Multigrid Methods for Elliptic Obstacle Problems on 2D Bisection Grids

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1 Introduction

In this paper, we develop and analyze an efficient multigrid method to solve the finite element systems from elliptic obstacle problems on two dimensional adaptive meshes. Adaptive finite element methods (AFEMs) based on local mesh refinement are an important and efficient approach when the solution is non-smooth. An optimality theory on AFEM for linear elliptic equations can be found in [8]. To achieve optimal complexity, an efficient solver for the discretization is indispensable.

The classical projected successive over-relaxation method by [5] converges but the convergence rate degenerates quickly as the mesh size approaches zero. To speed up the convergence, different multigrid and domain decomposition techniques have been developed, see the monograph [7] and the recent review [6]. In particular, the constraint decomposition method by [10] is proved to be convergent linearly with a rate which is *almost* robust with respect to the mesh size in $\mathbb{R}^2$; but the result is restricted to uniformly refined grids.

We shall extend the algorithm and theoretical results by [10] to an important class of adaptive grids obtained by newest vertex bisections; thereafter we call them *bisection grids* for short. This is new according to [6]: the existing work assumes quasi-uniformity of the underlying meshes. Based on a decomposition of bisection grids due to [3], we present an efficient constraint decomposition method on bisection grids and prove an almost uniform convergence

$$\mathcal{J}(u^k) - \mathcal{J}(u^*) \le C\Big(1 - \frac{1}{1 + |\log h_{\text{min}}|^2}\Big)^k, \tag{1}$$

where $\mathcal{J}(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - fu)\,\mathrm{d}x$ is the objective energy functional, u^k is the k-th iteration and u^* is the exact solution of the constrained minimization problem, $h_{\text{min}} = \min_{\tau\in\mathcal{T}} \operatorname{diam}(\tau)$ and the grid $\mathcal{T}$ is obtained by bisections from a suitable initial triangulation $\mathcal{T}_0$.

2 Constraint Decomposition Methods

The subspace correction framework [14] has been extended to nonlinear convex minimization problems [12]. This technique has also been applied to develop domain decomposition and multigrid methods for obstacle problems in [1, 11]. Furthermore, a constraint decomposition method (CDM) was introduced and proved to have a contraction factor which is almost independent of mesh size [10]. In this section, we briefly review the CDM for obstacle problems.

Let $\mathbb{V} \subset H_0^1$ be a finite dimensional Hilbert space and $\mathcal{J} : \mathbb{K} \to \mathbb{R}$ be a convex functional defined over the convex set $\mathbb{K} \subset \mathbb{V}$. We consider the energy minimization problem

$$\min_{v \in \mathbb{K}} \mathcal{J}(v). \tag{2}$$

In this paper, for simplicity, we only consider the case

$$\mathcal{J}(u) := \int_{\Omega} (\frac{1}{2}|\nabla u|^2 - fu)\,\mathrm{d}x \quad \text{and} \quad \mathbb{K} := \{v \in \mathbb{V} \,|\, v \geq 0\}, \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\mathcal{T}$ is a conforming triangulation of Ω, $\mathbb{V} = \mathbb{V}(\mathcal{T})$ is the continuous piecewise linear finite element space over $\mathcal{T}$. Let $\|\cdot\|$ be the norm associated to the energy $\mathcal{J}$. For our choice of $\mathcal{J}$ in (3), the energy norm is $\|u\| = \|\nabla u\|$. The algorithm discussed in this paper can be generalized to problems with more general energies and obstacles.

We decompose the space $\mathbb{V}$ into a sum of subspaces $\mathbb{V}_i \subset \mathbb{V}, i = 1, \ldots, m$:

$$\mathbb{V} = \mathbb{V}_1 + \cdots + \mathbb{V}_m = \textstyle\sum_{i=1}^m \mathbb{V}_i, \tag{4}$$

and further decompose the convex set $\mathbb{K}$ as follows

$$\mathbb{K} = \mathbb{K}_1 + \cdots + \mathbb{K}_m = \textstyle\sum_{i=1}^m \mathbb{K}_i \quad \text{with} \quad \mathbb{K}_i \subset \mathbb{V}_i \ (i = 1, \ldots, m), \tag{5}$$

where $\mathbb{K}_i$ are convex and closed in $\mathbb{V}_i$. Then we have the following abstract algorithm of successive subspace correction type.

