Numerical Solution of Linear Elliptic Problems with Robin Boundary Conditions by a Least-Squares/Fictitious Domain Method

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1 Introduction

Motivated by the numerical simulation of particulate flow with slip boundary conditions at the interface fluid/particles, our goal, in this publication, is to discuss a fictitious domain method for the solution of linear elliptic boundary value problems with Robin boundary conditions. The method is of the virtual control type and relies on a least-squares formulation making the problem solvable by a conjugate gradient algorithm operating in a well chosen control space. Numerical results are presented; they suggest optimal orders of convergence for the finite element implementation of our fictitious domain method. A (brief) history of fictitious domain methods can be found in, e.g., [[3], Chap. 8].

2 Formulation of the Boundary Value Problem

Let Ω and ω be two bounded domains of $\mathbf{R}^d$, such that $d \geq 1$ and $\overline{\omega} \subset \Omega$ (see Fig. 1). We denote by Γ and γ the boundaries of Ω and ω, respectively. The Robin–Dirichlet

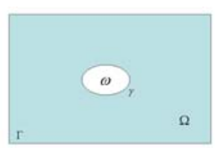

Fig. 1. Problem geometry.

problem under consideration reads as follows:

$$\begin{aligned}
&\alpha\psi - \mu\nabla^2\psi = f \quad \text{in} \quad \Omega \setminus \overline{\omega}, \\
&\psi = g_0 \quad \text{on} \quad \Gamma, \quad \mu\left(\frac{\partial\psi}{\partial n} + \frac{\psi}{l_s}\right) = g_1 \quad \text{on} \quad \gamma,
\end{aligned} \tag{1}$$

where: $\alpha \geq 0, \mu > 0, f \in L^2(\Omega \setminus \overline{\omega}), g_0 \in H^{3/2}(\Gamma), g_1 \in H^{1/2}(\gamma)$, $\mathbf{n}$ is the unit normal vector at γ pointing outward of $\Omega \setminus \overline{\omega}$ and l_s is a characteristic distance. We assume that Ω is convex and that γ is smooth. Problem (1) has a unique solution in $H^2(\Omega \setminus \overline{\omega})$ which is also the solution of the following linear variational problem:

$$\begin{aligned}
&\psi \in H^1(\Omega \setminus \overline{\omega}), \psi = g_0 \quad \text{on} \quad \Gamma, \\
&\alpha \int_{\Omega\setminus\overline{\omega}} \psi\varphi dx + \mu \int_{\Omega\setminus\overline{\omega}} \nabla\psi \cdot \nabla\varphi dx + \frac{\mu}{l_s}\int_\gamma \psi\varphi d\gamma \\
&= \int_{\Omega\setminus\overline{\omega}} f\varphi dx + \int_\gamma g_1\varphi d\gamma, \quad \forall\varphi \in V_0,
\end{aligned} \tag{2}$$

where $dx = dx_1...dx_d$ and $V_0 = \{\varphi | \varphi \in H^1(\Omega \setminus \overline{\omega}), \quad \varphi = 0 \quad \text{on} \quad \Gamma\}$.

3 A Least-Squares/Fictitious Domain Method for the Solution of Problem (1), (2)

3.1 A Fictitious Domain Formulation of Problem (1), (2)

We proceed as follows to define a fictitious domain variant of problem (1), (2):

(i) With $v \in L^2(\omega)$ we associate $\widetilde{f}(v)$ defined by

$$\widetilde{f}(v) \in L^2(\Omega), \quad \widetilde{f}(v)|_{\Omega\setminus\overline{\omega}} = f, \quad \widetilde{f}(v)|_\omega = v, \tag{3}$$

and then $\{\psi_1, \psi_2\}$ solution of the following elliptic system:

$$\alpha\psi_1 - \mu\nabla^2\psi_1 = \widetilde{f}(v) \quad \text{in} \quad \Omega, \quad \psi_1 = g_0 \quad \text{on} \quad \Gamma, \tag{4}$$

$$\alpha\psi_2 - \mu\nabla^2\psi_2 = v \quad \text{in} \quad \omega, \quad \mu\frac{\partial\psi_2}{\partial n} = \frac{\mu}{l_s}\psi_1 - g_1 \quad \text{on} \quad \gamma. \tag{5}$$

Both problems (4) and (5) have a unique solution in $H^1(\Omega)$ and $H^1(\omega)$, respectively (actually, ψ_1 and ψ_2 have both the H^2–regularity).

