A FETI-DP Formation for the Stokes Problem Without Primal Pressure Components

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Summary. A scalable FETI–DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithm for the Stokes problem is developed and analyzed. Advantages of this approach are a coarse problem without primal pressure unknowns and the use of a relatively cheap lumped preconditioner. Especially in three dimensions, these advantages provide a more robust and faster FETI-DP algorithm. In three dimensions, the velocity unknowns at subdomain corners and the averages of velocity unknowns over common faces are selected as the primal unknowns in the FETI-DP formulation. A condition number bound of the form $C(H/h)$ is established, where C is a positive constant which is independent of any mesh parameters and H/h is the number of elements across individual subdomains.

1 Introduction

FETI–DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms are known to be among the most scalable domain decomposition methods, which are iterative substructuring methods based on Lagrange multipliers, see [1, 2]. This family of algorithms was developed for the Stokes problem by [3, 5, 6, 7]. In all these works for the Stokes problem, a compatibility condition on the dual velocity unknowns is required for each subdomain. As a consequence of this requirement, the velocity averages over edges in addition to the velocity unknowns at the subdomain corners are selected as primal unknowns in two dimensions. In three dimensions, the introduction of face averages and more complicated primal unknowns related to edges are unavoidable. By enforcing a compatibility condition on the dual velocity unknowns in each subdomain, additional primal unknown pressure components, constant in each subdomain, are used to enforce these compatibility conditions in these algorithms. This gives an indefinite coarse problem with both primal velocity and primal pressure unknowns.

In our previous work [4], we developed a new FETI-DP algorithm for the Stokes problem in two dimensions. In this algorithm, only velocity unknowns at the subdomain corners are selected as primal variables to reduce complication of the implementation. The primal pressure components are not used in contrast to other approaches for the Stokes problem. In this formulation, we can eliminate all the pressure unknowns by solving local Stokes problems, since such a selection of the primal velocity unknowns results in dual velocity unknowns which guarantee the solvability of the local Stokes problems without eliminating spurious pressure components. The Dirichlet-type preconditioners are no longer relevant to the FETI-DP formulation and a lumped preconditioner is naturally employed. Its condition number bound $C(H/h)(1+\log(H/h))$ was proved with the constant C depending on the inf-sup constant of a certain pair of velocity and pressure spaces. Furthermore it was shown that the inf-sup constant is independent of any mesh parameters for rectangular subdomain partitions. This method can be considered as an extension of the work in [8] to the Stokes problem.

In this paper, we extend the FETI-DP algorithm without primal pressure unknowns to the three-dimensional Stokes problem. By relaxing the compatibility condition on the dual velocity unknowns, we can select a relatively small set of primal unknowns, which are the primal velocity unknowns at the subdomain corners. To ensure scalability of a method in three dimensions, our method involves only primal velocity unknowns, which are velocity averages over common faces.

The resulting coarse problem of our method consists of only the primal velocity unknowns and becomes symmetric and positive definite. This allows the use of a more practical Cholesky solver for the coarse problem in contrast to indefinite coarse problems that appear in [5, 6, 7, 9]. With a lumped preconditioner, a scalable condition number bound $C(H/h)$ is obtained for the FETI-DP algorithm.

2 FETI-DP Formulation

2.1 Model Problem

We consider the three-dimensional Stokes problem,

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \ \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \ \text{ in } \Omega, \\ \mathbf{u} &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbf{R}^3$ and $\mathbf{f} \in [L^2(\Omega)]^3$. We introduce a pair of inf-sup stable finite element spaces $(\widehat{X}, \overline{P})$ where $\widehat{X} \subset [H_0^1(\Omega)]^3$ and $\overline{P} \subset L_0^2(\Omega)$. Here $H_0^1(\Omega)$ is the Sobolev space with functions that have square integrable weak derivatives up to the first order and with zero traces on $\partial\Omega$, and $L_0^2(\Omega)$ consists of square integrable functions with average value zero over Ω. In addition, the velocity functions are continuous and the pressure functions can be discontinuous across element boundaries, and $\overline{P} = P \bigcap L_0^2(\Omega)$, where P is the space of pressure functions before enforcing the zero average condition.

2.2 FETI-DP Formulation Without Primal Pressure Components

We now decompose Ω into a non-overlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that each subdomain aligns to the finite element triangulation of Ω. The subdomain finite element spaces are then obtained as

$$X^{(i)} = \widehat{X}|_{\Omega_i},\ P^{(i)} = P|_{\Omega_i},$$

that are the restrictions of $\widehat{X}$ and P to the individual subdomains. Among the subdomain velocity unknowns, we select some unknowns at each subdomain boundary as primal unknowns and we denote each part of the subdomain velocity unknowns by $\mathbf{u}_I^{(i)}$, $\mathbf{u}_\Pi^{(i)}$, and $\mathbf{u}_\Delta^{(i)}$, where I, Π, and Δ denote unknowns of the subdomain interior, the primal unknowns, and the remaining dual unknowns at the subdomain boundary, respectively. In the present work, the velocity unknowns at the subdomain corners and the averages of the velocity unknowns over common faces are selected as the primal unknowns.

