A Simple Uniformly Convergent Iterative Method for the Non-symmetric Incomplete Interior Penalty Discontinuous Galerkin Discretization

Blanca Ayuso[1] and Ludmil T. Zikatanov[2]

[1] Departamento de Matemáticas, Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM, Universidad Autónoma de Madrid
Madrid 28049, Spain, blanca.ayuso@uam.es
[2] Department of Mathematics, Penn State University,
University Park, PA 16802, USA, ltz@math.psu.edu

We introduce a uniformly convergent iterative method for the systems arising from *non-symmetric* IIPG linear approximations of second order elliptic problems. The method can be viewed as a block Gauß–Seidel method in which the blocks correspond to restrictions of the IIPG method to suitably constructed subspaces. Numerical tests are included, showing the uniform convergence of the iterative method in an energy norm.

1 Introduction

In recent years, domain decomposition preconditioners and multilevel methods have been developed for the efficient solution of the linear systems that arise from Discontinuous Galerkin (DG) discretizations of elliptic problems (see [5] and the references therein). While most works deal with symmetric DG methods, very little is known for preconditioning the non-symmetric ones. However, designing efficient solvers for the resulting non-symmetric linear systems is of interest since they could be used as building blocks for preconditioning DG discretizations of non-symmetric PDEs (such as convection-diffusion problems). An important distinction between non-symmetric and symmetric DG schemes (even for discretizations of selfadjoint elliptic problems) is that the convergence analysis of the iterative methods is much more involved. As shown numerically in [1], the symmetric part of the preconditioned matrix of non-symmetric DG schemes (and in particular for the IIPG discretization considered here) can be indefinite and so the classical convergence theory for GMRES (see [6]) cannot be applied and new theoretical tools are needed.

In this paper, we design an efficient solver by using a space decomposition of the DG space introduced in [3]. We also extend some of the results from that work to the case of variable diffusion coefficient. The proposed iterative method is a successive subspace correction method for the non-symmetric IIPG discretization. We

demonstrate via numerical experiments uniform convergence with respect to both the penalty parameter and the number of degrees of freedom (dofs). In addition, as we will discuss, the method considered here is more efficient than those proposed and analyzed in [3]. However, the convergence of such method, although numerically evident, is much more difficult to analyze and we do not present such analysis here.

2 Interior Penalty Discontinuous Galerkin Methods

Given $f \in L^2(\Omega)$, we consider the following model problem

$$-\nabla \cdot (\mathbb{K}\nabla u) = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a convex polygon or polyhedron and $\mathbb{K} \in (L^\infty(\Omega))^{2d}_s$ is a given piecewise constant permeability symmetric positive-definite tensor satisfying: $0 < \mathfrak{c}_0\|\boldsymbol{\xi}\|^2 \le \boldsymbol{\xi}^t\mathbb{K}(x)\boldsymbol{\xi} \le \mathfrak{c}_1\|\boldsymbol{\xi}\|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d\ \forall\, x \in \Omega$.
Let $\mathcal{T}_h$ be a shape-regular family of partitions of Ω into d-dimensional simplexes T (triangles if $d = 2$ and tetrahedrons if $d = 3$) and let $h = \max_{T\in\mathcal{T}_h} h_T$ with h_T denoting the diameter of T for each $T \in \mathcal{T}_h$. We assume $\mathcal{T}_h$ does not contain hanging nodes and $\mathbb{K}$ is constant on each $T \in \mathcal{T}_h$.
Let V^{DG} denote the discontinuous finite element space defined by:

$$V^{DG} = \left\{ u \in L^2(\Omega) \ : \ u_{|_T} \ \in \ \mathbb{P}^1(T)\ \forall T \in \mathcal{T}_h \right\},$$

where $\mathbb{P}^1(T)$ denotes the space of linear polynomials on T. We denote by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ the sets of all interior faces and boundary faces (edges in $d = 2$), respectively, and we set $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$. Following [2], we define the *average* and *jump* trace operators. Let T^+ and T^- be two neighboring elements, and $\mathbf{n}^+$, $\mathbf{n}^-$ be their outward normal unit vectors, respectively ($\mathbf{n}^\pm = \mathbf{n}_{T^\pm}$). Let $\zeta^\pm$ and $\boldsymbol{\tau}^\pm$ be the restriction of ζ and $\boldsymbol{\tau}$ to $T^\pm$. We set:

$$\begin{aligned} 2\{\zeta\} &= (\zeta^+ + \zeta^-), \quad [\![\,\zeta\,]\!] = \zeta^+\mathbf{n}^+ + \zeta^-\mathbf{n}^- && \text{on } E \in \mathcal{E}_h^o, \\ 2\{\boldsymbol{\tau}\} &= (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-), \quad [\![\,\boldsymbol{\tau}\,]\!] = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- && \text{on } E \in \mathcal{E}_h^o, \end{aligned}$$

and on $E \in \mathcal{E}_h^\partial$ we set $[\![\,\zeta\,]\!] = \zeta\mathbf{n}$ and $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$. We will also use the notation

$$(u, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \int_T uw dx \qquad \langle u, w\rangle_{\mathcal{E}_h} = \sum_{E\in\mathcal{E}_h} \int_E uw \quad \forall\, u, w, \in V^{DG}.$$

The approximation to the solution of (1) reads:

$$\text{Find} \quad u \in V^{DG} \quad \text{such that} \quad \mathcal{A}(u, w) = (f, w)_{\mathcal{T}_h}\,, \quad \forall\, w \in V^{DG}. \tag{2}$$

$\mathcal{A}(\cdot,\cdot)$ is the bilinear form of the IIPG method (see [8]):

$$\mathcal{A}(u, w) = (\mathbb{K}\nabla u, \nabla w)_{\mathcal{T}_h} - \langle\{\mathbb{K}\nabla u\}, [\![\, w\,]\!]\rangle_{\mathcal{E}_h} + \langle S_E\{\mathbb{K}[\![\, u\,]\!]\}, [\![\, w\,]\!]\rangle_{\mathcal{E}_h}\,, \tag{3}$$

where $S_E = \alpha_E h_E^{-1}$ with $\alpha_E \geq \alpha^* > 0$ for all $E \in \mathcal{E}_h$ and h_E the length of the edge E in $d = 2$ and the diameter of the face E in $d = 3$. We denote by α^* a *fixed* value of the penalty parameter for which $\mathcal{A}(\cdot,\cdot)$ is coercive. In the numerical experiments we take α^* to be larger than (but close to) the critical value for which $\mathcal{A}(\cdot,\cdot)$ is only semidefinite. We point out that when α^* is close to that critical value one may expect less accurate DG solution due to the fact that the discrete problem is not stable. We present interative methods which are uniformly convergent independently of the value of α^* as long as, α^* is only just large enough to ensure coercivity of the bilinear forms, even though the discretizations in this case lead to "nearly singular" linear systems. This of course includes the cases of stable discretizations resulting in accurate DG solutions, and hence the methods that we propose are aplicable to the cases interesting from practical point of view. In general, α_E might vary from one face to another, but we assume that the possible variations on α are uniformly bounded, namely

$$\frac{\alpha_{\max}}{\alpha_{\min}} \approx 1, \quad \alpha_{\min} := \min_{E\in\mathcal{E}_h} \alpha_E \ , \quad \alpha_{\max} := \max_{E\in\mathcal{E}_h} \alpha_E \ . \tag{4}$$

Together with $\mathcal{A}(\cdot,\cdot)$, we consider also the bilinear form that results by computing all the integrals in (3) with the mid-point quadrature rule:

$$\mathcal{A}_0(u,w) = (\mathbb{K}\nabla u, \nabla w)_{\mathcal{T}_h} - \langle\{\mathbb{K}\nabla u\}, [\![w]\!]\rangle_{\mathcal{E}_h} + \langle S_E \mathcal{P}_E^0(\{\mathbb{K}[\![u]\!]\}), [\![w]\!]\rangle_{\mathcal{E}_h} \ , \tag{5}$$

where $\mathcal{P}_E^0 : L^2(E) \longrightarrow \mathbb{P}^0(E)$ is for each $E \in \mathcal{E}_h$, the L^2-orthogonal projection onto the constants: $\mathcal{P}_E^0(u) := \frac{1}{|E|}\int_E u, \forall\, u \in L^2(E)$. Continuity and coercivity can be shown for (3) in the DG energy norm (see [2]), which is equivalent to the norm induced by the symmetric part of $\mathcal{A}(\cdot,\cdot)$;

$$\|u\|_{\widetilde{\mathcal{A}}} := \widetilde{\mathcal{A}}(u,u) \quad \widetilde{\mathcal{A}}(u,w) = \frac{\mathcal{A}(u,w) + \mathcal{A}(w,u)}{2} \quad \forall\, u, w \in V^{DG} \ .$$

