Domain Decomposition Solvers for Frequency-Domain Finite Element Equations

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Summary. The paper is devoted to fast iterative solvers for frequency-domain finite element equations approximating linear and nonlinear parabolic initial boundary value problems with time-harmonic excitations. Switching from the time domain to the frequency domain allows us to replace the expensive time-integration procedure by the solution of a simple linear elliptic system for the amplitudes belonging to the sine- and to the cosine-excitation or a large nonlinear elliptic system for the Fourier coefficients in the linear and nonlinear case, respectively. The fast solution of the corresponding linear and nonlinear system of finite element equations is crucial for the competitiveness of this method.

1 Introduction

In many practical applications, for instance, in electromagnetics and mechanics, the excitation is time-harmonic. Switching from the time domain to the frequency domain allows us to replace the expensive time-integration procedure by the solution of a simple elliptic system for the amplitudes. This is true for linear problems, but not for nonlinear problems. However, due to the periodicity of the solution, we can expand the solution in a Fourier series. Truncating this Fourier series and approximating the Fourier coefficients by finite elements, we arrive at a large-scale coupled nonlinear system for determining the finite element approximation to the Fourier coefficients. In the literature, this approach is called multiharmonic FEM or harmonic-balanced FEM, and has been used by many engineers in different applications. see, e.g. [1] and the references therein.

Reference [2] provided the first rigorous numerical analysis for the eddy current problem. The practical aspects of the multiharmonic approach, including the construction of a fast multigrid preconditioned QMR solver for the Jacobi system arising in every Newton step and the implementation in an adaptive multilevel setting, are discussed in [3] by the same authors. There was no rigorous analysis of the multigrid preconditioned QMR solver, but the numerical results presented in this paper for academic and more practical problems indicated the efficiency of this solver.

The construction of fast solvers for such systems is very crucial for the overall efficiency of this multiharmonic approach. In this paper, we look at linear and nonlinear, time-harmonic potential problems. We construct and analyze an almost optimal preconditioned GMRes solver for the Jacobi systems arising from the Newton linearization of the large-scale coupled nonlinear system. This preconditioner is not robust with respect to the excitation frequency. In the linear case we are able to construct a robust preconditioner used in a MinRes solver. The multiharmonic approach is presented in Sect. 2, whereas the two different preconditioners and solvers are discussed in Sects. 3 and 4.

2 Frequency-Domain Finite Element Equations

Let us consider the following nonlinear, parabolic, scalar potential equation with a homogeneous Dirichlet boundary condition and an inhomogeneous initial condition as our model problem:

$$\begin{cases} \alpha\frac{\partial u}{\partial t} - \nabla\cdot(\nu(|\nabla u|)\nabla u) = f & \text{in } \Omega\times(0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{for } \mathbf{x}\in\overline{\Omega}, \\ u(\mathbf{x},t) = 0 & \text{for } (\mathbf{x},t)\in\partial\Omega\times[0,T], \end{cases} \tag{1}$$

where the right-hand side $f(\cdot,\cdot)$ is given by a time-harmonic excitation with the frequency ω, i.e.

$$f(\mathbf{x},t) = f^c(\mathbf{x})\cos(\omega t) + f^s(\mathbf{x})\sin(\omega t). \tag{2}$$

We assume that $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain, α is a given uniformly positive function in $L_\infty(\Omega)$, and $\nu : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuously differentiable function satisfying the properties

$$0 < \nu_{\min} \le \nu(s) \le \nu_{\max} \qquad \text{for } s \ge 0, \tag{3}$$

$$\text{and} \quad s\mapsto s\nu(s) \quad \text{is Lipschitz and strongly monotone for } s\ge 0. \tag{4}$$

These conditions ensure that there exists at least a unique weak solution to the initial boundary value problem (1), see [14]. In the linear case where the coefficient ν is independent of $|\nabla u|$, the solution $u(\mathbf{x},t) = u^c(\mathbf{x})\cos(\omega t) + u^s(\mathbf{x})\sin(\omega t)$ is time-harmonic as well, and we get an elliptic boundary value problem for defining the unknown amplitudes u^c and u^s which only depend on the spatial variable $\mathbf{x}$. This is not true in the nonlinear case. However, the solution u to (1) is still periodic in time, with frequency ω. Thus, we have the Fourier series representation

$$u(\mathbf{x},t) = \sum_{k=0}^{\infty} u_k^c(\mathbf{x})\cos(k\omega t) + u_k^s(\mathbf{x})\sin(k\omega t),$$

where the Fourier coefficients are given by

$$u_k^c(\mathbf{x}) = \frac{2}{T}\int_0^T u(\mathbf{x},t)\cos(k\omega t)\,dt \quad \text{and} \quad u_k^s(\mathbf{x}) = \frac{2}{T}\int_0^T u(\mathbf{x},t)\sin(k\omega t)\,dt.$$

