# A Posteriori Error Estimates for Semilinear Boundary Control Problems

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## 1 Introduction

In this paper we study the finite element approximation for boundary control problems governed by semilinear elliptic equations. Optimal control problems are very important model in science and engineering numerical simulation. They have various physical backgrounds in many practical applications. Finite element approximation of optimal control problems plays a very important role in the numerical methods for these problems. The approximation of optimal control by piecewise constant functions is well investigated by [7, 8]. The discretization for semilinear elliptic optimal control problems is discussed in [2]. Systematic introductions of the finite element method for optimal control problems can be found in [ 10].

As one of important kinds of optimal control problems, the boundary control problem is widely used in scientific and engineering computing. The literature on this problem is huge, see, e.g. [1, 9]. For some linear optimal boundary control problems, [11] investigates a posteriori error estimates and adaptive finite element methods. [ 3] discusses the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. Although a priori error estimates and a posteriori error estimates of finite element approximation are widely used in numerical simulations, it is not yet been utilized in semilinear boundary control problems.

Recently, in [4, 5, 6], we have derived a priori error estimates and superconvergence for linear optimal control problems using mixed finite element methods. A posteriori error analysis of mixed finite element methods for general convex optimal control problems has been addressed in [13].

In this paper, we derive a posteriori error estimates for a class of boundary control problems governed by semilinear elliptic equation. The problem that we are interested in is the following semilinear boundary control problems:

$$\min_{u\in K\subset U} \big\{g(y)+j(u)\big\} \tag{1}$$

subject to the state equation

$$-\mathrm{div}(A\nabla y) + \phi(y) = f, \ x \in \Omega, \tag{2}$$
$$(A\nabla y)\cdot n = Bu + z_0, \ x \in \partial\Omega, \tag{3}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a convex polygon, $g$ and $j$ are convex functionals, $f \in L^2(\Omega)$, $z_0 \in U = L^2(\partial\Omega)$, $B$ is a continuous linear operator from $U$ to $L^2(\Omega)$, and $K$ is a closed convex set of $U$. For any $R > 0$ the function $\phi(\cdot) \in W^{1,\infty}(-R, R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq 0$. We assume that the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\Omega))^{2\times 2}$ is a symmetric positive definite matrix and there are constants $c_0, \ c_1 > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $c_0\|\mathbf{X}\|^2_{\mathbb{R}^2} \leq \mathbf{X}^t A\mathbf{X} \leq c_1\|\mathbf{X}\|^2_{\mathbb{R}^2}$.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p} = \sum\limits_{|\alpha|\leq m} \|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p} = \sum\limits_{|\alpha|=m} \|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p$=2, we use the notation $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition $C$ or $c$ denotes a general positive constant independent of $h$.

The plan of this paper is as follows. In next section, we present the finite element discretization for semilinear boundary control problems. A posteriori error estimates are established for the boundary control problems in Sect. 3.

## 2 Finite Elements for Boundary Control Problems

We shall now describe the finite element discretization of general semilinear convex boundary control problems (1)–(3). Let $V = H^1(\Omega)$, $W = L^2(\Omega)$.

Let

$$a(y, w) = \int_\Omega (A\nabla y)\cdot\nabla w, \qquad \forall y, w \in V, \tag{4}$$
$$(f_1, f_2) = \int_\Omega f_1 f_2, \qquad \forall (f_1, f_2) \in W \times W, \tag{5}$$
$$(u, v)_U = \int_{\partial\Omega} uv, \qquad \forall (u, v) \in U \times U. \tag{6}$$

Then the boundary control problems (1)–(3) can be restated as

$$\min_{u\in K\subset U} \{g(y) + j(u)\} \tag{7}$$

subject to

$$a(y, w) + (\phi(y), w) = (f, w) + (Bu + z_0, w)_U, \qquad \forall w \in V, \tag{8}$$

where the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$ is indicated by $(\cdot,\cdot)$.