Algorithm 1 (CDM) Given an initial guess $u^0 \in \mathbb{K}$.
For $k = 0, 1, \ldots,$ till convergence
  Decompose $u^k = \sum_{i=1}^m u_i$, such that $u_i \in \mathbb{K}_i$; and let $w^0 = u^k$.
  For $i = 1 : m$
    $w^i = w^{i-1} + \text{argmin}_{d_i} \left\{ \mathcal{J}(w^{i-1} + d_i) \,|\, d_i \in \mathbb{V}_i \text{ and } u_i + d_i \in \mathbb{K}_i \right\}$.
  End For
  Let $u^{k+1} = w^m$.
End For

It is clear that each iteration w^i $(i = 1, \ldots, m)$ stays in the feasible set $\mathbb{K}$ due to (5). A linear convergence rate of Algorithm 1 has been established in [10] under the following assumptions:

Assumption 1 (Assumptions on Decomposition) (i) *Nonlinear Stability*: For any $u, v \in \mathbb{K}$, there exist a constant $C_1 > 0$ and decompositions $u = \sum_{i=1}^m u_i, v = \sum_{i=1}^m v_i$ with $u_i, v_i \in \mathbb{K}_i$ such that

$$\left(\sum_{i=1}^m \| u_i - v_i \|^2 \right)^{\frac{1}{2}} \leq C_1 \| u - v \| ;$$

(ii) *Nonlinear Strengthened Cauchy–Schwarz*: There exists $C_2 > 0$ such that

$$\sum_{i,j=1}^m | \langle \mathcal{J}'(w_{ij} + v_i) - \mathcal{J}'(w_{ij}), \widetilde{v}_j \rangle | \leq C_2 \left(\sum_{i=1}^m \| v_i \|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^m \| \widetilde{v}_j \|^2 \right)^{\frac{1}{2}},$$

for any $w_{ij} \in \mathbb{V}$, $v_i \in \mathbb{V}_i$, and $\widetilde{v}_j \in \mathbb{V}_j$.

Theorem 1 (Convergence Rate of CDM) *If Assumption 1 is satisfied, then Algorithm 1 converges linearly and*

$$\frac{\mathcal{J}(w) - \mathcal{J}(u^*)}{\mathcal{J}(u) - \mathcal{J}(u^*)} \leq 1 - \frac{1}{(\sqrt{1 + C_0} + \sqrt{C_0})^2}, \tag{6}$$

where u^* is the solution of (2) and $C_0 = 2C_2 + C_1^2 C_2^2$.

3 A Constraint Decomposition on Bisection Grids

In this section, we construct subspace decompositions of the linear finite element space $\mathbb{V}$, as well as a constraint decomposition of $\mathbb{K}$, on a bisection grid $\mathcal{T}$. Our new algorithm is based on a decomposition of bisection grids introduced in [3]; see also [13].

For each triangle $\tau \in \mathcal{T}$, we label one vertex of τ as the *newest vertex* and call it $V(\tau)$. The opposite edge of $V(\tau)$ is called the *refinement edge* and denoted by $E(\tau)$. This process is called *labeling* of $\mathcal{T}$. Given a labeled initial grid $\mathcal{T}_0$, newest vertex bisection follows two rules:

(i) a triangle (*father*) is bisected to obtain two new triangles (*children*) by connecting its newest vertex with the midpoint of its refinement edge;
(ii) the new vertex created at the midpoint of the refinement edge is labeled as the newest vertex of each child.

Therefore, refined grids $\mathcal{T}$ from a labeled initial grid $\mathcal{T}_0$ inherit labels according to the second rule and the bisection process can thus proceed. We define $\mathbb{C}(\mathcal{T}_0)$ as the set of conforming triangulations obtained from $\mathcal{T}_0$ by newest vertex bisection(s). It can be easily shown that all the descendants of a triangle in $\mathcal{T}_0$ fall into four similarity classes and hence any triangulation $\mathcal{T} \in \mathbb{C}(\mathcal{T}_0)$ is shape-regular.

Let $\mathcal{T}$ be a labeled conforming mesh. Two triangles sharing a common edge are called *neighbors* to each other. A triangle τ has at most three neighbors. The neighbor sharing the refinement edge of τ is called the *refinement neighbor* and denoted by

$F(\tau)$. Note that $F(\tau) = \varnothing$ if $E(\tau)$ is on the boundary of Ω. Although $E(\tau) \subset F(\tau)$, the refinement edge of $F(\tau)$ could be different than $E(\tau)$. An element τ is called *compatible* if $F(F(\tau)) = \tau$ or $F(\tau) = \varnothing$. We call a grid $\mathcal{T}$ *compatibly labeled* if every element in $\mathcal{T}$ is compatible and call such a labeling of $\mathcal{T}$ a *compatible labeling*.