(ii) We define $\mathbf{A} : L^2(\omega) \to H^1(\omega)$ by

$$\mathbf{A}(v) = (\psi_2 - \psi_1)\,|_\omega\,. \tag{6}$$

Operator $\mathbf{A}$ is clearly affine and continuous.

(iii) We observe that if v verifies $\mathbf{A}(v) = 0$, we then have $\psi_2 = \psi_1$ on ω and it is easy to see that the H^2–regularity of ψ_1 and ψ_2 implies that $\psi_1|_{\Omega\setminus\overline{\omega}} = \psi$, where ψ is the solution of problem (1), (2). The problem is now to solve the functional equation

$$\mathbf{A}(u) = 0. \tag{7}$$

Indeed, the functional Eq. (7) has infinitely many solutions, but among these solutions only one is of minimal norm in $L^2(\omega)$.

Remark 1. Problem (7) can be viewed as an exact controllability problem in the sense of [2] (and as a virtual control problem in the sense of [4]). If a conjugate gradient algorithm is applied to a least-squares variant of (7) starting with 0 as initial guess, we have convergence to the unique solution of problem (7) of minimal norm in $L^2(\omega)$.

3.2 A Least-Squares Formulation of Problem (7)

A "reasonable" least-squares formulation of (7) reads as follows:

$$\begin{aligned} &\text{Find} \quad u \in L^2(\omega) \quad \text{such that} \\ &J(u) \le J(v), \quad \forall \quad v \in L^2(\omega), \end{aligned} \tag{8}$$

with

$$J(v) = \frac{1}{2} \int_{\omega} \left[\alpha |\psi_2 - \psi_1|^2 + \mu |\nabla(\psi_2 - \psi_1)|^2 \right] dx, \tag{9}$$

ψ_1 and ψ_2 in (9) being obtained from v via the solution of the elliptic boundary value problems (4) and (5), respectively. The functional J is clearly convex and C^∞ over $L^2(\omega)$. Any solution of problem (7) is a solution of the minimization problem (of the *virtual control* type) (8). Such a solution is characterized by

$$DJ(u) = 0, \tag{10}$$

where $DJ(\cdot)$ is the differential of functional J. Using classical methods from Control Theory (see, e.g., [2]), we can show that

$$\forall v \in L^2(\omega), DJ(v) = (p_1 - \psi_1)|_\omega + \psi_2, \tag{11}$$

where in (11), ψ_1 and ψ_2 are defined from v via the solution of (4) and (5), respectively, and where p_1 is the the unique solution of the following adjunct equation (written directly in variational form, here):

$$\begin{aligned} &p_1 \in H_0^1(\Omega), \\ &\int_\Omega [\alpha p_1 \varphi + \mu \nabla p_1 \cdot \nabla \varphi] \, dx = \int_\omega [\alpha(\psi_1 - \psi_2)\varphi + \mu \nabla(\psi_1 - \psi_2) \cdot \nabla \varphi] \, dx \\ &\qquad + \frac{\mu}{l_s} \int_\gamma (\psi_2 - \psi_1) \varphi d\gamma, \quad \forall \quad \varphi \in H_0^1(\Omega). \end{aligned} \tag{12}$$