It is well known (see, e.g., [11]) that the optimal control problem has a solution $(y, u)$, and that a pair $(y, u)$ is the solution of (7)–(8) if and only if there is a co-state $p \in V$ such that triplet $(y, p, u)$ satisfies the following optimality conditions:

$$a(y, w) + (\phi(y), w) = (f, w) + (Bu + z_0, w)_U, \quad \forall w \in V, \tag{9}$$
$$a(q, p) + (\phi'(y)p, q) = (g'(y), q), \quad \forall q \in V, \tag{10}$$
$$(j'(u) + B^* p, v - u)_U \geq 0, \quad \forall v \in K \subset U, \tag{11}$$

where $B^*$ is the adjoint operator of $B$, $g'$ and $j'$ are the derivatives of $g$ and $j$. In the rest of the paper, we shall simply write the product as $(\cdot, \cdot)$ whenever no confusion should be caused.

Let us consider the finite element approximation of the control problem (7)–(8). Let $\mathrm{T}_h$ be regular partition of $\Omega$. Associated with $\mathrm{T}_h$ is a finite dimensional subspace $V_h$ of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomials of $m$-order ($m \geq 1$) $\forall \chi \in V_h$ and $\tau \in \mathrm{T}_h$. It is easy to see that $V_h \subset V$.

Let $\mathrm{T}_U^h$ be a partition of $\partial\Omega$ and $\partial\Omega = \bigcup\limits_{s \in \mathrm{T}_U^h} \bar{s}$. Associated with $\mathrm{T}_U^h$ is another finite dimensional subspace $U_h$ of $L^2(\partial\Omega)$, such that $\chi|_s$ are polynomials of $m$-order ($m \geq 0$) $\forall \chi \in U_h$ and $s \in \mathrm{T}_U^h$. Let $h_\tau$ $(h_s)$ denote the maximum diameter of the element $\tau$ $(s)$ in $\mathrm{T}_h$ $(\mathrm{T}_U^h)$, $h = \max\limits_{\tau \in \mathrm{T}_h}\{h_\tau\}$, and $h_U = \max\limits_{s \in \mathrm{T}_U^h}\{h_s\}$.

By the definition of finite element subspace, the finite element discretization of (7)–(8) is as follows: compute $(y_h, u_h) \in V_h \times U_h$ such that

$$\min_{u_h \in K_h \subset U_h} \left\{g(y_h) + j(u_h)\right\} \tag{12}$$
$$a(y_h, w_h) + (\phi(y_h), w_h) = (f, w_h) + (Bu_h + z_0, w_h)_U, \quad \forall w_h \in V_h, \tag{13}$$

where $K_h$ is a non-empty closed convex set in $U_h$.

Again, it follows that the optimal control problem (12)–(13) has a solution $(y_h, u_h)$, and that a pair $(y_h, u_h)$ is the solution of (12)–(13) if and only if there is a co-state $p_h \in V_h$ such that triplet $(y_h, p_h, u_h)$ satisfies the following optimality conditions:

$$a(y_h, w_h) + (\phi(y_h), w_h) = (f, w_h) + (Bu_h + z_0, w_h)_U, \quad \forall w_h \in V_h, \tag{14}$$
$$a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (g'(y_h), q_h), \quad \forall q_h \in V_h, \tag{15}$$
$$(j'(u_h) + B^* p_h, v_h - u_h)_U \geq 0, \quad \forall v_h \in K_h. \tag{16}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $u_h \in K$, we first define the state solution $(y(u_h), p(u_h))$ which satisfies

$$a(y(u_h), w) + (\phi(y(u_h)), w) = (f, w) + (Bu_h + z_0, w)_U, \quad \forall w \in V, \tag{17}$$
$$a(q, p(u_h)) + (\phi'(y(u_h))p(u_h), q) = (g'(y(u_h)), q), \quad \forall q \in V. \tag{18}$$

The following Lemma is important in deriving a posteriori error estimates of residual type.