For a compatible element τ, its refinement edge e is called a *compatible edge*, and $\omega_e = \tau \cup F(\tau)$ is called a *compatible patch*. By this definition, if e is a compatible edge, ω_e is either a pair of two triangles sharing the same refinement edge e or one triangle whose refinement edge e is on the boundary. In both cases, bisection of triangles in ω_e preserves mesh conformity; we call such a bisection a *compatible bisection*. Mathematically, we define the compatible bisection as a map $b_e : \omega_e \rightarrow \omega_p$, where ω_p consists of all triangles sharing the new point p introduced in the bisection. We then define the addition $\mathcal{T} + b_e := (\mathcal{T} \backslash \omega_e) \cup \omega_p$. For a sequence of compatible bisections $\mathcal{B} = (b_1, b_2, \cdots, b_m)$, we define

$$\mathcal{T} + \mathcal{B} := ((\mathcal{T} + b_1) + b_2) + \cdots + b_m,$$

whenever the addition is well defined.

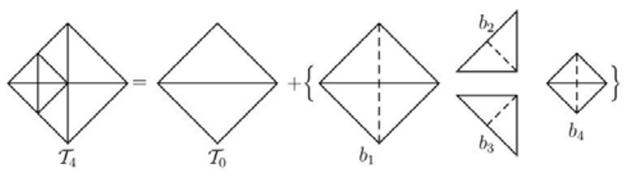

Fig. 1. A decomposition of a bisection grid.

Theorem 2 (Decomposition of Bisection Grids) *If $\mathcal{T}_0$ is conforming and compatibly labeled, then for any $\mathcal{T} \in \mathbb{C}(\mathcal{T}_0)$, there exists a compatible bisection sequence $\mathcal{B} = (b_1, b_2, \cdots, b_m)$, such that*

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{B}. \tag{7}$$

Remark 1. We only give a pictorial demonstration in Fig. 1 to illustrate the decomposition. For the proof of Theorem 2, we refer to [3, 13]. A practical decomposition algorithm has been developed and implemented in [4]. □

Throughout this paper, we will assume that $\mathcal{T} \in \mathbb{C}(\mathcal{T}_0)$ has been decomposed as in (7). We denote the intermediate grids by

$$\mathcal{T}_i := ((\mathcal{T}_0 + b_1) + b_2) \cdots + b_i \qquad i = 1, \ldots, m,$$

and observe that $\mathcal{T}_i \in \mathbb{C}(\mathcal{T}_0)$. Let $\mathcal{P}(\mathcal{T}_i)$ denote the set of interior vertices of the triangulation $\mathcal{T}_i$. Denote by $\psi_{i,p} \in \mathbb{V}(\mathcal{T}_i)$ the nodal basis function associated with a node $p \in \mathcal{P}(\mathcal{T}_i)$ and by $\omega_{i,p}$ the local patch (i.e. the support of $\psi_{i,p}$). The subspace corresponding to the compatible bisection b_i, which introduces the new vertex $p_i \in \mathcal{P}(\mathcal{T}_i)$, can be written as $\mathbb{V}_i := \text{span}\{\psi_{i,p}, p \in \mathcal{P}(\mathcal{T}_i) \cap \omega_{i,p_i}\}$. To enforce the homogenous Dirichlet boundary condition, we simply set $\mathbb{V}_i = \varnothing$ if p_i is a vertex on the boundary. Let $\mathbb{V}_0 = \mathbb{V}(\mathcal{T}_0)$ be the linear space corresponding to the initial mesh $\mathcal{T}_0$. Then we have a space decomposition $\mathbb{V} = \sum_{i=0}^m \mathbb{V}_i$.