4 On the Conjugate Gradient Solution of the Least-Squares Problem (8)

In order to solve the (linear) least-squares problem (8), we advocate a conjugate gradient algorithm operating in the space $L^2(\omega)$; this algorithm reads as follows:

$$u^0 \quad \text{is given in} \quad L^2(\omega); \tag{13}$$

solve

$$\psi_1^0 \in H^1(\Omega),$$
$$\alpha\psi_1^0 - \mu\nabla^2\psi_1^0 = \widetilde{f}(u^0) \quad \text{in} \quad \Omega, \quad \psi_1^0 = g_0 \quad \text{on} \quad \Gamma, \tag{14}$$

$$\psi_2^0 \in H^1(\omega),$$
$$\alpha\psi_2^0 - \mu\nabla^2\psi_2^0 = u^0 \quad \text{in} \quad \omega, \quad \mu\frac{\partial\psi_2^0}{\partial n} = \frac{\mu}{l_s}\psi_1^0 - g_1 \quad \text{on} \quad \gamma, \tag{15}$$

$$p_1^0 \in H_0^1(\Omega),$$
$$\int_\Omega \left[\alpha p_1^0\varphi + \mu\nabla p_1^0 \cdot \nabla\varphi\right] dx = \int_\omega \left[\alpha(\psi_1^0 - \psi_2^0)\varphi + \mu\nabla(\psi_1^0 - \psi_2^0)\cdot\nabla\varphi\right] dx$$
$$+ \frac{\mu}{l_s}\int_\gamma (\psi_2^0 - \psi_1^0)\varphi d\gamma, \quad \forall \quad \varphi \in H_0^1(\Omega), \tag{16}$$

and set

$$g^0 = (p_1^0 - \psi_1^0)|_\omega + \psi_2^0, \quad w^0 = g^0. \tag{17}$$

For $n \geq 0$, assuming that u^n, g^n and w^n are known, the last two different from 0, we compute u^{n+1}, g^{n+1} and w^{n+1} as follows (with χ_ω the characteristic function of ω):
Solve

$$\overline{\psi}_1^n \in H_0^1(\Omega),$$
$$\alpha\overline{\psi}_1^n - \mu\nabla^2\overline{\psi}_1^n = w^n\chi_\omega \quad \text{in} \quad \Omega, \quad \overline{\psi}_1^n = 0 \quad \text{on} \quad \Gamma, \tag{18}$$

$$\overline{\psi}_2^n \in H^1(\omega),$$
$$\alpha\overline{\psi}_2^n - \mu\nabla^2\overline{\psi}_2^n = w^n \quad \text{in} \quad \omega, \quad \mu\frac{\partial\overline{\psi}_2^n}{\partial n} = \frac{\mu}{l_s}\overline{\psi}_1^n \quad \text{on} \quad \gamma, \tag{19}$$

$$\overline{p}_1^n \in H_0^1(\Omega),$$
$$\int_\Omega \left[\alpha\overline{p}_1^n\varphi + \mu\nabla\overline{p}_1^n \cdot \nabla\varphi\right] dx = \int_\omega \left[\alpha(\overline{\psi}_1^n - \overline{\psi}_2^n)\varphi + \mu\nabla(\overline{\psi}_1^n - \overline{\psi}_2^n)\cdot\nabla\varphi\right] dx$$
$$+ \frac{\mu}{l_s}\int_\gamma (\overline{\psi}_2^n - \overline{\psi}_1^n)\varphi d\gamma, \quad \forall \quad \varphi \in H_0^1(\Omega), \tag{20}$$

and set

$$\overline{g}^n = (\overline{p}_1^n - \overline{\psi}_1^n)|_\omega + \overline{\psi}_2^n. \tag{21}$$

Next, compute

$$\rho_n = \frac{\int_\omega |g^n|^2 dx}{\int_\omega \overline{g}^n w^n dx} \tag{22}$$

and

$$u^{n+1} = u^n - \rho_n w^n, \quad (23)$$
$$g^{n+1} = g^n - \rho_n \overline{g}^n. \quad (24)$$