We will also use the equivalent form of the IIPG method obtained via the *weighted residual* approach introduced in [4]:

$$\begin{aligned}
\mathcal{A}(u,w) &= \langle [\![\mathbb{K}\nabla u]\!], \{w\}\rangle_{\mathcal{E}_h^o} + \langle [\![u]\!], S_E\{\mathbb{K}[\![w]\!]\}\rangle_{\mathcal{E}_h} && \forall u, w \in V^{DG}, \\
\mathcal{A}_0(u,w) &= \langle [\![\mathbb{K}\nabla u]\!], \{w\}\rangle_{\mathcal{E}_h^o} + \langle [\![u]\!], S_E\mathcal{P}_E^0(\{\mathbb{K}[\![w]\!]\})\rangle_{\mathcal{E}_h} && \forall u, w \in V^{DG},
\end{aligned}$$

where we have already discarded the term $(-\div(\mathbb{K}\nabla u), w)_{\mathcal{T}_h}$, since $u \in V^{DG}$ is linear and $\mathbb{K}$ is constant on each $T \in \mathcal{T}_h$, and therefore $(-\div(\mathbb{K}\nabla u), w)_{\mathcal{T}_h} = 0$. Next result guarantees the *spectral equivalence* of $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$.

Lemma 1. *Let $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ be the bilinear forms of the IIPG method defined in (3) and (5). Then, there exist $c_2 > 0$ depending only on the shape regularity of $\mathcal{T}_h$ and $c_0 = c_0(\alpha_{\max}, \mathfrak{c}_1) > 0$, such that:*

$$c_2\mathcal{A}_0(u,u) \leq \mathcal{A}(u,u) \leq c_0(\alpha_{\max}, \mathfrak{c}_1)\mathcal{A}_0(u,u) \quad \forall u \in V^{DG}.$$

3 Space Decomposition

In this section we introduce a new basis which provides a natural subspace splitting of the linear V^{DG}-space. We will show that if such basis is used the linear system associated to (2) has special properties that allow for simple derivation of efficient iterative methods for the non-symmetric IIPG method.

Let $\varphi_{E,T}$ denote the canonical Crouzeix-Raviart (CR) basis function on T, dual to the degree of freedom at the mass center m_E of the face E, and extended by zero outside T. Note that $\varphi_{E,T} \in L^2(\Omega)$ and the support of $\varphi_{E,T}$ is T. Then, for any $u \in V^{DG}$:

$$u(x) = \sum_{T\in\mathcal{T}_h} \sum_{E\in\partial T} u_T(m_E)\varphi_{E,T}(x) = \sum_{E\in\mathcal{E}_h} u^+(m_E)\varphi_E^+(x) + \sum_{E\in\mathcal{E}_h^o} u^-(m_E)\varphi_E^-(x).$$

with $\varphi_E^{\pm}(x) := \varphi_{E,T^{\pm}}(x)$. Let V^{CR} be the classical Crouziex-Raviart space:

$$V^{CR} = \left\{ v \in L^2(\Omega) \,:\, v_{|_T} \in \mathbb{P}^1(T)\, \forall T \in \mathcal{T}_h \text{ and } \mathcal{P}_E^0[\![v]\!] = 0 \;\forall\, E \in \mathcal{E}_h^o \right\}.$$

Note that $v = 0$ at the midpoint m_E of each $E \in \mathcal{E}_h^{\partial}$. We also define the space

$$\mathcal{Z} = \left\{ z \in L^2(\Omega) \,:\, z_{|_T} \in \mathbb{P}^1(T)\; \forall T \in \mathcal{T}_h \text{ and } \mathcal{P}_E^0\{v\} = 0 \;\forall\, E \in \mathcal{E}_h^o \right\}. \quad (6)$$