**Lemma 1.** *Let $\pi_h$ be the standard Lagrange interpolation operator. Then for $m = 0$ or $1$, $1 < q \le \infty$ and $\forall v \in W^{2,q}(\Omega)$,*

$$\|v - \pi_h v\|_{W^{m,q}(\tau)} \le C h_\tau^{2-m} |v|_{W^{2,q}(\tau)}. \tag{19}$$

## 3 A Posteriori Error Estimates

For given $u \in K$, let $S$ be the inverse operator of the state equation (9), such that $y(u) = SBu$ is the solution of the state equation (9). Similarly, for given $u_h \in K_h$, $y_h(u_h) = S_h B u_h$ is the solution of the discrete state Eq. (14). Let

$$S(u) = g(SBu) + j(u),$$

$$S_h(u_h) = g(S_h B u_h) + j(u_h).$$

It is clear that $S$ and $S_h$ are well defined and continuous on $K$ and $K_h$. Also the functional $S_h$ can be naturally extended on $K$. Then (7) and (12) can be represented as

$$\min_{u \in K} \{S(u)\}, \tag{20}$$

$$\min_{u_h \in K_h} \{S_h(u_h)\}. \tag{21}$$

It can be shown that

$$(S'(u), v) = (j'(u) + B^* p, v),$$
$$(S'(u_h), v) = (j'(u_h) + B^* p(u_h), v),$$
$$(S'_h(u_h), v) = (j'(u_h) + B^* p_h, v),$$

where $p(u_h)$ is the solution of the equations (17)–(18).

In many applications, $S(\cdot)$ is uniform convex near the solution $u$ (see, e.g., [12]). The convexity of $S(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. If $S(\cdot)$ is uniformly convex, then there is a $c > 0$, such that

$$(S'(u) - S'(u_h), u - u_h) \ge c\|u - u_h\|^2_{0,\partial\Omega}, \tag{22}$$

where $u$ and $u_h$ are the solutions of (7) and (12), respectively. We will assume the above inequality throughout this paper.

Now we establish the following a posteriori error estimates, which can be proved similarly to the proofs given in [12].

**Theorem 1.** *Let $u$ and $u_h$ be the solutions of (7) and (12), respectively. Assume that $K_h \subset K$ for any element $s \in \mathrm{T}_U^h$, $(B^* p_h + j'(u_h))|_s \in H^1(s)$ and that there exists a $v_h \in K_h$ such that*

$$|(B^*p_h + j'(u_h), v_h - u)_U| \le C \sum_s h_s |B^*p_h + j'(u_h)|_{1,s} \|u - u_h\|_{0,s}. \quad (23)$$

*Then*

$$\|u - u_h\|^2_{0,\partial\Omega} \le C \left(\eta_1^2 + \|p_h - p(u_h)\|^2_{0,\partial\Omega}\right), \quad (24)$$

*where*

$$\eta_1^2 = \sum_s h_s^2 |B^*p_h + j'(u_h)|^2_{1,s}.$$

*Proof.* It follows from (22) that for all $v_h \in K_h$,

$$\begin{aligned} c\|u - u_h\|^2_{0,\partial\Omega} \le & S'(u)(u - u_h) - S'(u_h)(u - u_h) \\ = & (j'(u) + B^*p, u - u_h)_U - (j'(u_h) + B^*p(u_h), u - u_h)_U \\ \le & -(j'(u_h) + B^*p(u_h), u - u_h)_U \\ \le & (j'(u_h) + B^*p_h, u_h - v_h)_U + (j'(u_h) + B^*p_h, v_h - u)_U \\ & + (B^*(p_h - p(u_h)), u - u_h)_U \\ \le & (j'(u_h) + B^*p_h, v_h - u)_U + (B^*(p_h - p(u_h)), u - u_h)_U. \end{aligned} \quad (25)$$