If $\frac{\int_\omega |g^{n+1}|^2 dx}{\max\{\int_\omega |g^0|^2 dx, \int_\omega |u^{n+1}|^2 dx\}} \le tol$, take $u = u^{n+1}$ and $\psi = \psi_1^{n+1}|_{\Omega\setminus\overline{\omega}}$; else compute

$$\gamma_n = \frac{\int_\omega |g^{n+1}|^2 dx}{\int_\omega |g^n|^2 dx} \quad (25)$$

and set

$$w^{n+1} = g^{n+1} + \gamma_n w^n. \quad (26)$$

Do $n+1 \to n$ and return to (18).

5 On the Finite Element Implementation of the Least-Squares/ Fictitious Domain Methodology

5.1 Generalities

We (briefly) discuss in this section the finite element implementation of the least-squares/fictitious domain methodology described in Sects. 3 and 4. We will assume that $\overline{\omega} \subset \Omega \subset \mathbf{R}^2$ and that Ω is convex and/or has a smooth boundary. We assume also that γ is smooth. For simplicity, we still denote by ω and Ω the polygonal approximations of the above domains. From the triangulations $\mathcal{T}_{h_1}$ of Ω and $\mathcal{T}_{h_2}$ of ω, we define the following finite dimensional spaces:

$$V_{h_1} = \{\varphi | \varphi \in C^0(\overline{\Omega}), \varphi|_T \in P_1, \forall T \in \mathcal{T}_{h_1}\}, \quad (27)$$
$$V_{0h_1} = \{\varphi | \varphi \in V_{h_1}, \varphi = 0 \quad \text{on} \quad \Gamma\}, \quad (28)$$

and

$$V_{h_2} = \{\varphi | \varphi \in C^0(\overline{\omega}), \varphi|_T \in P_1, \forall T \in \mathcal{T}_{h_2}\}, \quad (29)$$

P_1 being the space of the polynomials of two variables of degree ≤ 1. We will use $\mathbf{h}$ to denote the pair $\{h_1, h_2\}$. The finite element spaces V_{h_1}, V_{0h_1} and V_{h_2} are finite dimensional approximations to $H^1(\Omega), H_0^1(\Omega)$ and $H^1(\omega)$, respectively. Similarly, we will use V_{h_2} to approximate the "control" space $L^2(\omega)$. As usual, h_1 (resp., h_2) denotes the length of the longest edge(s) of $\mathcal{T}_{h_1}$ (resp., $\mathcal{T}_{h_2}$).

5.2 Finite Element Approximation of the Least-Squares Problem (8)

We approximate the least-squares problem (8) by

$$\begin{aligned}&\text{Find} \quad u_{\mathbf{h}} \in V_{h_2} \quad \text{such that}\\ &J_{\mathbf{h}}(u_{\mathbf{h}}) \leq J_{\mathbf{h}}(v), \quad \forall \quad v \in V_{h_2},\end{aligned} \tag{30}$$

where

$$J_{\mathbf{h}}(v) = \frac{1}{2}\int_\omega \left[\alpha|\psi_2 - \pi_2\psi_1|^2 + \mu|\nabla(\psi_2 - \pi_2\psi_1)|^2\right] dx. \tag{31}$$

In (31), ψ_1 is the solution of the following fully discrete Dirichlet problem:

$$\begin{aligned}&\psi_1 \in V_{h_1}, \quad \psi_1 = g_{0h_1} \quad \text{on} \quad \Gamma,\\ &\int_\Omega [\alpha\psi_1\varphi + \mu\nabla\psi_1 \cdot \nabla\varphi]\, dx = \int_\Omega f_{h_1}\varphi dx + \int_\omega v\pi_2\varphi dx, \quad \forall \quad \varphi \in V_{0h_1},\end{aligned} \tag{32}$$

where: (i) g_{0h_1} is an approximation of g_0. (ii) $f_{h_1} \in V_{h_1}$; it approximates f over $\Omega \setminus \overline{\omega}$ and vanishes over ω. (iii) $\pi_2 : C^0(\overline{\Omega}) \to V_{h_2}$ is the interpolation operator defined as follows

$$\pi_2\varphi = \sum_{i=1}^{N_{h_2}} \varphi(Q_i)w_{2i}, \quad \forall \quad \varphi \in C^0(\overline{\Omega}), \tag{33}$$