Observe that functions from $\mathcal{Z}$ have nonzero jumps on each internal face and so they can be deemed as highly oscillatory. Defining now:

$$\varphi_E^{CR} = \varphi_{E,T^+} + \varphi_{E,T^-} = 2\{\varphi_{E,T^{\pm}}\} \quad \forall\, E \in \mathcal{E}_h^o \quad E = T^+ \cap T^-,$$

$$\begin{cases} \psi^z_{E,T^{\pm}} = \varphi_{E,T^{\pm}} - \varphi_{E,T^{\mp}} & \forall\, E \in \mathcal{E}_h^o \quad E = T^+ \cap T^-, \\ \psi^z_{E,T} = \varphi_{E,T} & \forall\, E \in \mathcal{E}_h^{\partial}, \quad E = T \cap \partial\Omega, \end{cases}$$

we obtain a decomposition of the functions $\{\varphi_{E,T}\}$ which provides following representation for the spaces V^{CR} and $\mathcal{Z}$:

$$\begin{array}{rcl} V^{CR} & = & \operatorname{span}\{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o} \\ \mathcal{Z} & = & \operatorname{span}\{\psi^z_{E,T}\}_{E\in\mathcal{E}_h^o} \oplus \operatorname{span}\{\psi^z_{E,T}\}_{E\in\mathcal{E}_h^{\partial}}. \end{array} \quad (7)$$

Next result summarizes these observations.

Proposition 1. *For any $u \in V^{DG}$ there exists a unique $v \in V^{CR}$ and a unique $z \in \mathcal{Z}$ such that $u = v + z$, that is: $V^{DG} = V^{CR} \oplus \mathcal{Z}$.*

Thus, for all $u \in V^{DG}$ we have $u = v + z$ with unique v and z, given by

$$v = \sum_{E\in\mathcal{E}_h^o} \mathcal{P}_E^0(\{u\})\varphi_E^{CR}(x), \qquad z = \sum_{E\in\mathcal{E}_h} \mathcal{P}_E^0\left(\frac{[\![u]\!]\cdot \mathbf{n}^+}{2}\right)\psi^z_{E,T^+}(x).$$

Next Lemma gives a $\mathcal{A}$-"orthogonality" property of the subspace splitting.

Lemma 2. *Let $u \in V^{DG}$ be such that $u = v + z$ with $v \in V^{CR}$ and $z \in \mathcal{Z}$. Let $\mathcal{A}_0(\cdot,\cdot)$ be the bilinear form defined in* (5)*. Then,*

$$\mathcal{A}_0(v, z) = 0 \qquad \forall\, v \in V^{CR}, \quad \forall\, z \in \mathcal{Z}.$$

3.1 Matrix Representation of the DG Bilinear Forms

We denote by A the discrete operator $(Au, w) = \mathcal{A}(u, w)$ (resp. $(A_0 u, w) = \mathcal{A}_0(u, w)$). Let $\mathbb{A}$ (resp. $\mathbb{A}_0$) be the matrix representation of A (resp. A_0) in certain basis. The solution of (2) amounts to the solution of the linear system

$$\mathbb{A}\mathbf{u} = \mathbf{f}, \tag{8}$$

where $\boldsymbol{u}, \boldsymbol{f}$ are the vector representations of the unknown u and the source f. If the basis (7) is used for all these representations, we have:

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{v} \end{bmatrix}, \qquad \mathbb{A}_0 = \begin{bmatrix} \mathbb{A}_0^{zz} & \mathbf{0} \\ \mathbb{A}_0^{vz} & \mathbb{A}_0^{vv} \end{bmatrix}, \qquad \mathbb{A} = \begin{bmatrix} \mathbb{A}^{zz} & \mathbb{A}^{zv} \\ \mathbb{A}^{vz} & \mathbb{A}^{vv} \end{bmatrix}. \tag{9}$$