Here, the period is $T = 2\pi/\omega$. Similarly, the potential

$$\Psi[u](\mathbf{x},t) := \nu(|\nabla u|)\nabla u(\mathbf{x},t)$$

can be expressed as a Fourier series

$$\Psi[u](\mathbf{x},t) = \sum_{k=0}^{\infty} \Psi_k^c[u](\mathbf{x})\cos(k\omega t) + \Psi_k^s[u](\mathbf{x})\sin(k\omega t)$$

with vector-valued Fourier coefficients Ψ_k^c and Ψ_k^s. Approximating u and Ψ by the truncated series

$$u(\mathbf{x},t) \approx \tilde{u}(\mathbf{x},t) := \sum_{k=0}^{N} u_k^c(\mathbf{x})\cos(k\omega t) + u_k^s(\mathbf{x})\sin(k\omega t) \tag{5}$$

and

$$\Psi[u](\mathbf{x},t) \approx \tilde{\Psi}[\tilde{u}](\mathbf{x},t) := \sum_{k=0}^{N} \Psi_k^c[\tilde{u}](\mathbf{x})\cos(k\omega t) + \Psi_k^s[\tilde{u}](\mathbf{x})\sin(k\omega t)$$

yields the following system of nonlinear equations for the Fourier coefficients:

$$\alpha\omega \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & N \\ & & & -N & 0 \end{pmatrix} \begin{pmatrix} u_1^c \\ u_1^s \\ \vdots \\ u_N^c \\ u_N^s \end{pmatrix} - \nabla\cdot \begin{pmatrix} \Psi_1^c[\tilde{u}] \\ \Psi_1^s[\tilde{u}] \\ \vdots \\ \Psi_N^c[\tilde{u}] \\ \Psi_N^s[\tilde{u}] \end{pmatrix} = \begin{pmatrix} f^c \\ f^s \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

Throughout this paper, we denote by $\mathbf{u} := (u_1^c, u_1^s, \ldots, u_N^c, u_N^s)^T$ the vector of $2N$ Fourier coefficients and by $\tilde{u}$ the approximation to u given by the finite series (5). We shall solve a variational problem for $\mathbf{u}$ in $H_0^1(\Omega)^{2N} := (H_0^1(\Omega))^{2N}$, where $H_0^1(\Omega)$ is the Sobolev space of order 1 on Ω, with vanishing trace on the boundary of Ω. Note that the Fourier coefficients corresponding to $k = 0$ need not be solved for due to the initial condition, cf. [4].

The finite element approximation to (6) leads to a large nonlinear system of finite element equations of the form

$$\mathbf{F_h}(\underline{\mathbf{u}}_\mathbf{h}) = \underline{\mathbf{f}}_\mathbf{h} \tag{7}$$

for determining the finite element solution

$$\mathbf{S}_h^1 := \left(\mathsf{span}\{\phi_j\}_{j=1}^{N_h}\right)^{2N} \ni \tilde{\mathbf{u}}_h \leftrightarrow \underline{\mathbf{u}}_\mathbf{h} = (\underline{u}_{1,h}^c, \underline{u}_{1,h}^s, \ldots, \underline{u}_{N,h}^c, \underline{u}_{N,h}^s)^T \in \mathbb{R}^{2N\cdot N_h}$$

to the Fourier coefficients $\mathbf{u} \in H_0^1(\Omega)^{2N}$. Here, ϕ_j are piecewise linear basis functions in $H_0^1(\Omega)$. Thus the multiharmonic approach yields a time-independent nonlinear system for the solution of which highly parallel solvers can be constructed.

Following [2] we can show that under standard regularity assumptions, the discretization error behaves like $O(h + N^{-1})$ with respect to the $L_2((0,T), H^1(\Omega))$ norm.

