On Domain Decomposition Algorithms for Contact Problems with Tresca Friction

Julien Riton[1], Taoufik Sassi[1], and Radek Kučera[2]

[1] LMNO, University of Caen, Caen, France, riton.julien@math.unicaen.fr; Taoufik.Sassi@math.unicaen.fr
[2] Technical University of Ostrava, Ostrava, Czech Republic, radek.kucera@vsb.cz

1 Introduction

Development of numerical methods for the solution of contact problems is a challenging task whose difficulty lies in the non-linear conditions for non-penetration and friction. Recently, many authors proposed to use various numerical algorithms combined with multigrid or domain decomposition techniques; see, e.g., the primal-dual active set algorithm [8], the non-smooth multiscale method [10], or the augmented Lagrangian based algorithm [3]. Another alternative consists in the formulation of suitable iterations solving the elasticity equations for each sub-body separately with certain boundary conditions [5]. In [1], the authors proposed a Dirichlet-Neumann algorithm which takes into account the natural interface for frictionless contact problems. Another improvement has led to a Neumann–Neumann algorithm in which they added two Neumann sub-problems in order to ensure the continuity of normal stresses [2]. Later, various numerical implementations of this approach was given in [7, 9]. In this contribution, we extend the algorithm to two-body contact problems with Tresca friction. The advantage consists in decoupling the non-penetration and friction conditions between the bodies so that they are treated separately by smaller subproblems that may be solved in parallel. With respect to existing (global) algorithms, our method is suitable in situations when material or geometrical qualities of the bodies are considerably different. By numerical experiments, we illustrate that the algorithm is mesh independent for a suitable choice of parameters.

2 Contact Problems with Tresca Friction

Let us consider two elastic bodies occupying bounded domains $\Omega^\alpha \in \mathbb{R}^2$, $\alpha = 1, 2$. Each boundary $\Gamma^\alpha := \partial\Omega^\alpha$ is assumed piecewise continuous and composed of three disjoint, non-empty parts Γ_u^α, Γ_ℓ^α, and Γ_c^α. Each body Ω^α is fixed on Γ_u^α and subject to surface tractions $\boldsymbol{\phi}^\alpha \in \mathbf{L}^2(\Gamma_\ell^\alpha)$. The body forces are denoted by $\boldsymbol{f}^\alpha \in \mathbf{L}^2(\Omega^\alpha)$. In the initial configuration, the bodies possess the common contact interface $\Gamma_c := {\Gamma_c}^1 = {\Gamma_c}^2$, where the unilateral contact with Tresca friction is considered. The problem consists in finding the displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$, $\boldsymbol{u}^\alpha := \boldsymbol{u}_{|\Omega^\alpha}$, and the stress tensor $\sigma = (\sigma(\boldsymbol{u}^1), \sigma(\boldsymbol{u}^2))$ such that for $\alpha = 1, 2$:

$$\left.\begin{aligned} \mathbf{div}\sigma(\mathbf{u}^\alpha) + \boldsymbol{f}^\alpha = \ & \mathbf{0} \text{ in } \Omega^\alpha, \\ \sigma(\boldsymbol{u}^\alpha)\boldsymbol{n}^\alpha = \ & \boldsymbol{\phi}^\alpha \text{ on } \Gamma_\ell^\alpha, \\ \boldsymbol{u}^\alpha = \ & \mathbf{0} \text{ on } \Gamma_u^\alpha \end{aligned}\right\} \tag{1}$$

and σ is related to the strain tensor $e(\boldsymbol{u}^\alpha) = \left(\nabla \boldsymbol{u}^\alpha + (\nabla \boldsymbol{u}^\alpha)^T\right)/2$ by Hooke's law for linear isotropic materials:

$$\sigma_{ij}(\boldsymbol{u}^\alpha) = \mathsf{E}^\alpha_{ijkh} e_{kh}(\boldsymbol{u}^\alpha),$$

where $\mathsf{E} = (\mathsf{E}^\alpha_{ijkh})_{1 \le i,j,k,h \le 2} \in (L^\infty(\Omega^\alpha))^{16}$ is the fourth-order tensor satisfying the symmetry and ellipticity conditions.