The blocks $\mathbb{A}^{zz}$, $\mathbb{A}_0^{zz}$ and $\mathbb{A}^{vv}$, $\mathbb{A}_0^{vv}$ correspond, respectively, to the stiffness matrices that result when approximating the solution to (1) with the IIPG method restricted to the $\mathcal{Z}$ and V^{CR} subspaces. The block lower triangular form of $\mathbb{A}_0$ in (9) is a consequence of Lemma 2. The solution of (8) with the block matrix $\mathbb{A}$ (or $\mathbb{A}_0$) as in (9) will certainly involve solutions of smaller systems with $\mathbb{A}^{zz}$ (or $\mathbb{A}_0^{zz}$) and $\mathbb{A}^{vv}$ (or $\mathbb{A}_0^{vv}$). Since, such systems are also solved in every iteration in the method we propose, we next comment on methods for their solution:

Solution in V^{CR}: Restricting the IIPG to the V^{CR} space, we get:

$$\mathcal{A}_0(v, \varphi) = (\mathbb{K}\nabla v, \nabla\varphi)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} (\mathbb{K}\nabla v, \nabla\varphi)_T \qquad \forall\, v\,, \varphi \in V^{CR}\,,$$
$$\mathcal{A}(v, \varphi) = (\mathbb{K}\nabla v, \nabla\varphi)_{\mathcal{T}_h} + \langle S_E[\![\, v\,]\!], \{\mathbb{K}[\![\,\varphi\,]\!]\}\rangle_{\mathcal{E}_h^o} \qquad \forall v, \varphi \in V^{CR}\,.$$

Hence, both $\mathbb{A}_0^{vv}$ and $\mathbb{A}^{vv}$ are s.p.d. Moreover, note that $\mathcal{A}_0$ is the standard non-conforming CR finite element method for the solution of (1). From the spectral equivalence in Lemma 1, any system with $\mathbb{A}_0^{vv}$ or $\mathbb{A}^{vv}$ can be efficiently solved by using any of the known solvers for the CR approximation of (1); as those proposed in [7] or [9].

Solution in the $\mathcal{Z}$-space: Using the weighted residual formulation together with the definition (6) of the $\mathcal{Z}$ space, it follows that $\forall\ z,\ \psi \in \mathcal{Z}$:

$$\mathcal{A}_0(z, \psi) = \langle S_E \mathcal{P}_E^0([\![\, z\,]\!]), \{\mathbb{K}[\![\,\psi\,]\!]\}\rangle_{\mathcal{E}_h} \quad \mathcal{A}(z, \psi) = \langle S_E[\![\, z\,]\!], \{\mathbb{K}[\![\,\psi\,]\!]\}\rangle_{\mathcal{E}_h}\,. \tag{10}$$

Thus, restricted to $\mathcal{Z}$, both $\mathcal{A}_0$ and $\mathcal{A}$ are symmetric and coercive (since we have set $\alpha_E \geq \alpha^* > 0$ for all $E \in \mathcal{E}_h$). Therefore the blocks $\mathbb{A}_0^{zz}$ and $\mathbb{A}^{zz}$ are both s.p.d. Next Lemma establishes upper and lower bounds on their eigenvalues showing that $\mathbb{A}_0^{zz}$ and $\mathbb{A}_{zz}$ are well conditioned.

Lemma 3. *Let $\mathcal{Z}$ be the space defined in* (6)*. Then for all $z \in \mathcal{Z}$, it holds*

$$c_1(\alpha_{\min})h^{-2}\|\mathbb{K}^{1/2}z\|_{0,\mathcal{T}_h}^2 \leq \mathcal{A}_0(z, z) \leq \mathcal{A}(z, z) \leq c_2(\alpha_{\max})h^{-2}\|\mathbb{K}^{1/2}z\|_{0,\mathcal{T}_h}^2\,,$$

where c_1 and c_2 depend on the mesh regularity and $\alpha_{\min}, \alpha_{\max}$ are as in (4)*.*

The proof of similar result can be found in [3]. By virtue of this lemma and (4), denoting by κ the condition number, we have that $\kappa(\mathbb{A}^{zz}) = O(1)$ and $\kappa_2(\mathbb{A}_0^{zz}) = O(1)$ independent of the mesh size. Clearly, then a linear system with $\mathbb{A}^{zz}$ can be solved using the method of Conjugate Gradients (CG) and the number of CG iterations needed for achieving a prescribed tolerance is also independent of the mesh size. It is also easy to show that the matrix $\mathbb{A}_0^{zz}$ is diagonal, and hence $\mathbb{A}_0^{zz}$ can be also used as a preconditioner for $\mathbb{A}^{zz}$.