Based on this space decomposition, there are infinitely many possibilities to decompose the feasible set $\mathbb{K}$. We do not consider the optimal way to choose such a constraint decomposition. We simply choose

$$\mathbb{K} = \sum_{i=0}^m \mathbb{K}_i \quad \text{with} \quad \mathbb{K}_i := \{v \in \mathbb{V}_i \,|\, v \geq 0\}, \tag{8}$$

and focus on how to decompose $u \in \mathbb{V}$ at each iteration in Algorithm 1. Let $\mathbb{W}_j = \sum_{i=0}^j \mathbb{V}_i$, $j = 1, \ldots, m$. For $i = m, m-1, \ldots, 1$, we first define $I_i^{i-1} : \mathbb{W}_i \to \mathbb{W}_{i-1}$ such that

$$I_i^{i-1} v(p) = \begin{cases} \min\{v(p), v(p_i)\}, & \text{if } p \in \mathcal{P}(\mathcal{T}_{i-1}) \cap \omega_{i,p_i} \\ v(p), & \text{if } p \in \mathcal{P}(\mathcal{T}_{i-1}) \backslash \omega_{i,p_i}. \end{cases}$$

We then define $Q_i : \mathbb{V} \to \mathbb{W}_{i-1}$ to be $Q_i := I_i^{i-1} I_{i+1}^i \cdots I_m^{m-1}$. Notice that Q_i's are nonlinear operators, i.e. $Q_i u - Q_i v \neq Q_i(u - v)$. Finally we define a decomposition $u = \sum_{i=0}^m u_i$, with

$$u_m := u - Q_m u, \quad u_i := Q_{i+1} u - Q_i u \ (i = m-1, \ldots, 1), \quad u_0 = Q_1 u. \tag{9}$$

Comparing these with the definitions of $\mathbb{V}_i$ and $\mathbb{K}_i$, we can easily see that $u_i \in \mathbb{K}_i$, for $i = 0, 1, \ldots, m$.

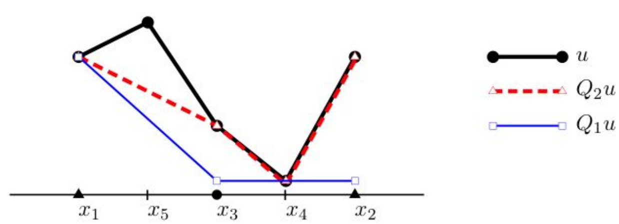

Fig. 2. A one-dimensional example for the decomposition of u. Suppose the initial grid $\mathcal{T}_0 = \{(x_1, x_3), (x_3, x_2)\}$. And the final grid $\mathcal{T}$ can be viewed as $\mathcal{T}_0 + b_1 + b_2$ where b_1 bisects the element (x_3, x_2) and introduces x_4 and b_2 bisects (x_1, x_2) and introduces x_5. As we discussed above $\mathcal{T}_1 = \mathcal{T}_0 + b_1$ and $\mathcal{T} = \mathcal{T}_2 = \mathcal{T}_1 + b_2$. From the definition of Q_i, we can easily obtain a decomposition of u.

Now we prove the convergence rate of the proposed algorithm.

Lemma 1 (Stability of Q_i). *Let $u, v \in \mathbb{V}$. For $i = 0, 1, \ldots, m$ and any element $\tau \in \mathcal{T}_i$, we have*

$$h_\tau^{-1}\|Q_{i+1}u - Q_{i+1}v\|_{L^2(\tau)} \le C\big(1 + |\log(h_\tau/h_{\min})|\big)^{\frac{1}{2}}\|u - v\|_{H^1(\omega_{i,\tau})},$$

where C is a generic constant independent of the meshsize.

Proof. From the definition of Q_i, for any $u, v \in \mathbb{V}$, we have that

$$\|Q_{i+1}u - Q_{i+1}v\|_{L^2(\tau)} \le C \sum_{p\in\mathcal{P}(\mathcal{T}_i)\cap\tau} \|u-v\|_{L^\infty(\omega_{i,p})}|\tau|^{\frac{1}{2}} \le Ch_\tau\|u-v\|_{L^\infty(\omega_{i,\tau})}.$$

The result then follows directly from the discrete Sobolev inequality between L^∞ and H^1 in two dimensions; see [2]. □

We introduce the *generation* of elements and compatible bisections. The generation of each element in the initial grid $\mathcal{T}_0$ is defined to be 0, and the generation of a child is 1 plus that of the father. In [13] we proved that all triangles in a compatible patch ω_e are of the same generation, which can be used to define the generation, gen$(\cdot)$, for a compatible bisection b_e and the corresponding new vertex. For two different compatible bisections, b_{e_1} and b_{e_2}, of the same generation, their patches are disjointed, i.e., $\omega_{e_1} \cap \omega_{e_2} = \varnothing$.