We will use the usual notation for the normal and tangential components of the displacement and stress vectors on Γ_c:

$$u_N^\alpha = \boldsymbol{u}^\alpha.\boldsymbol{n}^\alpha, \ \ u_T^\alpha = \boldsymbol{u}^\alpha.\boldsymbol{t}^\alpha, \ \ \sigma_N^\alpha = (\sigma(u^\alpha)\boldsymbol{n}^\alpha).\boldsymbol{n}^\alpha, \ \ {\sigma_T}^\alpha = (\sigma(\boldsymbol{u}^\alpha)\boldsymbol{t}^\alpha).\boldsymbol{n}^\alpha,$$

where $\boldsymbol{n}^\alpha$ denotes the unit outer normal vector to Γ_c^α and $\boldsymbol{t}^\alpha$ is the unit tangential vector satisfying $\boldsymbol{t}^\alpha.\boldsymbol{n}^\alpha = 0$ and $\boldsymbol{t}^1 = -\boldsymbol{t}^2$. On Γ_c, the unilateral contact law is given by

$$\sigma_N^1 = \sigma_N^2 := \sigma_N, \qquad \sigma_T^1 = \sigma_T^2 := \sigma_T, \tag{2}$$

$$[u_N] \le 0, \quad \sigma_N \le 0, \quad \sigma_N[u_N] = 0, \tag{3}$$

where $[u_N] := u_N^1 + u_N^2$ is the jump in the normal direction across the interface Γ_c. The Tresca law of friction is given by

$$\left.\begin{aligned} & |\sigma_T| \le g, \\ & |\sigma_T| < g \Rightarrow [u_T] = 0, \\ & |\sigma_T| = g \Rightarrow \exists \kappa \ge 0 : \ [u_T] = -\kappa \sigma_T \text{ on } \Gamma_c, \end{aligned}\right\} \tag{4}$$

where $g \in \mathrm{L}^2(\Gamma_c)$, $g \ge 0$, is the given slip bound on Γ_c and $[u_T] := u_T^1 + u_T^2$.

Remark 1. In the Coulomb law of friction, g replaces $\mathcal{F}|\sigma_N|$, i.e., the product of the coefficient of friction $\mathcal{F}$ and à-priori unknown absolute value the normal contact stress σ_N.

The problem of finding the couple $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ satisfying (1), (2), (3), and (4) will be called $(\mathcal{P})$. Its existence and uniqueness is established in [6].

3 Algorithms and the Implementation

We start with the algebraic formulation of the non-decomposed problem. Let p_α denote the dimension of the finite element space $\mathbb{V}^\alpha_{0,h}$ defined on the triangulation

$\mathcal{T}_h^\alpha$ of Ω^α, $\alpha = 1, 2$, and $p := p_1 + p_2$. Further, let q be the number of contact nodes of Ω^1, i.e., the nodes of $\mathcal{T}_h^1$ lying on $\overline{\Gamma}_c \setminus \overline{\Gamma}_u^1$. As we consider matching grids, the contact nodes of Ω^1 and Ω^2 coincide. By $\mathbf{A} \in \mathbb{R}^{p \times p}$ and $\mathbf{b} \in \mathbb{R}^p$, we denote the stiffness matrix and the load vector, respectively, of the whole structure. Let us note that $\mathbf{A}$, $\mathbf{b}$ can be naturally decomposed into blocks corresponding to Ω^1 and Ω^2 so that $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2)$, $\mathbf{b} = (\mathbf{b}_1^\top, \mathbf{b}_2^\top)^\top$, where $\mathbf{A}_\alpha \in \mathbb{R}^{p_\alpha \times p_\alpha}$ are symmetric, positive definite and $\mathbf{b}_\alpha \in \mathbb{R}^{p_\alpha}$, $\alpha = 1, 2$. We introduce the matrices $\mathbf{N}_\alpha, \mathbf{T}_\alpha \in \mathbb{R}^{q \times p_\alpha}$, $\alpha = 1, 2$, projecting contact displacements to the directions of $\boldsymbol{n}^\alpha$, $\boldsymbol{t}^\alpha$, respectively, i.e., each row of $\mathbf{N}_\alpha, \mathbf{T}_\alpha$ contains the two components of the corresponding normal $\boldsymbol{n}^\alpha$ and tangential $\boldsymbol{t}^\alpha$ vectors. For sake of simplicity we denote by $\mathbf{B}_\alpha = (\mathbf{N}_\alpha^\top, \mathbf{T}_\alpha^\top)^\top$ that are matrices with orthonormal rows. Finally, the vector $\mathbf{g} \in \mathbb{R}^q$ is determined by the nodal values of g.

