_i\nabla U_i^k) + c_i\, U_i^k = \mathcal{P}^i f \text{ in } \Omega_i\times(0,T), \tag{10}$$
$$\big(\nu_i\partial_{n_i} - \frac{b_i\cdot n_i}{2}\big)\, U_i^k + S_{ij} U_i^k = \mathcal{P}^i\big((\nu_j\partial_{n_i} - \frac{b_j\cdot n_i}{2})\, U_j^{k-1} + \widetilde{S}_{ij} U_j^{k-1}\big) \text{ on } \Gamma_{ij},$$

with $S_{ij}U = p_{ij}\,U + q_{ij}\,(\partial_t(\mathcal{I}^i U) + r_{ij}\partial_y U - s_{ij}\partial_{yy}U)$ and $\widetilde{S}_{ij}U = p_{ij}\,U + q_{ij}\,(\partial_t(\mathcal{I}^j U) + r_{ij}\partial_y U - s_{ij}\partial_{yy}U)$. If the algorithm converges, it converges to the solution of

$$\partial_t(\mathcal{I}^i U_i) + \nabla\cdot(b_i U_i - \nu_i\nabla U_i) + c_i\, U_i = \mathcal{P}^i f \text{ in } \Omega_i\times(0,T),$$
$$\big(\nu_i\partial_{n_i} - \frac{b_i\cdot n_i}{2}\big)\, U_i + S_{ij}\, U_i = \mathcal{P}^i\big((\nu_j\partial_{n_i} - \frac{b_j\cdot n_i}{2})\, U_j + \widetilde{S}_{ij}\, U_j\big) \text{ on } \Gamma_{ij}. \tag{11}$$

Theorem 3. *Assume $p_{ij} = p > 0$. If $q_{ij} = 0$, or if $q_{ij} = q > 0$ with $r_{ij} = 0$, $s_{ij} = s > 0$ and $b_i = 0$, Problem* (11) *has a unique solution $(U_i)_{i\in J}$, and U_i is the limit of the iterates of algorithm* (10).

The proof is based on energy estimates (see Halpern et al. [2009]).

We now state the error estimate.

Theorem 4. *Let $k = \sup_n k_n$. If $p_{ij} = p > 0$ and $q_{ij} = 0$, the error between u and the solution U_i of* (11) *is estimated by:*

$$\sum_{i=1}^{I} \|u - U_i\|^2_{L^\infty(0,T,L^2(\Omega_i))} \leq C k^{2(d+1)} \|\partial_t^{d+1} u\|^2_{L^2(0,T;H^2(\Omega))}. \tag{12}$$

Proof. We introduce the projection operator P_i^- as

$$\forall n \in \{1, ..., N_i\},\ P_i^- \varphi \in \mathbf{P}_d(I_n^i),$$
$$P_i^- \varphi(t_{n+1}^i) = \varphi(t_{n+1}^i), \quad \forall \psi \in \mathbf{P}_{d-1}(I_n^i), \int_{I_n^i} (P_i^- \varphi - \varphi)(t)\psi(t)\, dt = 0.$$

We define $W_i = P_i^-(u|_{\Omega_i})$, $\Theta_i = U_i - W_i$ and $\rho_i = W_i - u|_{\Omega_i}$. Classical projection estimates in Thomee [1997] yield the estimate on ρ_i:

$$\sum_{i=1}^{I} \|\rho_i\|^2_{L^\infty(0,T,L^2(\Omega_i))} \leq C k^{2(d+1)} \|\partial_t^{d+1} u\|^2_{L^2(0,T;L^2(\Omega))}.$$

Since $U_i - u|_{\Omega_i} = \Theta_i + \rho_i$, it suffices to prove estimate (12) for Θ_i. Now, using the equations of u and U_i, and the identity $\frac{d}{dt}\mathcal{I}^i P_i^- = \mathcal{P}^i \frac{d}{dt}$, Θ_i satisfies:

$$\begin{aligned}
&\partial_t(\mathcal{I}^i\,\Theta_i) + \nabla\cdot(b_i\Theta_i) - \nu\Delta\Theta_i + c_i\Theta_i = \mathcal{P}^i(-\nabla\cdot(b_i\rho_i) + \nu\Delta\rho_i - c_i\rho_i) \\
&\qquad\qquad + (1-\mathcal{P}^i)\partial_t u \text{ in } \Omega_i \times (0,T), \\
&\big(\nu_i\,\partial_{n_i} - \frac{b_i\cdot n_i}{2}\big)\Theta_i + p\,\Theta_i = \mathcal{P}^i\big((\nu_j\,\partial_{n_i} - \frac{b_j\cdot n_i}{2})\Theta_j + p\,\Theta_j\big) \\
&\qquad\qquad - (1-\mathcal{P}^i)\big((\nu_j\,\partial_{n_i} - \frac{b_j\cdot n_i}{2})W_j + p\,W_j\big) \text{ on } \Gamma_{ij}\times(0,T).
\end{aligned} \tag{13}$$