It is easy to see that

$$\begin{aligned} (B^*(p_h - p(u_h)), u - u_h)_U \le & C\|B^*(p_h - p(u_h))\|^2_{0,\partial\Omega} + \frac{\delta}{2}\|u - u_h\|^2_{\partial\Omega} \\ \le & C\|p_h - p(u_h)\|^2_{0,\partial\Omega} + \frac{\delta}{2}\|u - u_h\|^2_{\partial\Omega}, \end{aligned} \quad (26)$$

where $\delta$ is an arbitrary positive constant. Then (25)-(26) and (23) imply that

$$\begin{aligned} c\|u - u_h\|^2_{0,\partial\Omega} \le & C\sum_s h_s^2 |B^*p_h + j'(u_h)|^2_{1,s} \\ & + C\|p_h - p(u_h)\|^2_{0,\partial\Omega} + \frac{\delta}{2}\|u - u_h\|^2_{\partial\Omega} \\ = & C\eta_1^2 + C\|p_h - p(u_h)\|^2_{0,\partial\Omega} + \frac{\delta}{2}\|u - u_h\|^2_{\partial\Omega}. \end{aligned} \quad (27)$$

Then (24) follows from (27).

Now, we are able to derive the following result.

**Theorem 2.** *Let $(y(u_h), p(u_h))$ and $(y_h, p_h)$ be the solutions of (17)–(18) and (14)–(15), respectively. Assume that $g'$ is Lipschitz continuous in a neighborhood of $y$. Then*

$$\|p(u_h) - p_h\|^2_{1,\Omega} \le C(\eta_2^2 + \eta_3^2) + C\|y(u_h) - y_h\|^2_{0,\Omega}, \quad (28)$$

*where*

$$\eta_2^2 = \sum_{\tau\in \mathrm{T}_h} h_\tau^2 \int_\tau \left(g'(y_h) + \mathrm{div}(A^*\nabla p_h) - \phi'(y_h)p_h\right)^2,$$

$$\eta_3^2 = \sum_{l\cap\partial\Omega=\phi} h_l \int_l [A^*\nabla p_h \cdot n]^2 + \sum_{l\subset\partial\Omega} h_l \int_l (A^*\nabla p_h \cdot n)^2,$$

*where $l$ is a face of an element $\tau$, $[(A^*\nabla p_h \cdot n)]$ is the $A$-normal derivative jump over the interior face $l$, defined by*

$$[(A^*\nabla p_h \cdot n)]_l = (A^*\nabla p_h|_{\tau_l^1} - A^*\nabla p_h|_{\tau_l^2}) \cdot n,$$

*where $n$ is the unit normal vector on $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ outwards $\tau_l^1$.*

Analogously to Theorem 2, we show the following estimate:

**Theorem 3.** *Let $(y(u_h), p(u_h))$ and $(y_h, p_h)$ be the solutions of (17)–(18) and (14)–(15), respectively. Assume that $g'$ is Lipschitz continuous in a neighborhood of $y$. Then*

$$\|y(u_h) - y_h\|_{1,\Omega}^2 \le C(\eta_4^2 + \eta_5^2), \tag{29}$$

*where*

$$\eta_4^2 = \sum_{\tau\in \mathrm{T}_h} h_\tau^2 \int_\tau \left(f + \mathrm{div}(A\nabla y_h) - \phi(y_h)\right)^2,$$

$$\eta_5^2 = \sum_{l\cap\partial\Omega=\phi} h_l \int_l [A\nabla y_h \cdot n]^2 + \sum_{l\subset\partial\Omega} h_l \int_l (A\nabla y_h \cdot n - Bu_h - z_0)^2,$$

*where $l$ is a face of an element $\tau$, $[(A\nabla y_h \cdot n)]$ is the $A$-normal derivative jump over the interior face $l$, defined by*

$$[(A\nabla y_h \cdot n)]_l = (A\nabla y_h|_{\tau_l^1} - A\nabla y_h|_{\tau_l^2}) \cdot n,$$

*where $n$ is the unit normal vector on $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ outwards $\tau_l^1$.*

Now, we are able to derive our main result.