The finite element approximation of $(\mathcal{P})$ leads to the following algebraic problem:

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} - \mathbf{u}^\top \mathbf{b} + \sum_{i=1}^{q} g_i |\mathbf{T}_1\mathbf{u}_1 + \mathbf{T}_2\mathbf{u}_2|_i \\ &\text{subject to} \ \ \mathbf{N}_1\mathbf{u}_1 + \mathbf{N}_2\mathbf{u}_2 \le \mathbf{0}, \end{aligned} \tag{5}$$

where $\mathbf{u} = (\mathbf{u}_1^\top, \mathbf{u}_2^\top)^\top$, $\mathbf{u}_\alpha \in \mathbb{R}^{p_\alpha}$, $\alpha = 1, 2$ and $|\mathbf{v}| = (|v_1|, |v_2|, ..., |v_q|)^\top$ for $v = (v_1, ..., v_q)^\top$.

The problem (5) can be solved by ALGORITHM 1 and ALGORITHM 2 which are discrete versions of our domain decomposition methods .

Algorithm 1 Let $\boldsymbol{\lambda}^{(0)} = (\boldsymbol{\lambda}_\nu^{(0)\top}, \boldsymbol{\lambda}_\tau^{(0)\top})^\top \in \mathbb{R}^{2q}$ and $\theta > 0$ be given. For $k \ge 1$ compute $\mathbf{u}_\alpha^{(k)}, \mathbf{w}_\alpha^{(k)} \in \mathbb{R}^{p_\alpha}$, $\alpha = 1, 2$, and $\boldsymbol{\lambda}^{(k)} = (\boldsymbol{\lambda}_\nu^{(k)\top}, \boldsymbol{\lambda}_\tau^{(k)\top})^\top \in \mathbb{R}^{2q}$ as follows:

(*Step 1*) $\{$*Normal bilateral contact with Tresca friction for* Ω^1.$\}$

$$\begin{aligned} \mathbf{u}_1^{(k)} := \text{argmin}\ &\frac{1}{2}\mathbf{u}_1^\top \mathbf{A}_1\mathbf{u}_1 - \mathbf{u}_1^\top \mathbf{b}_1 + \sum_{i=1}^{q} g_i |\mathbf{T}_1\mathbf{u}_1 - \boldsymbol{\lambda}_\tau^{(k-1)}|_i \\ &\text{subject to } \mathbf{N}_1\mathbf{u}_1 = \boldsymbol{\lambda}_\nu^{(k-1)}; \end{aligned}$$

(*Step 2*) $\{$*Normal unilateral and tangential bilateral contact for* Ω^2.$\}$

$$\begin{aligned} \mathbf{u}_2^{(k)} := \text{argmin}\ &\frac{1}{2}\mathbf{u}_2^\top \mathbf{A}_2\mathbf{u}_2 - \mathbf{u}_2^\top \mathbf{b}_2 \\ &\text{subject to } \boldsymbol{\lambda}_\nu^{(k-1)} + \mathbf{N}_2\mathbf{u}_2 \le \mathbf{0},\ \mathbf{T}_2\mathbf{u}_2 = -\boldsymbol{\lambda}_\tau^{(k-1)}; \end{aligned}$$

(*Step 3*) $\{$*Residual deformation of* Ω^1.$\}$

$$\mathbf{A}_1\mathbf{w}_1^{(k)} = \frac{1}{2}\mathbf{B}_1^\top(\mathbf{B}_1(\mathbf{b}_1 - \mathbf{A}_1\mathbf{u}_1^{(k)}) - \mathbf{B}_2(\mathbf{b}_2 - \mathbf{A}_2\mathbf{u}_2^{(k)}));$$

(*Step 4*) $\{$*Residual deformation of* Ω^2.$\}$

$$\mathbf{A}_2\mathbf{w}_2^{(k)} = \frac{1}{2}\mathbf{B}_2^\top(\mathbf{B}_1(\mathbf{b}_1 - \mathbf{A}_1\mathbf{u}_1^{(k)}) - \mathbf{B}_2(\mathbf{b}_2 - \mathbf{A}_2\mathbf{u}_2^{(k)}));$$

(*Step 5*) $\{$*Relaxation of the contact displacements.*$\}$

$$\boldsymbol{\lambda}^{(k)} = \boldsymbol{\lambda}^{(k-1)} + \theta(\mathbf{B}_1\mathbf{w}_1^{(k)} + \mathbf{B}_2\mathbf{w}_2^{(k)}).$$

In *Step 3* and *Step 4*, we compute deformations of the bodies induced by the non-equilibria of contact stresses on Γ_c. These deformations vanish in the solution due to the transfer condition (2). Below, we will show that the dual formulation simplifies considerably the implementation of the algorithm.

