A Domain Decomposition Method Combining a Boundary Element Method with a Meshless Local Petrov-Galerkin Method

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Summary. A non-overlapping domain decomposition algorithm combining boundary element method with meshless local Petrov-Galerkin method is presented for solving the boundary value problem with discontinuous coefficient in this paper. The static relaxation parameter is employed to speed up the convergence rate. The convergence range and the optimal value of static relaxation parameter are studied, but the numerical results show that the optimal static relaxation parameter is different for different problems. Therefore, a dynamic relaxation parameter is presented for the algorithm. The numerical results show that the number of iteration with the dynamic relaxation parameter is less than that with the static relaxation parameter.

Key words: boundary element method, meshless local Petrov–Galerkin method, domain decomposition method, relaxation parameter

1 Introduction

As we know, the meshless methods and boundary element method (BEM) are widely employed as two of the main numerical methods for the solution of a wide variety of science and engineering problems. However, they exhibit different advantages when applied to different classes of problems. The main feature of the meshless methods is the absence of an explicit mesh, and the approximate solutions are constructed entirely based on a cluster of scattered nodes. Therefore, the meshless methods are well suited to problems with extremely large deformation, dynamic fracturing, or explosion [1, 2]. On the other hand, the main advantage of BEM is that the dimensionality of the problem is reduced by one, and the BEM is very efficient for the analysis of homogenous linear problems in unbounded domains.

It is attractive to divide a computational domain into sub-domains and to use the most appropriate method for each sub-domain. The idea of coupling the meshless methods and BEM is by now well known as an efficient analysis tool, which makes use of their advantages. A great number of articles on the topic, such as combining element-free Galerkin method (EFGM) with BEM [10, 11, 12, 21], reproducing kernel particle method (RKPM) with BEM [16], the mesh-free finite cloud method (MFCM) with BEM [17], meshless local Petrov-Galerkin (MLPG) method with BEM [9], can be found. The above coupling methods deduce an entire unified system for the whole domain, by combining the discretized equations for the BEM and different meshless methods in sub-domains. However, the algorithm for constructing a large entire system for the whole domain is complicated and time-consuming for computation when compared with that for each single equation, and may destroy the desirable features originally existing in the meshless methods matrices, namely, symmetry and sparsity.

The domain decomposition methods (DDM) combining FEM-BEM or BEM-BEM have been developed [3, 4, 5, 6, 7, 8, 13, 14, 15, 18]. The DDM is better than the above coupling methods when the domains under consideration are governed by different differential equations or constructed of different materials, especially in the case of large domain with complicated boundary manifold. Therefore, a non-overlapping DDM combining BEM with MLPG method is presented in order to make use of their advantages and preserve the nature of the both methods in this paper.

This paper is arranged as follows: Sect. 2 gives a non-overlapping domain decomposition algorithm combining BEM and MLPG method for solving the boundary value problem with discontinuous coefficient. Then, the dynamic relaxation parameter is presented for the algorithm in the next section. In Sect. 4, the convergence range and the optimal value of the static relaxation parameter are studied and the validity of the dynamic relaxation parameter is verified by numerical results. Finally, the conclusions are given in Sect. 5.

2 A DDM Combining BEM with the MLPG Method

Consider the following boundary value problem with discontinuous coefficient

$$\begin{cases} \nabla \cdot (\gamma(x)\nabla u(x)) = 0, \ x \in \Omega \\ u(x) = f(x), \ x \in \Gamma_u \\ q(x) = \gamma(x) \cdot \partial u(x)/\partial n = g(x), \ x \in \Gamma_q \end{cases} \tag{1}$$

where $\Omega \subset \mathrm{R}^2$, Γ is its boundary. $x = (x_1, x_2)$ denotes the point in R^2. $\gamma(x)$ is the conductivity coefficient. $f(x), g(x)$ are the given boundary data. The problem (1) often appears in engineering problems, e.g., heat conduction and electric conduction models with mixed materials, Darcy flow in porous media, etc. Many methods, such as FEM [19] and BEM [20], have been successfully used to solving the problem (1), however, the domain decomposition is suitable for the problems with discontinuity conductivity coefficients. In this paper, we assume that the conductivity coefficient is as follows

$$\gamma(x) = \begin{cases} 1, \ x \in \Omega_B \subset \Omega \\ \gamma_M(x), x \in \Omega_M = \Omega \setminus \Omega_B \end{cases} \tag{2}$$

Then the domain of the original problem can be decomposed into BEM sub-domain Ω_B and MLPG sub-domain Ω_M, let $\Gamma^I = \partial\Omega_B \cap \partial\Omega_M$ be the BEM/MLPG interface (Fig.1). Apparently, the continuity and equilibrium conditions should be satisfied at the interface, that is

$$u_B(x) = u_M(x), \quad \frac{\partial u_B(x)}{\partial n_B} + \gamma_M(x)\frac{\partial u_M(x)}{\partial n_M}=0,\ x \in \Gamma^I \tag{3}$$

where $u_B(x) = u(x)|_{\Omega_B}$, $u_M(x) = u(x)|_{\Omega_M}$, n_B and n_M are the unit outward normal vectors for the BEM and MLPG sub-domains, respectively.