We set $\|\varphi\|_i = \|\varphi\|_{L^2(\Omega_i)}$ and $|||\varphi|||_i^2 = \nu_i\|\nabla\varphi\|^2_{L^2(\Omega_i)} + c\|\varphi\|^2_{L^2(\Omega_i)}$. Multiply the first equation of (13) by Θ_i, integrate on $(t_n^i, t_{n+1}^i)\times\Omega_i$, using (7) and integrate by parts in space. Terminate with Cauchy Schwarz inequality:

$$\begin{aligned}
\frac{1}{2}\|\Theta_i((t_{n+1}^i)^-)\|_i^2 + \int_{I_n^i} |||\Theta_i(t,\cdot)|||_i^2\, dt - \int_{I_n^i}\int_{\Gamma_i} (\nu_i\partial_{n_i}\Theta_i - \frac{b_i\cdot n_i}{2}\Theta_i)\,\Theta_i dy\, dt \\
\leq \frac{1}{2}\|\Theta_i((t_n^i)^-)\|_i^2 + C\int_{I_n^i} \|\rho_i(t,\cdot)\|^2_{H^2(\Omega_i)}\, dt.
\end{aligned}$$

Rewriting the boundary integral, we obtain:

$$\begin{aligned}
&\frac{1}{2}\|\Theta_i((t_{n+1}^i)^-)\|_i^2 + \int_{I_n^i} |||\Theta_i(t,\cdot)|||_i^2\, dt \\
&\qquad + \frac{1}{4p}\sum_{j\in\mathcal{N}_i}\int_{I_n^i}\int_{\Gamma_{ij}} (\nu_i\partial_{n_i}\Theta_i - \frac{b_i\cdot n_i}{2}\Theta_i - p\Theta_i)^2\, dy\, dt \\
&\leq \frac{1}{4p}\sum_{j\in\mathcal{N}_i}\int_{I_n^i}\int_{\Gamma_{ij}} (\nu_i\partial_{n_i}\Theta_i - \frac{b_i\cdot n_i}{2}\Theta_i + p\Theta_i)^2\, dy \\
&\qquad + \frac{1}{2}\|\Theta_i((t_n^i)^-)\|_i^2 + C\int_{I_n^i} \|\rho_i(t,\cdot)\|^2_{H^2(\Omega_i)}\, dt.
\end{aligned}$$

Using the transmission condition in (13) and the fact that $\mathcal{P}^i$ and $1-\mathcal{P}^i$ are orthogonal to each other and have norm 1, we get by a trace theorem:

$$\frac{1}{2}\|\Theta_i((t^i_{n+1})^-)\|_i^2 + \int_{I^i_n} |||\Theta_i(t,\cdot)|||_i^2 \, dt$$
$$+ \frac{1}{4p}\sum_{j\in\mathcal{N}_i}\int_{I^i_n}\int_{\Gamma_{ij}} (\nu_i\partial_{n_i}\Theta_i - \frac{b_i\cdot n_i}{2}\Theta_i - p\Theta_i)^2\, dy\, dt$$
$$\leq \frac{1}{4p}\sum_{j\in\mathcal{N}_i}\int_{I^i_n}\int_{\Gamma_{ij}} (\nu_j\partial_{n_j}\Theta_j - \frac{b_j\cdot n_j}{2}\Theta_j - p\Theta_j)^2\, dy + \frac{1}{2}\|\Theta_i((t^i_n)^-)\|_i^2$$
$$+ C\int_{I^i_n}\|\rho_i(t,\cdot)\|^2_{H^2(\Omega_i)}\, dt + C\int_{I^i_n}\|(1-\mathcal{P}^i)(u|_{\Omega_i})(t,\cdot)\|^2_{H^2(\Omega_i)}\, dt. \quad (14)$$

Classical error estimates in Thomee [1997] imply:

$$\int_0^T \|\rho_i(t,\cdot)\|^2_{H^2(\Omega_i)}\, dt + \int_0^T \|(1-\mathcal{P}^i)(u|_{\Omega_i})(t,\cdot)\|^2_{H^2(\Omega_i)}$$
$$\leq Ck^{2(d+1)}\|\partial_t^{d+1}u\|^2_{L^2(0,T;H^2(\Omega_i))}. \quad (15)$$

Summing (14) in j and n, and using the previous equation yields (12).

4 Numerical results

The above analysis deals with continuous problems and problems semi-discretized in time. We have implemented the algorithm with $d = 1$ and $\mathbf{P}_1$ finite elements in space in each subdomain using mortar methods like in Gander et al. [2005], in order to permit non-matching grids in time and space on the boundary. Time windows are used in order to reduce the number of iterations of the algorithm. In the first example, the coefficients are optimized numerically using the convergence factor. In the second one, formulas from Bennequin et al. [2009] are used.