The minimization in *Step 1* is equivalent to the saddle-point problem:

Find $(\mathbf{u}_1, \mathbf{s}_1) \in \mathbb{R}^{p_1} \times \Lambda(\mathbf{g})$ *such that*

$$\mathcal{L}_1(\mathbf{u}_1, \mathbf{s}_1) = \min_{\mathbf{v}_1 \in \mathbb{R}^{p_1}} \max_{\mathbf{r}_1 \in \Lambda(\mathbf{g})} \mathcal{L}_1(\mathbf{v}_1, \mathbf{r}_1) = \max_{\mathbf{r}_1 \in \Lambda(\mathbf{g})} \min_{\mathbf{v}_1 \in \mathbb{R}^{p_1}} \mathcal{L}_1(\mathbf{v}_1, \mathbf{r}_1),$$

where $\mathcal{L}_1 : \mathbb{R}^{p_1} \times \Lambda(\mathbf{g}) \mapsto \mathbb{R}$ is the Lagrangian defined by

$$\mathcal{L}_1(\mathbf{v}_1, \mathbf{r}_1) := \frac{1}{2}\mathbf{v}_1^\top\mathbf{A}_1\mathbf{v}_1 - \mathbf{v}_1^\top\mathbf{b}_1 + \mathbf{r}_1^\top(\mathbf{B}_1\mathbf{v}_1 - \boldsymbol{\lambda}^{(k-1)})$$

with $\Lambda(\mathbf{g}) := \{\mathbf{r}_1 = (\mathbf{r}_{1\nu}^\top, \mathbf{r}_{1\tau}^\top)^\top \in \mathbb{R}^{2q} : |\mathbf{r}_{1\tau}| \le \mathbf{g}\}$. Eliminating $\mathbf{u}_1$ from the max-min formulation we arrive at the quadratic programming problem:

$$\text{minimize } \frac{1}{2}\,\mathbf{s}_1^\top\mathbf{C}_1\mathbf{s}_1 - \mathbf{s}_1^\top\mathbf{h}_1 \quad \text{subject to } \mathbf{s}_1 \in \Lambda(\mathbf{g}), \tag{6}$$

where $\mathbf{C}_1 := \mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{B}_1^\top$ is symmetric, positive definite and $\mathbf{h}_1 := \mathbf{B}_1\mathbf{A}_1^{-1}\mathbf{b}_1 - \boldsymbol{\lambda}^{(k-1)}$. After computing $\mathbf{s}_1$ from (6) one can obtain $\mathbf{u}_1^{(k)}$ in *Step 1* by $\mathbf{u}_1^{(k)} = \mathbf{A}_1^{-1}(\mathbf{b}_1 - \mathbf{B}_1^\top\mathbf{s}_1)$.

The minimization problem in *Step 2* can be handled analogously. It is equivalent to the saddle-point problem:

Find $(\mathbf{u}_2, \mathbf{s}_2) \in \mathbb{R}^{p_2} \times \Lambda_+$ *such that*

$$\mathcal{L}_2(\mathbf{u}_2, \mathbf{s}_2) = \min_{\mathbf{v}_2 \in \mathbb{R}^{p_2}} \max_{\mathbf{r}_2 \in \Lambda_+} \mathcal{L}_2(\mathbf{v}_2, \mathbf{r}_2) = \max_{\mathbf{r}_2 \in \Lambda_+} \min_{\mathbf{v}_2 \in \mathbb{R}^{p_2}} \mathcal{L}_2(\mathbf{v}_2, \mathbf{r}_2),$$

where $\mathcal{L}_2 : \mathbb{R}^{p_2} \times \Lambda_+ \mapsto \mathbb{R}$ is the Lagrangian defined by

$$\mathcal{L}_2(\mathbf{v}_2, \mathbf{r}_2) := \frac{1}{2}\mathbf{v}_2^\top\mathbf{A}_2\mathbf{v}_2 - \mathbf{v}_2^\top\mathbf{b}_2 + \mathbf{r}_2^\top(\boldsymbol{\lambda}^{(k-1)} + \mathbf{B}_2\mathbf{v}_2)$$