4 A Uniformly Convergent Iterative Method

The general setting is a linear iterative algorithm with a given $\mathcal{B}(\cdot,\cdot) \approx \mathcal{A}(\cdot,\cdot)$:

Algorithm 1 *Given initial guess u_0, let u_k, $k \geq 0$ be the current approximation to the solution. The next iterate u_{k+1} is then defined by*

1. *Solve* $\mathcal{B}(e_k, w) = (f, w)_{\mathcal{T}_h} - \mathcal{A}(u_k, w) \quad \forall\, w \in V^{DG}$.
2. *Update* $u_{k+1} = u_k + e_k$.

In [3], the uniform convergence of Algorithm 1 is shown with $\mathcal{B} = \widetilde{\mathcal{A}}$, the symmetric part of $\mathcal{A}$. Here we propose more efficient (in terms of computational work) algorithm. It is suggested by the fact that $\mathbb{A}_0$ is lower triangular and the symmetric parts of $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ are spectrally equivalent. This suggest to take the "block lower triangular part" of $\mathcal{A}(\cdot,\cdot)$ as $\mathcal{B}(\cdot,\cdot)$, namely:

$$\mathcal{B}(z + v, \psi^z + \varphi) := \mathcal{A}(z, \psi^z) + \mathcal{A}(z, \varphi) + \mathcal{A}(v, \varphi), \tag{11}$$

$\forall\, v, \varphi \in V^{CR}$ and $\forall\, z, \psi^z \in \mathcal{Z}$. The restrictions of this bilinear form on V^{CR} and $\mathcal{Z}$ are easy to find and the resulting iterative method can be then written in terms of solution of the problems on the subspaces as follows:

Algorithm 2 *Let u_0 be a given initial guess. For $k \geq 0$, and given $u_k = z_k + v_k$, the next iterate $u_{k+1} = z_{k+1} + v_{k+1}$ is defined via the two steps:*

1. *Solve* $\mathcal{A}(z_{k+1}, \psi^z) = (f, \psi^z)_{\mathcal{T}_h} - \mathcal{A}(v_k, \psi^z) \quad \forall\, \psi^z \in \mathcal{Z}$.
2. *Solve* $\mathcal{A}(v_{k+1}, \varphi) = (f, \varphi)_{\mathcal{T}_h} - \mathcal{A}(z_{k+1}, \varphi) \quad \forall\, \varphi \in V^{CR}$.

Observe that algorithm 2 requires two solutions of smaller s.p.d problems: one solution in $\mathcal{Z}$-space (step 1 of the algorithm 2), and one solution in V^{CR}-space (step 2 of algorithm 2). The solution of the subproblems on $\mathcal{Z}$ and on V^{CR} can be done in an efficient way as discussed in the previous section.

5 Numerical Results

We present a set of numerical experiments aimed at assessing the performance of the proposed iterative method. We consider the model problem (1) on the unit square in $\mathbb{R}^2$ triangulated with a family of unstructured triangulations $\mathcal{T}_h$. In the tables given

below $J = 1$ corresponds to the coarsest grid and each refined triangulation on level J, $J = 2, \ldots, 4$ is obtained by subdividing each of the triangles forming the grid on level $(J-1)$ into four congruent triangles. We set the permeability coefficient $\mathbb{K} = \mathbb{I}$, to ease the comparison with the iterative methods for IIPG given in [3], where the symmetric part of $\mathcal{A}$, which we denote by $\widetilde{\mathcal{A}}$, is used as preconditioner. The coarsest grid has $n_1 = 480$ dofs and the finest ($J = 4$) has approximately $n_4 = 30,720$ dofs. We have set $\alpha = K\alpha^*$, and $\alpha^* = 0.9$ for $J = 1$ and $\alpha^* = 1.3$ for $J \geq 2$. We denote by $\mathbb{B}$ the matrix representation of $\mathcal{B}(\cdot,\cdot)$, as given in (11), and by $\widetilde{\mathbb{A}}$ the one of $\widetilde{\mathcal{A}}(\cdot,\cdot)$. The latter being s.p.d. induces a norm $\mathbb{R}^{n_J}$ denoted here by $\|\cdot\|_{\widetilde{\mathbb{A}}}$. The corresponding matrix norm is denoted below by the same symbol.