Solving (7) by Newton's method ($\tau_n = 1$)

$$\underline{\mathbf{u}}_{\mathbf{h}}^{n+1} = \underline{\mathbf{u}}_{\mathbf{h}}^{n} + \tau_n \underline{\mathbf{w}}_{\mathbf{h}}^{n} = \underline{\mathbf{u}}_{\mathbf{h}}^{n} + \tau_n \mathbf{F}'_{\mathbf{h}}(\underline{\mathbf{u}}_{\mathbf{h}}^{n})^{-1}(\underline{\mathbf{f}}_{\mathbf{h}} - \mathbf{F}_{\mathbf{h}}(\underline{\mathbf{u}}_{\mathbf{h}}^{n})), \tag{8}$$

we have to solve the large-scale linear system

$$\mathbf{F}'_{\mathbf{h}}(\underline{\mathbf{u}}_{\mathbf{h}}^{n})\underline{\mathbf{w}}_{\mathbf{h}}^{n} = \underline{\mathbf{r}}_{\mathbf{h}}^{n} := \underline{\mathbf{f}}_{\mathbf{h}} - \mathbf{F}_{\mathbf{h}}(\underline{\mathbf{u}}_{\mathbf{h}}^{n}), \tag{9}$$

with the Jacobi matrix ${\mathbf{F}_{\mathbf{h}}}'(\underline{\mathbf{u}}_{\mathbf{h}}^{n})$ as system matrix and the residual $\underline{\mathbf{r}}_{\mathbf{h}}^{n}$ as right-hand side.

Reference [4] show that the Jacobi-systems (9) can successfully be solved by the preconditioned GMRes method using a special domain decomposition preconditioner. We will explain the construction of this preconditioner for the corresponding linear problem in the next section, but the results remain valid for the Jacobi-systems (9) too.

In the remainder of this paper, we discuss preconditioned iterative methods for solving linear systems of the form

$$\begin{pmatrix} K_h & \sigma M_h \\ -\sigma M_h & K_h \end{pmatrix} \begin{pmatrix} \underline{u}_h^c \\ \underline{u}_h^s \end{pmatrix} = \begin{pmatrix} \underline{f}_h^c \\ \underline{f}_h^s \end{pmatrix}, \tag{10}$$

arizing from the time-domain finite element discretization of the initial-boundary value problem (1) with the time-harmonic excitation (2) in the linear case where the coefficient ν is independent of $|\nabla u|$. The coefficient σ is equal to $\alpha\omega$. Here and in the following, we assume that α is a positive constant. The stiffness matrix K_h and the mass matrix M_h are computed from the bilinear forms

$$\int_\Omega \nu(\mathbf{x})\nabla\phi(\mathbf{x}) \cdot \nabla\psi(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad \int_\Omega \phi(\mathbf{x})\psi(\mathbf{x})\, d\mathbf{x},$$

respectively. The system matrix $\mathbf{D_h}$ in (10) is obviously positive definite and non-symmetric (block skew-symmetric).

3 Domain Decomposition Solver

Following [13], we propose a non-symmetric two-level Schwarz preconditioner for (10) of the form

$$\mathbf{C_h^{-1}} = \mathbf{I_H^h D_H^{-1} I_h^H} + \beta \mathbf{B_h^{-1}}, \tag{11}$$

where $\mathbf{D_H}$ is a coarse grid version of $\mathbf{D_h}$, $\mathbf{I_h^H}$ and $\mathbf{I_H^h}$ are appropriate restriction and prolongation operators, $\mathbf{B_h}$ is a symmetric positive definite (SPD) preconditioner for the SPD part $\mathbf{A_h} = \text{blockdiag}(K_h, K_h)$ of $\mathbf{D_h}$, and β is a positive scaling constant. [5] proposed a wire-basket-based domain decomposition method that gives an effective preconditioner $\mathbf{B_h}$ for the symmetric positive definite matrix $\mathbf{A_h}$, with a condition number estimate which is independent of jumps in the coefficient and depends only polylogarithmically on H/h, see also [11]. Using this wire-basket domain decomposition preconditioner $\mathbf{B_h}$ in (11), we arrive at the following convergence estimate for the GMRes preconditioned by the Xu-Cai preconditioner (11):

Theorem 1 *Assume that the adjoint linear problem is $H^{1+s}(\Omega)^2$-coercive with some $s \in (0,1]$, and H is sufficiently small, specifically $H^s < c(1+\log(H/h))^{-2}$. Then the GMRes method preconditioned by the preconditioner (11) with the wire-basket component $\mathbf{B_h}$ converges and the convergence estimate*

$$\|\underline{\mathbf{r}}_h^m\|_{\mathbf{A_h}} \leq \left(1 - c\, c_{\log}^{-4}\left(1+c_{\log}^2\right)^{-2}\right)^{m/2} \|\underline{\mathbf{r}}_h^0\|_{\mathbf{A_h}} := \gamma(H/h)^{m/2}\|\underline{\mathbf{r}}_h^0\|_{\mathbf{A_h}}$$

holds for the preconditioned residual $\underline{\mathbf{r}}_h^m = \mathbf{C_h^{-1}}(\underline{\mathbf{f}}_h - \mathbf{D_h}\underline{\mathbf{u}}_h^m)$ at the m-th iteration, where $c_{\log} := 1+\log(H/h)$, $0 < \gamma(H/h) < 1$, and the constant c depends on ν and σ, but not on H and h.