and $\Lambda_+ := \{\mathbf{r}_2 = (\mathbf{r}_{2\nu}^\top, \mathbf{r}_{2\tau}^\top)^\top \in \mathbb{R}^{2q} : \mathbf{r}_{2\nu} \ge \mathbf{0}\}$. Analogously, this max–min problem leads to the quadratic programming problem:

$$\text{minimize} \frac{1}{2}\,\mathbf{s}_2^\top\mathbf{C}_2\mathbf{s}_2 - \mathbf{s}_2^\top\mathbf{h}_2 \quad \text{subject to } \mathbf{s}_2 \in \Lambda_+, \tag{7}$$

where $\mathbf{C}_2 := \mathbf{B}_2\mathbf{A}_2^{-1}\mathbf{B}_2^\top$ is again symmetric, positive definite and $\mathbf{h}_2 := \mathbf{B}_2\mathbf{A}_2^{-1}\mathbf{b}_2 - \mathbf{c} + \boldsymbol{\lambda}^{(k-1)}$. After solving (7) one can obtain $\mathbf{u}_2^{(k)}$ in *Step 2* as $\mathbf{u}_2^{(k)} = \mathbf{A}_2^{-1}(\mathbf{b}_2 - \mathbf{B}_2^\top \mathbf{s}_2)$.

As (6) and (7) are the minimization problems with strictly quadratic functions constrained by simple inequality bounds, it is appropriate to solve them by the conjugate gradient method combined with the projected gradient technique [4]. Since both problems are independent, one can solve them in parallel.

Step 3 and *Step 4* may be simplified. Let $\mathbf{s}_1^{(k)}$, $\mathbf{s}_2^{(k)}$ be the solutions to (6), (7), respectively, in the k-th step. Since $\mathbf{A}_\alpha\mathbf{u}_\alpha^{(k)} - \mathbf{b}_\alpha + \mathbf{B}_\alpha^\top\mathbf{s}_\alpha^{(k)} = \mathbf{0}$, we get $\mathbf{s}_\alpha^{(k)} = \mathbf{B}_\alpha(\mathbf{b}_\alpha - \mathbf{A}_\alpha\mathbf{u}_\alpha^{(k)})$. Using these results, we arrive at: $\mathbf{A}_\alpha\mathbf{w}_\alpha^{(k)} = \frac{1}{2}\mathbf{B}_\alpha^\top(\mathbf{s}_1^{(k)} - \mathbf{s}_2^{(k)})$, so that the computations of $\mathbf{u}_\alpha^{(k)}$, $\alpha = 1, 2$, can be omitted.

In the second algorithm we obtain the same structure as before, only *Step 1* and *Step 2* are different.

Algorithm 2 (different steps)

(*Step 1*) $\{$*Linear elasticity for* Ω^1.$\}$

$$\begin{aligned}\mathbf{u}_1^{(k)} := \operatorname{argmin}\ & \frac{1}{2}\mathbf{u}_1^\top\mathbf{A}_1\mathbf{u}_1 - \mathbf{u}_1^\top\mathbf{b}_1 \\ \text{subject to } & \mathbf{B}_1\mathbf{u}_1 = \boldsymbol{\lambda}^{(k-1)};\end{aligned}$$

(*Step 2*) $\{$*Unilateral contact with Tresca friction for* Ω^2.$\}$

$$\begin{aligned}\mathbf{u}_2^{(k)} := \operatorname{argmin}\ & \frac{1}{2}\mathbf{u}_2^\top\mathbf{A}_2\mathbf{u}_2 - \mathbf{u}_2^\top\mathbf{b}_2 + \sum_{i=1}^{q} g_i|\boldsymbol{\lambda}_\tau^{(k-1)} + \mathbf{T}_2\mathbf{u}_2|_i \\ \text{subject to } & \boldsymbol{\lambda}_\nu^{(k-1)} + \mathbf{N}_2\mathbf{u}_2 \le \mathbf{0};\end{aligned}$$