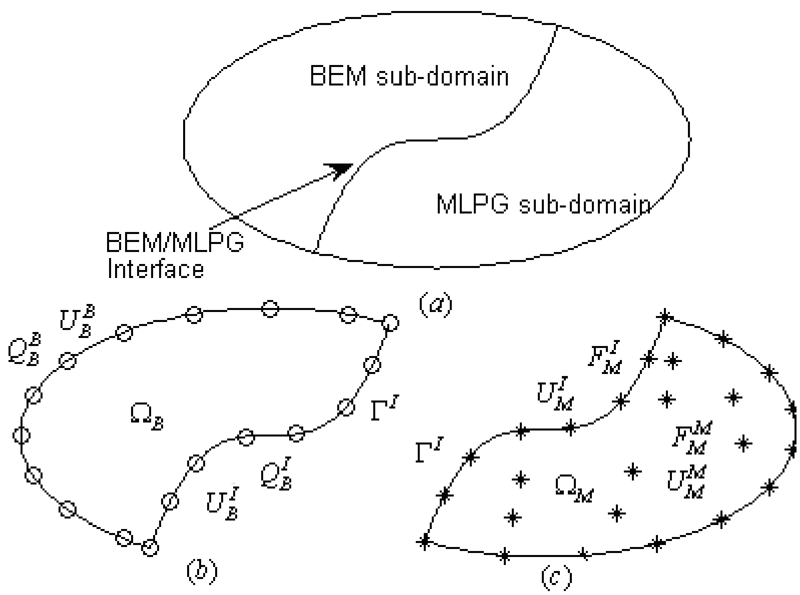

Fig. 1. Domain decomposed into BEM and MLPG sub-domains.

On the one hand, we can obtain the following boundary integral equation for the BEM sub-domain Ω_B

$$c(y)u_B(y) + \int\limits_{\partial\Omega_B} u_B(x)\frac{\partial u^*(x,y)}{\partial n_B}\mathrm{d}\Gamma = \int\limits_{\partial\Omega_B} u^*(x,y)\frac{\partial u_B(x)}{\partial n_B}\mathrm{d}\Gamma,\ y \in \partial\Omega_B \tag{4}$$

where $u^*(x,y) = -\frac{1}{2\pi}\ln|x-y|$ is the fundamental solution of Laplace equation, $c(y)$ depends on the geometry shape at point y. The boundary integral equation (4) can be rewritten as the following matrix form

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{Bmatrix} U_B^B \\ U_B^I \end{Bmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{Bmatrix} Q_B^B \\ Q_B^I \end{Bmatrix} \tag{5}$$

where U_B^B and Q_B^B are column vectors containing the non-interface nodal potentials and fluxes values, respectively, U_B^I and Q_B^I are column vectors containing the interface nodal potentials and fluxes values, respectively. H and G are the corresponding coefficient matrices.

On the other hand, we have the following local weak form for the MLPG sub-domain Ω_M by means of the MLPG method

$$\begin{aligned}&\int\limits_{\Omega_{Mi}} (\gamma_M \cdot \nabla u \cdot \nabla v)\,\mathrm{d}\Omega + \int\limits_{\Gamma_{ui}} \left(\alpha u v - \gamma_M \cdot \frac{\partial u}{\partial n_M} \cdot v\right) \mathrm{d}\Gamma \\ &= \alpha \int\limits_{\Gamma_{ui}} f v \mathrm{d}\Gamma + \int\limits_{\Gamma_{qi}} g v \mathrm{d}\Gamma + \int\limits_{\Gamma_{Ii}} \left(\gamma_M \cdot \frac{\partial u}{\partial n_M}\right) v \mathrm{d}\Gamma\end{aligned} \tag{6}$$

where $\Omega_{Mi} \subset \Omega_M$ is a local sub-domain, v is a weight function, α is a penalty factor, $\Gamma_{Ii} = \partial\Omega_{Mi} \cap \Gamma^I$, $\Gamma_{ui} = \partial\Omega_{Mi} \cap \Gamma_u$, $\Gamma_{qi} = \partial\Omega_{Mi} \cap \Gamma_q$. Then the assembled linear equations are given by