$\{Q_i\}_{i=1}^{N_{h_2}}$ being the set of the vertices of $\mathcal{T}_{h_2}$ and w_{2i} the P_1– shape function associated with the vertex Q_i (we clearly have N_{h_2} = dimension of V_{h_2}). Returning to (31), the function ψ_2 there is the solution of the following discrete Neumann problem

$$\begin{aligned}&\psi_2 \in V_{h_2},\\ &\int_\omega [\alpha\psi_2\varphi + \mu\nabla\psi_2 \cdot \nabla\varphi]\, dx = \int_\omega v\varphi dx\\ &+ \frac{\mu}{l_s}\int_\gamma (\pi_2\psi_1 - g_{1h_2})\varphi d\gamma, \quad \forall \quad \varphi \in V_{h_2},\end{aligned} \tag{34}$$

g_{1h_2} being an approximation of g_1. As a discrete analogue of problem (8), the finite dimensional least-squares problem (30) is also of the virtual control type and well-suited to solution by a conjugate gradient algorithm operating in the space V_{h_2}. Due to page limitation we cannot describe this algorithm here; actually, this discrete analogue of algorithm (13)–(26) will be fully described in [1].

6 Numerical Experiments

As test problem, we consider the particular case of problem (1) associated with: (i) $\Omega = (0,4) \times (0,4), \omega = \left\{\{x_1, x_2\} \mid \left(\frac{x_1 - G_1}{a}\right)^2 + \left(\frac{x_2 - G_2}{b}\right)^2 < 1\right\}$ with $G_1 = G_2 = 2, a = 1/4$ and $b = 1/8$. (ii) $f(x_1, x_2) = \alpha(x_1^3 - x_2^3) - 6\mu(x_1 - x_2), \forall\{x_1, x_2\} \in \Omega\backslash\overline{\omega}$. (iii) $g_0 = x_1^3 - x_2^3, g_1 = \mu\left[3(n_1x_1^2 - n_2x_2^2) + (x_1^3 - x_2^3)/l_s\right]$, $\{n_1, n_2\} = \mathbf{n}$ being the unit normal vector at γ pointing to ω. (iv) $\alpha = 100, \mu = 0.1$

and $l_s = 0.1$. The unique solution of problem (1), associated with the above data, in $H^1(\Omega\backslash\overline{\omega})$ is given by

$$\psi(x_1, x_2) = x_1^3 - x_2^3.$$

Concerning the finite element implementation of the least-squares/fictitious domain methodology discussed in Sects. 3 and 4, we employed for $\mathcal{T}_{h_1}$ (resp., $\mathcal{T}_{h_2}$) uniform triangulations of Ω (resp., triangulations of ω) like the one shown in Fig. 2(*left*) (resp., Fig. 2(*right*)). We used $u^0 = 0$ to initialize the discrete analogue of the conjugate gradient algorithm (13)–(26), and took $tol = 10^{-10}$ for the stopping criterion.

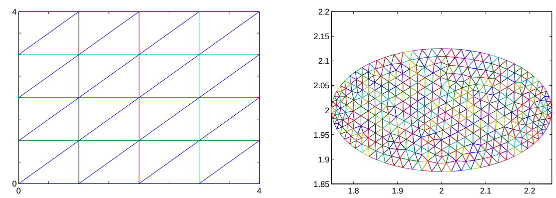

Fig. 2. A uniform triangulation of Ω(*left*); A triangulation of ω(*right*).