Let us denote the relative precision of the k-th iterative step of ALGORITHM 1,2 by

$$\varepsilon_\lambda^{(k)} := \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^{(k-1)}\| / \|\boldsymbol{\lambda}^{(k)}\|,$$

where $\|\cdot\|$ stands for the approximation of the $\mathbf{L}^2(\Gamma_c)$-norm. We terminate if $\varepsilon_\lambda^{(k)} \le$ *tol* for a prescribed tolerance *tol* > 0. In order to increase the efficiency of the algorithm, we initialize the inner iterative solvers in *Step 1* and *Step 2* by the respective results from the previous outer iterate, i.e., by $\mathbf{s}_1^{(k-1)}$ and $\mathbf{s}_2^{(k-1)}$, and we terminate them by an adaptive (inner) terminating tolerance $tol_{in}^{(k)} > 0$. The idea is to choose $tol_{in}^{(k)}$ in such a way that it respects the precision $\varepsilon_\lambda^{(k-1)}$ achieved in the outer loop: $tol_{in}^{(k)} := r_{tol} \times \varepsilon_\lambda^{(k-1)}$, where $0 < r_{tol} < 1$, $\varepsilon_\lambda^{(0)} := 1$.

4 Numerical Experiments

We consider two plane elastic bodies $\Omega^1 = (0,3) \times (1,2)$ and $\Omega^2 = (0,3) \times (0,1)$ made of an isotropic, homogeneous material characterized by Young modulus

21.19×10^{10} and Poisson ratio 0.277 (steel); see Fig.1.(a). The decompositions of Γ^1 and Γ^2 are as follows:

$$\Gamma_u^1 = \{0\} \times (1,2),\ \Gamma_c^1 = (0,3) \times \{1\},\ \Gamma_\ell^1 = \Gamma^1 \setminus \overline{\Gamma_u^1 \cup \Gamma_c^1},$$
$$\Gamma_u^2 = \{0\} \times (0,1),\ \Gamma_c^2 = (0,3) \times \{1\},\ \Gamma_\ell^2 = \Gamma^2 \setminus \overline{\Gamma_u^2 \cup \Gamma_c^2}.$$

The volume forces vanish for both bodies. The non-vanishing surface tractions $\boldsymbol{\phi}^1 = (\phi_1^1, \phi_2^1)$ act on Γ_ℓ^1 so that

$$\begin{aligned}
&\phi_1^1(x,2) = 0,\ \phi_2^1(x,2) = \phi_{2,L}^1 + \phi_{2,R}^1 x,\ x \in (0,3),\\
&\phi_1^1(3,y) = \phi_{1,B}^1(2-y) + \phi_{1,U}^1(y-1),\\
&\phi_2^1(3,y) = \phi_{2,B}^1(2-y) + \phi_{2,U}^1(y-1),\ \ y \in (1,2),
\end{aligned}$$

where $\phi_{2,L}^1 = -6e7$, $\phi_{2,R}^1 = -1e7$, $\phi_{1,B}^1 = 2e7$, $\phi_{1,U}^1 = 2e7$, $\phi_{2,B}^1 = 4e7$, and $\phi_{2,U}^1 = 2e7$. The slip bound is $g = 1.7e7$. Fig.1.(b–d) show results of the computations.

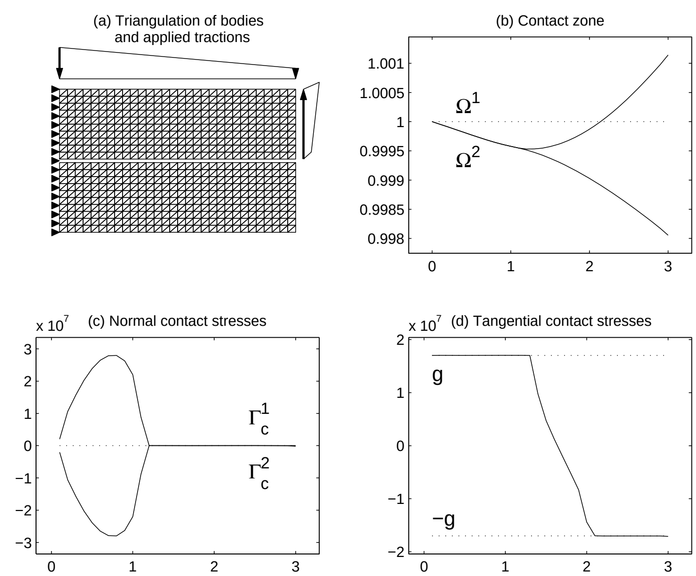

Fig. 1. Geometry and results.