We introduce the velocity spaces, $X_I^{(i)}$, $X_\Pi^{(i)}$, and $X_\Delta^{(i)}$ corresponding to the unknowns $\mathbf{u}_I^{(i)}$, $\mathbf{u}_\Pi^{(i)}$, and $\mathbf{u}_\Delta^{(i)}$, respectively. We also introduce a space $X_r^{(i)}$ of both the interior and the dual velocity unknowns,

$$X_r^{(i)} = X_I^{(i)} \times X_\Delta^{(i)},$$

and use the notation $\mathbf{u}_r^{(i)}$ for the velocity unknowns in the space $X_r^{(i)}$.

Throughout the paper, for given spaces $W^{(i)}$ associated with Ω_i we denote by W the product space of $W^{(i)}$ and by $\widetilde{W}$ the subspace of W, where the strong continuity at the primal unknowns is enforced. The subspace of W, where continuity at all interface unknowns is enforced, will be denoted by $\widehat{W}$. The unknowns at these spaces W, $\widetilde{W}$, and $\widehat{W}$ are then decoupled, partially coupled, and fully coupled across the subdomain interface, respectively. We also use the same notational convention for the velocity unknowns; $\mathbf{u}_r$ denotes $(\mathbf{u}_r^{(1)}, \dots, \mathbf{u}_r^{(N)})$ and $\widetilde{\mathbf{u}}$ denotes velocity unknowns in the space $\widetilde{X}$. We will use the same notation $\mathbf{u}$ to denote velocity unknowns and the corresponding finite element function.

We now obtain a discrete form of the Stokes problem (1) in the finite element space $(\widetilde{X}, \overline{P})$ by enforcing the pointwise continuity on the remaining part of the interface unknowns using Lagrange multipliers $\boldsymbol{\lambda} \in M$:

find $((\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi), \overline{p}, \boldsymbol{\lambda}) \in \widetilde{X} \times \overline{P} \times M$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & \overline{B}_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & \overline{B}_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & \overline{B}_\Pi^T & 0 \\ \overline{B}_I & \overline{B}_\Delta & \overline{B}_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \widehat{\mathbf{u}}_\Pi \\ \overline{p} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \\ 0 \\ 0 \end{pmatrix}, \tag{2}$$

where $\overline{B}_I$, $\overline{B}_\Delta$, and $\overline{B}_\Pi$ are defined by

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q\, dx, \quad \forall q \in \overline{P},$$

J_Δ is a boolean matrix that computes jump of the dual unknowns across the subdomain interface Γ_{ij},

$$J_\Delta \mathbf{u}_\Delta|_{\Gamma_{ij}} = \mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(j)},$$

and the other operators are defined by

$$\sum_i \int_{\Omega_i} \nabla \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{v}}\, dx.$$

The common interface Γ_{ij} can be an edge or a face of subdomains Ω_i and Ω_j. We introduce fully redundant Lagrange multipliers in our FETI-DP formulation. For $\boldsymbol{\lambda} \in M$, $\boldsymbol{\lambda}|_{F_{ij}}$ denotes the Lagrange multipliers which are related to the continuity constraints $\mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(j)} = 0$ on the common face $\overline{F}_{ij}$. Similarly, $\boldsymbol{\lambda}|_{E_{ik}}$ denotes the Lagrange multipliers related to the continuity constraints $\mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(k)} = 0$ on the common edge E_{ik}, which is the only common part of the two subdomains Ω_i and Ω_k. We call $\boldsymbol{\lambda}|_{F_{ij}}$ face-based Lagrange multipliers and $\boldsymbol{\lambda}|_{E_{ik}}$ edge-based Lagrange multipliers, respectively.