We first give an example of a multidomain solution with time windows. The physical domain is $\Omega = (0,1)\times(0,2)$, the final time is $T = 4$. The initial value and the right hand side are $u_0 = f = e^{-100((x-0.55)^2+(y-1.7)^2)}$. The domain Ω is split into two subdomains $\Omega_1 = (0,0.5)\times(0,2)$ and $\Omega_2 = (0.5,1)\times(0,2)$. The reaction c is zero, the advection and diffusion coefficients are $b_1 = (0,-1)$, $\nu_1 = 0.05$, and $b_2 = (-0.1,0)$, $\nu_2 = 0.1$. The mesh size and time step in Ω_1 are $h_1 = 3.93.10^{-2}$ and $k_1 = 2.5.10^{-2}$, while in Ω_2, $h_2 = 8.84.10^{-2}$ and $k_2 = 6.25.10^{-2}$. On Figure 1, we observe, at final time $T = 4$, that the approximate solution computed using 4 uniform time windows, with 3 iterations in the first time window, and then 2 iterations in the next ones (right figure), is close to the reference solution computed in one time window on a conforming finer space-time grid (left figure).

We analyze now the precision for continuous coefficients. The advection field is $b = (-\sin(\pi(y-\frac{1}{2}))\cos(\pi(x-\frac{1}{2})), \cos(\pi(y-\frac{1}{2}))\sin(\pi(x-\frac{1}{2})))$, and the diffusion is $\nu = 1$. The exact solution is given by $u(x,t) =$

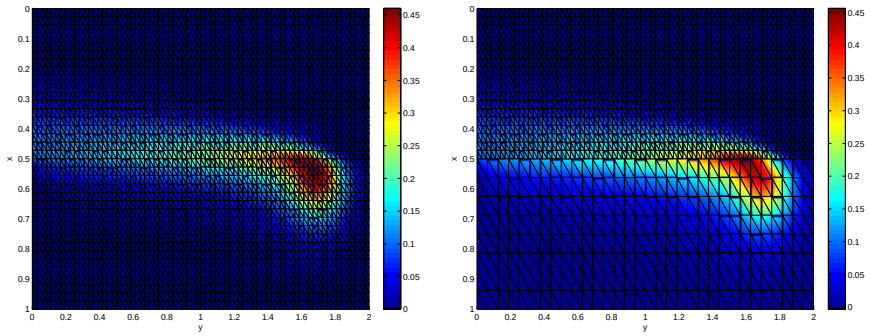

Fig. 1. Computation using discontinuous Galerkin with time windows

$cos(\pi x)sin(\pi y)cos(\pi t)$, in the unit square. The domain is decomposed into 2 subdomains with the interface at $x = 0.3$. The space grid is fixed and non conforming with mesh sizes $h_1 = 0.0074$ and $h_2 = 0.011$. We start with four time grids : time grids 1 and 2 are the conforming finner and coarser ones with respectively 7 and 5 grid points in each domain. Time grid 3 is nonconforming with 5 grid points in Ω_1 and 7 grid points in Ω_2, and time grid 4 is nonconforming with 7 grid points in Ω_1 and 5 grid points in Ω_2. Thereafter the time steps are divided by 2 several times. Figure 2 shows the norms of the error in $L^\infty(I; L^2(\Omega_i))$ versus the number of time refinements, for subdomain 1 on the left, and subdomain 2 on the right. First we observe the order 2 in time for conforming and nonconforming cases. They fit the theoretical estimates, even though we have theoretical results only for Robin transmission conditions and the space continuous problem. Moreover, the error obtained in the nonconforming case, in the subdomain where the grid is finer, is nearly the same as the error obtained in the conforming finer case.

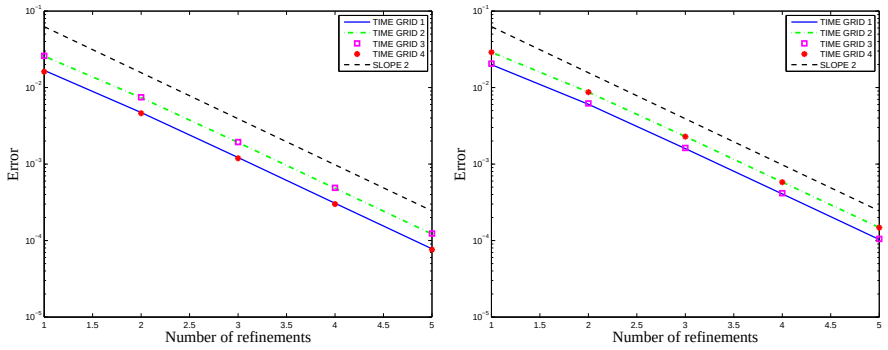

Fig. 2. Error curves versus the refinement in time, for Ω_1 (left) and Ω_2 (right)

5 Conclusions

We have extended the numerical method proposed in Halpern and Japhet [2008] to higher dimensions and analyzed it for heterogeneous advection-reaction-diffusion problems. It relies on the splitting of the time interval into time windows, in which a few iterations of an OSWR algorithm are performed by a discontinuous Galerkin method in time, with projection between space-time grids on the interfaces. We have shown both theoretically and numerically that the method preserves the order of the discontinuous Galerkin method.

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