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{Bmatrix} U_M^M \\ U_M^I \end{Bmatrix} = \begin{Bmatrix} F_M^M \\ F_M^I \end{Bmatrix} \tag{7}$$

where U_M^I is the interface nodal potentials vectors, U_M^M is the all nodal potentials vectors except the interface nodal potentials, K and F are the corresponding coefficient matrix and right side vector, respectively. Note that F_M^I is a vector containing $\int\limits_{\Gamma_{Ii}} \left(\gamma_M \cdot \frac{\partial u}{\partial n_M}\right) v \mathrm{d}\Gamma$, then the conditions (3) can be rewritten as

$$U_B^I = U_M^I, \quad F_M^I = P Q_M^I = -P Q_B^I, \tag{8}$$

where P is a transition matrix, which depends on the weight function v and the shape function in the moving least square approximation, Q_M^I is a column vector containing the interface nodal fluxes values.

Therefore, a parallel Dirichlet-Neumann BEM-MLPG algorithm is as follows:

Step 1: assign the initial potential vector $U_{B,0}^I$ and flux vector $Q_{M,0}^I$, and n:=0.

Step 2: solve

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{Bmatrix} U_{B,n}^B \\ U_{B,n}^I \end{Bmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{Bmatrix} Q_{B,n}^B \\ Q_{B,n}^I \end{Bmatrix} \text{ for } Q_{B,n}^I \tag{9}$$

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{Bmatrix} U_{M,n}^M \\ U_{M,n}^I \end{Bmatrix} = \begin{Bmatrix} F_{M,n}^M \\ P Q_{M,n}^I \end{Bmatrix} \text{ for } U_{M,n}^I \tag{10}$$

Step 3: apply

$$U_{B,n+1}^I = \beta U_{M,n}^I + (1-\beta) U_{B,n}^I \tag{11}$$

$$Q_{M,n+1}^I = -Q_{B,n}^I \tag{12}$$

Step 4: check if $\left\|U_{B,n+1}^I - U_{B,n}^I\right\| + \left\|Q_{M,n+1}^I - Q_{M,n}^I\right\| \le \varepsilon \max(||U_{B,n+1}^I||, ||Q_{M,n+1}^I||)$, if yes then stop, otherwise set n:=n+1, and go to Step 2.

here β is a relaxation parameter to ensure and/or accelerate convergence, ε is the user specified error allowance.

3 A Dynamic Relaxation Parameter

If the relaxation parameter β is assigned as a constant for every iteration, an optimal $\bar{\beta}$ can be obtained by testing with different values. However, we find that the optimal static value is different for the different problems in numerical test, therefore we couldn't find a suitable optimal value for all problems. Fortunately, a dynamic relaxation parameter has been obtained for the sequential FEM-BEM algorithm [18], that is to say, the iterative procedure can be facilitated by allowing the relaxation parameter to change dynamically with each iteration. In this section, a dynamic relaxation parameter will be presented for the parallel Dirichlet-Neumann BEM-MLPG iterative algorithm.

By minimizing the square error functional

$$G(\beta) = \left\| U^I_{B,n+1}(\beta) - U^I_{B,n}(\beta) \right\|^2 + \left\| Q^I_{M,n+1}(\beta) - Q^I_{M,n}(\beta) \right\|^2 \tag{13}$$

with respect to the relaxation parameter β, one gets an optimal dynamic value for the next iteration, i.e.,

$$\beta_n = \frac{\langle e_{B,n}, e_{B,n} - e_{M,n} \rangle + \langle e_{B,n-1}, e_{B,n-1} - e_{M,n-1} \rangle}{\left\| e_{B,n} - e_{M,n} \right\|^2 + \left\| e_{B,n-1} - e_{M,n-1} \right\|^2}, n \geq 3 \tag{14}$$

$$e_{B,n} = U^I_{B,n} - U^I_{B,n-1}, \quad e_{M,n} = U^I_{M,n} - U^I_{M,n-1}, n \geq 2 \tag{15}$$

where $\langle a, b \rangle$ is a inner product, and $\|a\|^2 = \langle a, a \rangle$.

4 Numerical Examples

To illustrate the convergence results of the iterative algorithm, a numerical example is presented in this section. Moreover, the accelerated convergence of the dynamic relaxation parameter will also be shown. In this section, we choose the error bound as $\varepsilon = 10^{-4}$.

Example (Potential flow problem [6, 8]) We consider the mixed boundary value problem (1), and assume that $\Omega_B = [0,1] \times [0,1]$ and $\Omega_M = [1,2] \times [0,1]$, the conductivity coefficient $\gamma_M(x) = 2$, the boundary conditions are selected such that $u(0, x_2) = 0$, $u(2, x_2) = 200$ and zero flux elsewhere (Fig. 2a).