We recall that the pressure finite element space,

$$\overline{P} = P \bigcap L_0^2(\Omega),$$

where $P = \prod_{i=1}^N P^{(i)}$. These local pressure spaces $P^{(i)}$ do not satisfy the zero average condition. In order to eliminate all the pressure unknowns by solving independent local Stokes problems, we will use the pressure space P instead of $\overline{P}$ in our FETI-DP formulation. By adding a constant pressure component, we extend the pressure space $\overline{P}$ to the space P. The added constant component will give us an additional condition on $\widetilde{\mathbf{u}}$,

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q\, dx = 0, \quad q = c, \tag{3}$$

which is equivalent to

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, c\, dx = c \sum_{ij} \int_{F_{ij}} (\mathbf{u}_\Delta^{(i)} - \mathbf{u}_\Delta^{(j)}) \cdot \mathbf{n}_{ij}\, ds = 0.$$

Here F_{ij} denotes the common face of two subdomains Ω_i and Ω_j. The above equation can be obtained as a linear combination of the continuity constraints on $\mathbf{u}_\Delta$,

$$J_\Delta \mathbf{u}_\Delta = 0.$$

Since $J_\Delta \mathbf{u}_\Delta = 0$ has been already enforced in (2), by adding (3) to the algebraic system (2), we obtain an extended algebraic system which is equivalent to (2).

We write the extended algebraic system with the pressure space P as follows: find $((\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi), p, \boldsymbol{\lambda}) \in (\widetilde{X}, P, M)$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & B_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & B_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & B_\Pi^T & 0 \\ B_I & B_\Delta & B_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \widehat{\mathbf{u}}_\Pi \\ p \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

Here B_I, B_Δ, and B_Π are defined by

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\mathbf{u}}\, q\, dx, \quad \forall q \in P,$$

and the other terms are the same as those in (2).

In the new algebraic form, the unknowns $(\mathbf{u}_I, \mathbf{u}_\Delta, p)$ can be eliminated by solving independent local problems. The advantage of the extended algebraic system is that no pressure unknowns are left and only the primal velocity unknowns remain after solving the local problems. The primal velocity unknowns can be eliminated by solving a global coarse problem, which is smaller and more practical than those of other domain decomposition algorithms for the Stokes problem [5, 6, 7, 9]. As a result a linear system on $\boldsymbol{\lambda}$ will be obtained,

$$F_{DP}\boldsymbol{\lambda} = d.$$

The introduction of fully redundant Lagrange multipliers and the extension of the pressure space make the resulting system singular.

The extension of the pressure space introduces one more null space component which is given by

$$\boldsymbol{\mu}_0|_{F_{ij}} = \zeta_{ij}\mathbf{n}_{ij},\ \forall F_{ij} \ \text{ and } \ \boldsymbol{\mu}_0|_{E_{lk}} = 0,\ \forall E_{lk}.$$

Here $\boldsymbol{\mu}_0|_{F_{ij}}$ and $\boldsymbol{\mu}_0|_{E_{lk}}$ are face-based and edge-based Lagrange multipliers, respectively, $\mathbf{n}_{ij}$ is the unit normal to the face F_{ij}, and at each nodal point $x_l \in \overline{F}_{ij}$, $\zeta_{ij}(x_l)$ is given by

$$\zeta_{ij}(x_l) = \int_{F_{ij}} \phi_l(x(s), y(s), z(s))\, ds,$$

where ϕ_l is the velocity basis function related to the node x_l. This can be shown by observing that $(\mathbf{u}_I, \mathbf{u}_\Delta, \widehat{\mathbf{u}}_\Pi) = 0$, $p = c$, and $\boldsymbol{\lambda} = c\boldsymbol{\mu}_0$ are solutions of (4) for the zero force terms $(\mathbf{f}_I, \mathbf{f}_\Delta, \mathbf{f}_\Pi) = 0$ with c an arbitrary constant.

Let Range(J_Δ) be the range space of J_Δ. We then have

$$M = \text{Null}(J_\Delta^T) \bigoplus \text{Range}(J_\Delta).$$

We now introduce a subspace of M, which is orthogonal to the null space components of F_{DP},

$$M_c = \left\{ \boldsymbol{\mu} \in \mathrm{Range}(J_\Delta) \, : \, \boldsymbol{\mu}^t \boldsymbol{\mu}_0 = 0 \right\}.$$

We then perform the preconditioned conjugate gradient iteration in the subspace M_c by employing a lumped preconditioner $\widehat{M}^{-1}$,

$$\widehat{M}^{-1} = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T \begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} = J_\Delta K_{\Delta\Delta} J_\Delta^T.$$