In tables below we compare the performance of ALGORITHMS 1 and 2 for various values of θ and degrees of freedom p and q. We set $tol = 10^{-4}$, $r_{tol} = 0.1$ and we report the number of outer and inner iterations (*out*/*inn*). Since *inn* is proportional to computing time, it characterizes the total complexity of the algorithm. Here the symbol "-" means that the terminating tolerance is not achieved after the 100th iteration. The numerical experiments show higher efficiency of Algorithm 1 in which the non-linear conditions of non-penetration and friction are decoupled into *Step 1* and *Step 2*.

Table 1. Algorithm 1, *out/inn* for various θ.

p/q	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$
12672/384	69/835	36/537	26/473	–
19680/480	69/845	37/574	25/445	–
23760/528	70/805	37/585	25/469	–
28224/576	69/845	37/591	25/479	–
38304/672	69/890	36/598	26/490	–
49920/768	70/881	36/610	25/497	–

Table 2. Algorithm 2, *out/inn* for various θ.

p/q	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$
12672/384	86/959	47/571	91/921	–
19680/480	86/961	48/587	98/983	–
23760/528	86/961	47/587	99/1021	–
28224/576	87/991	47/600	–	–
38304/672	86/979	48/603	–	–
49920/768	87/1000	47/588	91/952	–

5 Conclusions and Comments

We have presented two different ways of decomposing unilateral contact problems with Tresca friction. According to the previous analysis, one can say that the variant with the decoupled non-penetration and friction conditions is more efficient. The theoretical proof of the convergence will be presented elsewhere. It is based on the Banach fixed point theorem applied to an appropriate mapping that is Lipschitzian and contractive in a suitable norm equivalent to the norm of the trace space $\mathbf{H}^{1/2}(\Gamma_c)$ (see [7] for the frictionless case).

The algorithm can be easily extend to the solution of problems with Coulomb friction as well as for 3D problems. In 3D, the inner minimization will be performed by the method of [11] that treats circular constraints arising from the friction law.

Acknowledgement. The third author acknowledges the support of the project MSM 6198910027 of the Ministry of Education of the Czech Republic and of the project GAČR 101/08/0574 of the Grant Agency of the Czech Republic.

References

1. G. Bayada, J. Sabil, and T. Sassi. Neumann-Dirichlet algorithm for unilateral contact problems: Convergence results. *C.R. Acad. Sci. Paris, Ser.I*, 335:381–386, 2002.
2. G. Bayada, J. Sabil, and T. Sassi. A Neumann–Neumann domain decomposition algorithm for the Signorini problem. *Appl. Math. Lett.*, 17:1153–1159, 2004.

3. Z. Dostál, T. Kozubek, P. Horyl, T. Brzobohatý, and A. Markopoulos. Scalable TFETI algorithm for two dimensional multibody contact problems with friction. *J. Comput. Appl. Math. (accepted)*, 2010.
4. Z. Dostál and J. Schőberl. Minimizing quadratic functions over non-negative cone with the rate of convergence and finite termination. *Comput. Optim. Appl.*, 30(1):23–44, 2005.
5. C. Eck and B.I. Wohlmuth. Convergence of a contact-Neumann iteration for the solution of two-body contact problems. *Math. Mod. Meth. Appl. Sci.*, 13(8):1103–1118, 2003.
6. J. Haslinger, I. Hlaváček, and J. Nečas. Numerical methods for unilateral problems in solid mechanics. In P.G. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis*, pp. 313–485. Elsevier Science, Amsterdam, 1996.
7. J. Haslinger, R. Kučera, and T. Sassi. A domain decomposition algorithm for contact problems: analysis and implementation. *Math. Model. Nat. Phenom.*, 1:123–146, 2009.
8. S. Hűeber, G. Stadler, and B.I. Wohlmuth. A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction. *SIAM J. Sci. Comput.*, 30(2):527–596, 2008.
9. M.A. Ipopa. *Domain Decomposition Algorithms for Contact Problems: Convergence and Numerical Nimulations*. PhD thesis, University of Caen, Caen 2008.
10. R. Krause. A non-smooth multiscale method for solving frictional two-body contact problems in 2D and 3D with multigrid efficiency. *SIAM J. Sci. Comput.*, 31(2):1399–1423, 2009.
11. R. Kučera. Convergence rate of an optimization algorithm for minimizing quadratic functions with separable convex constraints. *SIAM J. Optim.*, 19(2):846–862, 2008.