Lemma 2 (Stable Decomposition). *For any $u, v \in \mathbb{K}$, the decompositions $u = \sum_{i=0}^m u_i, v = \sum_{i=0}^m v_i$ given by (9) satisfy*

$$\left(\sum_{i=0}^m \|u_i - v_i\|^2\right)^{\frac{1}{2}} \le C\big(1 + |\log h_{\min}|\big)\,\|u - v\|\,;$$

Proof. First note that the support of $\mathbb{V}_i$ is restricted to the extended patch $\widetilde{\omega}_{i,p_i} := \cup_{x\in\omega_{i,p_i}}\omega_{i,x}$. Using an inverse inequality and stability of Q_i, we have

$$\|u_i - v_i\|^2_{\widetilde{\omega}_{i,p_i}} \le C\|h_\tau^{-1}(u_i - v_i)\|^2_{L^2(\widetilde{\omega}_{i,p_i})} \le C(1 + |\log h_{\min}|)\,\|u - v\|^2_{\widetilde{\omega}_{i,p_i}}.$$

For bisections with the same generation k, the extended patches, $\widetilde{\omega}_{i,p_i}$, have finite overlapping and $\cup_{p,\mathrm{gen}(p)=k}\,\widetilde{\omega}_{i,p_i} \le C|\Omega|$. Let $L = \max_{\tau\in\mathcal{T}}\mathrm{gen}(\tau)$. Then

$$\sum_{i=1}^m \|u_i - v_i\|^2_{\widetilde{\omega}_{i,p_i}} = \sum_{k=1}^L \sum_{p_i,\mathrm{gen}(p_i)=k} \|u_i - v_i\|^2_{\widetilde{\omega}_{i,p_i}} \le CL(1+|\log h_{\min}|)\,\|u - v\|^2_\Omega\,.$$

The result then follows from the observation that $L \le C|\log h_{\min}|$. □

The proof of the following Strengthened Cauchy–Schwarz (SCS) inequality can be found in [13]. The idea of the proof is to apply standard SCS for each compatible decomposition and then rearrange the sum by generations.

Lemma 3 (Strengthened Cauchy Schwarz Inequality). *For any $u_i, v_i \in \mathbb{V}_i, i = 0, \ldots, m$, we have*

$$\Big|\sum_{i=0}^m\sum_{j=0}^m(\nabla u_i, \nabla v_j)\Big| \le C\Big(\sum_{i=0}^m |u_i|_1^2\Big)^{1/2}\Big(\sum_{i=1}^m |v_i|_1^2\Big)^{1/2}. \tag{10}$$

Applying the abstract theory (Theorem 1) and Lemma 2 and Lemma 3, we get the following rate of convergence.

Theorem 3 (Convergence Rate) *Let u^k be the k-th iteration of Algorithm 1 with the decomposition (9). We then have the following convergence rate*

$$\mathcal{J}(u^k) - \mathcal{J}(u^*) \leq C\Big(1 - \frac{1}{1 + |\log h_{\min}|^2}\Big)^k. \tag{11}$$

4 Numerical Experiments

In this section, we use a numerical example by [10] to test the proposed algorithm: Let $\Omega = (-2, 2)^2$, $f = 0$ and the obstacle $\chi(x) = \sqrt{1 - |x|^2}$ if $|x| \leq 1$ and -1, otherwise. In this case, the exact solution is known to be

$$u_*(x) = \begin{cases} \sqrt{1 - |x|^2} & \text{if } |x| \leq r_* \\ -r_*^2 \ln(|x|/2)\sqrt{1 - r_*^2} & \text{otherwise,} \end{cases}$$

where $r_* \approx 0.6979651482$. We give the Dirichlet boundary condition according to the exact solution above.

Table 1. The reduction factors for the CDM algorithm on adaptively refined meshes. The reduction factor is the ratio of energy error between two consecutive iterations.

Adaptive mesh	Degrees of freedom	$h_{\min}$	Reduction factor
1	719	1.563e-2	0.508
2	1,199	1.105e-2	0.599
3	2,107	7.813e-3	0.660
4	3,662	5.524e-3	0.651
5	6,560	3.901e-3	0.691
6	1,1841	2.762e-3	0.701

The contraction factors are computed and reported in Table 1 for a sequence of adaptive meshes, where the adaptive mesh refinement is driven by a posteriori error estimators starting from a uniform initial mesh; such adaptive algorithms and estimators can be found in [9] for example. The linear convergence rate is confirmed by our numerical experiments and the reduction rate is evaluated when the convergence becomes linear; there is a superlinear region in the beginning.

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