In Table 1, we report on the mesh sizes $h_2 = 1/40$ and $h_1 = 1/5, 1/10, 1/20$ and $1/40$: (i) The number of conjugate gradient iterations necessary to achieve convergence. (ii) Various norms of the approximation error. These results suggest: (a) For h_1 small enough, the number of iterations varies slowly with h_1. (b) $\| \psi_{\mathbf{h}} - \psi \|_{L^\infty(\Omega\backslash\overline{\omega})} \approx O(h_1^2)$, $\| \psi_{\mathbf{h}} - \psi \|_{L^2(\Omega\backslash\overline{\omega})} \approx O(h_1^2)$ and $\| \psi_{\mathbf{h}} - \psi \|_{H^1(\Omega\backslash\overline{\omega})} \approx O(h_1)$. Concerning the decay of the cost function $J_{\mathbf{h}}$, we have, if $\mathbf{h} = \{1/10, 1/40\}$ (resp., $\mathbf{h} = \{1/20, 1/40\}$), $J_{\mathbf{h}}(u^0) = 3.67$ (resp., $J_{\mathbf{h}}(u^0) = 3.35$) and $J_{\mathbf{h}}(u^{61}) = 4.20 \times 10^{-9}$ (resp., $J_{\mathbf{h}}(u^{59}) = 5.11 \times 10^{-9}$), showing clearly that the computed approximations of ψ_1 and ψ_2 match quite well over ω. In order to further investigate the convergence properties of the methodology discussed in the above sections we performed computations with $h_2 = 1/20$ and $h_1 = 1/10, 1/20, 1/40$ and $1/80$. The corresponding results have been reported in Table 2. From these results we observe that: (i) If $h_1 \geq h_2$, the number of iterations necessary to achieve convergence does not vary significantly with h_1; on the other hand this number of iterations increases sharply when h_1 decreases below h_2. (ii) The various approximation errors vary as expected (that is as in Table 1) if $h_1 \geq h_2$; on the other hand, they vary quite differently if $h_1 < h_2$, the only one behaving "nicely" being $\| \psi_{\mathbf{h}} - \psi \|_{H^1(\Omega\backslash\overline{\omega})}$, which shows a text-book $O(h_1)$ behavior as h_1 varies over the interval $[1/80, 1/10]$. From these results we suggest to take $h_1 = h_2$ to be on the safe side.

Remark 2. The results reported in [1] show a sharp decrease of the number of iteration when the methodology discussed here is applied to the solution of parabolic problems, including situations where ω is moving.

Table 1. Summary of numerical results ($h_2 = 1/40$).

h_1	Number of iterations	$\| \psi - \psi_h \|_{L^\infty(\Omega\setminus\overline{\omega})}$	$\| \psi - \psi_h \|_{L^2(\Omega\setminus\overline{\omega})}$	$\| \psi - \psi_h \|_{H^1(\Omega\setminus\overline{\omega})}$
1/5	34	0.1046	7.8370E-03	0.2855
1/10	61	2.1845E-02	1.9028E-03	0.1423
1/20	59	4.5840E-03	4.7015E-04	7.1089E-02
1/40	68	1.1385E-03	1.1708E-04	3.5518E-02

Table 2. Summary of numerical results ($h_2 = 1/20$).

h_1	Number of iterations	$\| \psi - \psi_h \|_{L^\infty(\Omega\setminus\overline{\omega})}$	$\| \psi - \psi_h \|_{L^2(\Omega\setminus\overline{\omega})}$	$\| \psi - \psi_h \|_{H^1(\Omega\setminus\overline{\omega})}$
1/10	33	2.1845E-02	1.9038E-03	0.1424
1/20	36	4.5840E-03	4.6807E-04	7.1063E-02
1/40	114	2.1385E-03	1.0163E-04	3.5434E-02
1/80	85	3.1514E-03	5.2854E-05	1.7532E-02

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