In Table 1 are given the rates of convergence measured in $\|\cdot\|_{\widetilde{\mathbb{A}}}$-norm and in Table 2 the asymptotic convergence rates (the spectral $\rho(\mathbb{I} - \mathbb{B}^{-1}\mathbb{A})$).

Table 1. Rate of convergence: $\|\mathbb{I} - \mathbb{B}^{-1}\mathbb{A}\|_{\widetilde{\mathbb{A}}}$ for different levels and different values of the penalty parameter $\alpha = K\alpha^*$.

	$J=1$	$J=2$	$J=3$	$J=4$
$K=1$	0.541	0.530	0.571	0.595
$K=2$	0.529	0.566	0.574	0.579
$K=4$	0.576	0.610	0.616	0.619
$K=8$	0.616	0.641	0.645	0.648

Table 2. Asymptotic convergence rate: $\rho(\mathbb{I} - \mathbb{B}^{-1}\mathbb{A})$ for different levels and different values of the penalty parameter $\alpha = K\alpha^*$.

	$J=1$	$J=2$	$J=3$	$J=4$
$K=1$	0.448	0.465	0.469	0.470
$K=2$	0.451	0.465	0.469	0.470
$K=4$	0.454	0.466	0.469	0.470
$K=8$	0.456	0.467	0.470	0.470

The conclusion that we may draw from the above experiments is that the convergence rate in $\|\cdot\|_{\widetilde{\mathbb{A}}}$ norm is uniform with respect to the mesh size, and deteriorates when increasing the value of the penalty parameter. The asymptotic convergence rate is uniformly bounded with respect to both K and the mesh size. The numerical tests clearly show that the $\|\cdot\|_{\widetilde{\mathbb{A}}}$ norm could be used to theoretically analyze the convergence behavior of this iterative method. However, as we have already mentioned, obtaining quantitative theoretical results that reflect and match the convergence rates presented in Table 1 could be quite involved and is subject of a current research.

Acknowledgments First author was supported by MEC grants MTM2008-03541 and HI2008-0173. The work of the second author was supported in part by the National Science Foundation NSF-DMS 0511800 and NSF-DMS 0810982.

References

1. P.F. Antonietti and B. Ayuso. Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: non-overlapping case. *Math. Model. Numer. Anal.*, 41(1):21–54, 2007.
2. D.N. Arnold, F. Brezzi, B. Cockburn, and L. Donatella Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02.
3. B. Ayuso de Dios and L. Zikatanov. Uniformly convergent iterative methods for discontinuous Galerkin discretizations. *J. Sci. Comput.*, 40(1–3):4–36, 2009.
4. F. Brezzi, B. Cockburn, L.D. Marini, and E. Süli. Stabilization mechanisms in discontinuous Galerkin finite element methods. *Comput. Methods Appl. Mech. Eng.*, 195(25–28): 3293–3310, 2006.
5. M. Dryja, J. Galvis, and M. Sarkis. BDDC methods for discontinuous Galerkin discretization of elliptic problems. *J. Complex.*, 23(4–6):715–739, 2007.
6. S.C. Eisenstat, H.C. Elman, and M.H. Schultz. Variational iterative methods for nonsymmetric systems of linear equations. *SIAM J. Numer. Anal.*, 20(2):345–357, 1983.
7. M. Sarkis. Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements. *Numer. Math.*, 77(3):383–406, 1997.
8. S. Sun and M.F. Wheeler. Symmetric and nonsymmetric discontinuous Galerkin methods for reactive transport in porous media. *SIAM J. Numer. Anal.*, 43(1):195–219 (electronic), 2005.
9. P.S. Vassilevski and J. Wang. An application of the abstract multilevel theory to nonconforming finite element methods. *SIAM J. Numer. Anal.*, 32(1):235–248, 1995.