Using the proposed iterative algorithm in Sect. 2, the problem is solved by three different discretization types denoted as Fig. 2 (b–d).

Figure 3 shows the convergence ranges and the optimal static values with the static relaxation parameter for the different discretization types. Beyond the values shown in Fig. 3, the iterative algorithm will not converge. From Fig. 3, we know that convergence ranges and optimal static values are different for the different discretization types. Therefore it is impossible to select a suitable optimal static value for all cases.

Table 1 shows the numbers of iterations with optimal static values and dynamic values for the different discretization types. From Table 1, obviously, the number of iterations with dynamic value is less than or equal to that with the static value. Therefore, we can say that the dynamic values is the optimal relaxation parameter.

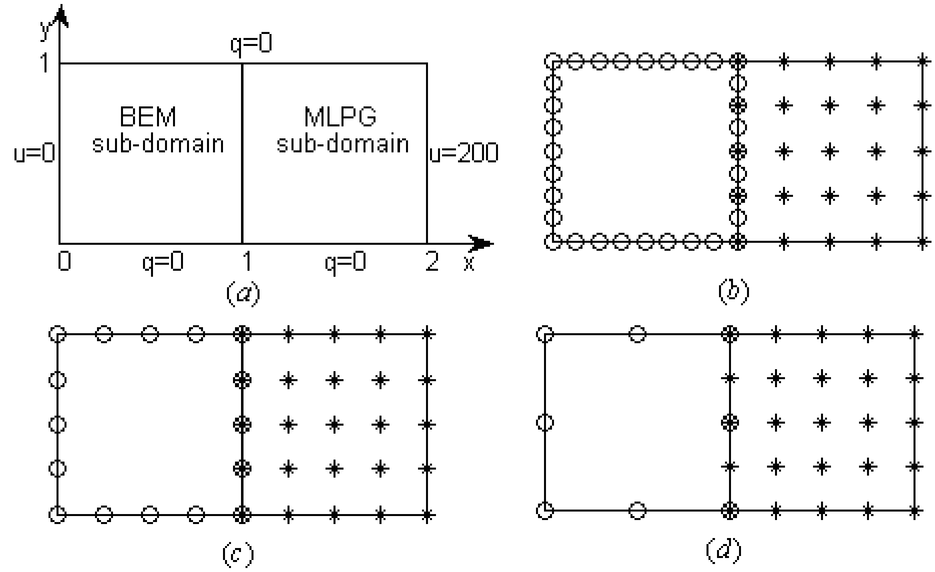

Fig. 2. Potential flow problem and discretization types.

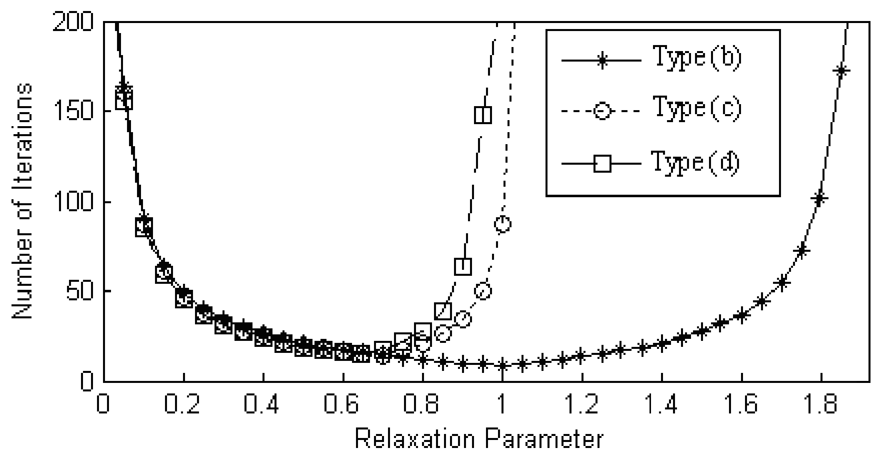

Fig. 3. Convergence ranges and optimal static values for the different discretization types.

Table 1. Number of iterations for the different discretization types.

Relaxation parameter	Type (b)	Type (c)	Type (d)
Optimal static value β	8	14	16
Dynamic values β_n	8	11	9

5 Conclusions

Generally speaking, the static relaxation parameter is always employed to speed up the convergence rate of domain decomposition methods. However the convergence ranges and optimal values of the static relaxation parameter are different for different problems. Therefore, the dynamic relaxation parameter is used in the proposed domain decomposition algorithm in this paper, the numerical results show that the dynamic relaxation parameter is the optimal relaxation parameter which is well suited to all cases.

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