**Theorem 4.** *Let $(y, p, u)$ and $(y_h, p_h, u_h)$ be the solutions of (9)–(11) and (14)–(16), respectively. Assume that all the conditions of Theorems 1–3 hold. Then*

$$\|u - u_h\|_{0,\partial\Omega}^2 + \|y - y_h\|_{1,\Omega}^2 + \|p(u_h) - p_h\|_{1,\Omega}^2 \le C\sum_{i=1}^5 \eta_i^2, \tag{30}$$

*where $\eta_i$, $i = 1, 2, 3, 4, 5$ are defined in Theorem 1, Theorem 2, and Theorem 3.*

*Proof.* From Theorems 1–3 and the trace theorem we can see that

$$\begin{aligned}\|u - u_h\|_{0,\partial\Omega}^2 &\le C\left(\eta_1^2 + \|p_h - p(u_h)\|_{0,\partial\Omega}^2\right) \le C\left(\eta_1^2 + \|p_h - p(u_h)\|_{1,\Omega}^2\right)\\ &\le C\left(\eta_1^2 + \eta_2^2 + \eta_3^2 + \|y_h - y(u_h)\|_{0,\Omega}^2\right) \le C\sum_{i=1}^5 \eta_i^2.\end{aligned} \tag{31}$$

Note that

$$\|y - y_h\|_{1,\Omega} \le \|y_h - y(u_h)\|_{1,\Omega} + \|y_h - y(u_h)\|_{1,\Omega}, \tag{32}$$

$$\|p - p_h\|_{1,\Omega} \le \|p_h - p(u_h)\|_{1,\Omega} + \|p_h - p(u_h)\|_{1,\Omega}. \tag{33}$$

It follows from (9) and (17) that

$$a(y - y(u_h), w) + (\phi(y) - \phi(y(u_h)), w) = (B(u - u_h), w), \quad \forall w \in V. \tag{34}$$

Let $w = y - y(u_h)$, we have that

$$\|y - y(u_h)\|_{1,\Omega} \le C\|B(u - u_h)\|_{0,\Omega} \le C\|u - u_h\|_{0,\partial\Omega}. \tag{35}$$

Similarly, from (10) and (18) imply that

$$\begin{aligned} &a(q, p - p(u_h)) + (\phi'(y)(p - p(u_h)), q) \\ &= (g'(y) - g'(y(u_h)), q) + ((\phi'(y(u_h)) - \phi'(y))p(u_h), q), \quad \forall q \in V. \end{aligned} \tag{36}$$

Let $q = p - p(u_h)$, using (31), (35), and the trace theorem, we have that

$$\begin{aligned} \|p - p(u_h)\|_{1,\Omega}^2 &\le (g'(y) - g'(y(u_h)), p - p(u_h)) \\ &\quad + ((\phi'(y(u_h)) - \phi'(y))p(u_h), p - p(u_h)) \\ &\le \|g'(y) - g'(y(u_h))\|_{0,\Omega}\|p - p(u_h)\|_{0,\Omega} \\ &\quad + \|\phi'(y(u_h)) - \phi'(y)\|_{0,\Omega}\|p(u_h)\|_{0,4,\Omega}\|p - p(u_h)\|_{0,4,\Omega} \\ &\le C\|y - y(u_h)\|_{0,\Omega}^2 + C\delta\|p - p(u_h)\|_{1,\Omega}^2 \\ &\le C\|u - u_h\|_{0,\partial\Omega}^2 + C\delta\|p - p(u_h)\|_{1,\Omega}^2 \\ &\le C\sum_{i=1}^{5}\eta_i^2 + C\delta\|p - p(u_h)\|_{1,\Omega}^2. \end{aligned}$$

Then, for $\delta$ sufficiently small,

$$\|p - p(u_h)\|_{1,\Omega}^2 \le C\sum_{i=1}^{5}\eta_i^2, \tag{37}$$

and thus (30) follows from (31), (35), and (37).

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