3 Analysis of a Bound of Condition Number

In this section, we will provide a condition number bound of the FETI-DP operator with the lumped preconditioner by proving the following inequalities:

$$C_1 \beta^2 \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \le \langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \le C_2 \left(\frac{H}{h} \right) \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle, \quad \forall \boldsymbol{\lambda} \in M_c, \tag{5}$$

where β is the inf-sup constant of a certain pair of velocity and pressure finite element spaces, $(\widehat{E}_{I,\Pi}, \overline{P})$. In a more detail, $\widehat{E}_{I,\Pi} = X_I + \widehat{E}_\Pi$ and $\widehat{E}_\Pi$ consists of functions $\mathbf{v}$ in $\widehat{X}$, which minimize the H^1-discrete seminorm and satisfy

$$\begin{aligned} \mathbf{v}(V) \qquad\qquad &= \mathbf{a}_V, \\ \int_F I^h(\theta_F \mathbf{v})(x)\, dx(s) \quad &= \mathbf{a}_F, \\ \int_F I^h(\theta_E \mathbf{v})(x) \cdot \mathbf{n}_F \, dx(s) &= a_F^E, \quad \forall F \in \mathcal{F}(E), \end{aligned}$$

with given values of $\mathbf{a}_V$, $\mathbf{a}_F$, and a_F^E, which are provided for all vertices $V \in \mathcal{V}$, all faces $F \in \mathcal{F}$, and all edges $E \in \mathcal{E}$. Here $\mathcal{F}(E)$ is the set of faces with the edge E in common, and θ_F and θ_E are face and edge cut-off functions, which are one at the interior nodes of the face F, and those of the edge E, respectively, and zero at the other unknowns. In addition, $I^h(\mathbf{v})$ is the nodal interpolant of $\mathbf{v}$ to the velocity finite element space $\widehat{X}$.

These inequalities in (5) yield the following condition number bound,

$$\kappa(\widehat{M}^{-1} F_{DP}) \le C \frac{1}{\beta^2} \left(\frac{H}{h} \right).$$

3.1 Lower Bound

Lemma 1. *For any $\boldsymbol{\mu} \in M_c$, there exists $\mathbf{u} \in \widetilde{X}$ such that*
1. $J_\Delta \mathbf{u}_\Delta = \boldsymbol{\mu}$,
2. $\sum_i \int_{\Omega_i} \nabla \cdot \mathbf{u} \, q \, dx = 0, \quad \forall q \in P$,
3. $\langle K\mathbf{u}, \mathbf{u} \rangle \le C \frac{1}{\beta^2} \langle K_{\Delta\Delta} J_\Delta^T \boldsymbol{\mu}, J_\Delta^T \boldsymbol{\mu} \rangle$, *where β is the inf-sup constant of the pair* $(\widehat{E}_{I,\Pi}, \overline{P})$.

We introduce

$$\widetilde{X}(\mathrm{div}) = \left\{ \mathbf{v} \in \widetilde{X} \ : \ \int_{\Omega_i} \nabla \cdot \mathbf{v} q \, dx = 0, \quad \forall q \in P \right\}.$$

We then have the identity,

$$\langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{v} \in \widetilde{X}(\mathrm{div})} \frac{\langle J_\Delta \mathbf{v}_\Delta, \boldsymbol{\lambda} \rangle^2}{\langle K \mathbf{v}, \mathbf{v} \rangle}, \tag{6}$$

where K is the stiffness matrix given by

$$K = \begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} \end{pmatrix}.$$

The following lower bound can be obtained from Lemma 1 and (6):

Theorem 1. *For any $\boldsymbol{\lambda} \in M_c$, we have*

$$C_1 \beta^2 \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq \langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle,$$

where β is the inf-sup constant of the pair $(\widehat{E}_{I,\Pi}, \overline{P})$ and C_1 is a positive constant that does not depend on any mesh parameters.

3.2 Upper Bound

The following result is obtained from a Poincaré inequality, see [8, Lemma 4]:

Lemma 2. *Let Ω_i be a three-dimensional subdomain. For any function $v \in H^1(\Omega_i)$,*

$$\|v - c_F\|_{L^2(F)}^2 \leq CH |v|_{H^1(\Omega_i)}^2,$$

where F is a face of the subdomain Ω_i and c_F is given by

$$c_F = \frac{\int_F I^h(\theta_F v) dx(s)}{\int_F dx(s)}.$$

By using the above lemma, we obtain:

Lemma 3. *There exists a constant C such that*

$$\langle K_{\Delta\Delta} J_\Delta^T J_\Delta \mathbf{u}_\Delta, J_\Delta^T J_\Delta \mathbf{u}_\Delta \rangle \leq C \frac{H}{h} \langle K \mathbf{u}, \mathbf{u} \rangle, \quad \textit{for any } \mathbf{u} \in \widetilde{X}.$$

The identity in (6) combined with Lemma 3 gives the upper bound:

Theorem 2. *For any $\boldsymbol{\lambda} \in M_c$, we have*

$$\langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq C_2 \frac{H}{h} \langle \widehat{M} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle,$$

where C_2 is a positive constant that does not depend on any mesh parameters.

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