The proof of this theorem can be found in [4]. In the same paper we present our numerical results which show that our preconditioned GMRes method is a quite efficient solver for the linear system (10) and can efficiently be used for solving the Jacobi-systems (9) as well. The number of iterations depends only polylogarithmically on H/h. In order to clarify the dependence on σ, [7] performed a Fourier analysis of the preconditioned matrix $\mathbf{C_h^{-1}D_h}$ for the corresponding one-dimensional problem with constant ν, where the exact SPD part $\mathbf{A_h}$ was used as $\mathbf{B_h}$, and $H = 2h$. This analysis shows that this preconditioner is not robust with respect to σ, see also the second line of Table 2. In the next section we present a robust preconditioner for the linear system (10) in an equivalent symmetric, but indefinite setting.

4 A Symmetric and Indefinite Reformulation

The non-symmetric and positive definite system (10) can be reformulated in the following equivalent form

$$\begin{pmatrix} M_h & K_h \\ K_h & -\sigma^2 M_h \end{pmatrix} \begin{pmatrix} \underline{u}_h^s \\ \frac{1}{\sigma}\underline{u}_h^c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma}\underline{f}_h^c \\ \underline{f}_h^s \end{pmatrix} \tag{12}$$

with a symmetric but indefinite system matrix $\mathbf{D_h}$. For simplicity, we use the same notation $\mathbf{D_h}$ for the system matrix in (10) and (12). It follows from [10] that the block-diagonal preconditioner

$$\mathbf{C_h} = \frac{1}{\sigma}\begin{pmatrix} \sigma M_h + K_h & 0 \\ 0 & \sigma^2(\sigma M_h + K_h) \end{pmatrix} \tag{13}$$

is robust with respect to both the discretization parameter h and the bad parameter σ. More precisely, the condition number

$$\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) = \|\mathbf{C_h}^{-1}\mathbf{D_h}\|_{\mathbf{C_h}} \|\mathbf{D_h}^{-1}\mathbf{C_h}\|_{\mathbf{C_h}} = |\lambda_{2N_h}| \,/\, |\lambda_1| \leq c = const \quad (14)$$

can be estimated by a positive constant c that is independent of both h and σ, where the eigenvalues of the preconditioned matrix $\mathbf{C_h}^{-1}\mathbf{D_h}$ are ordered in such a way that $|\lambda_{2N_h}| \geq |\lambda_{2N_h-1}| \geq \cdots \geq |\lambda_1| > 0$. Therefore, solving

$$\mathbf{C_h}^{-1}\mathbf{D_h}\underline{\mathbf{u}}_h = \mathbf{C_h}^{-1}\underline{\mathbf{f}}_h$$

by means of the MinRes method proposed by [8], we can ensure that the preconditioned residual $\underline{\mathbf{r}}_h^{2m} = \mathbf{C_h}^{-1}\underline{\mathbf{f}}_h - \mathbf{C_h}^{-1}\mathbf{D_h}\underline{\mathbf{u}}_h^{2m}$ of the $2m$-th MinRes iterate satisfies the iteration error estimate

$$\|\underline{\mathbf{r}}_h^{2m}\|_{\mathbf{C_h}} \leq \frac{2q^m}{1-q^{2m}} \|\underline{\mathbf{r}}_h^0\|_{\mathbf{C_h}} \quad (15)$$

with $q = (\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) - 1)/(\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) + 1)$, see e.g. [12] or [6]. Thus, the number of MinRes iterations required for reducing the initial error by some fixed factor $\varepsilon \in (0, 1)$ is independent of both h and σ. Of course, in practice, the diagonal blocks $\sigma M_h + K_h$ in the preconditioner (13) should be replaced by appropriate preconditioners, e.g. by appropriate domain decompostion or multigrid preconditioners, see e.g. [11].

Applying again the Fourier Analysis (FA) to our one-dimensional problem gives quantitative rates which are displayed in Table 1, for σ ranging from 10^{-10} to 10^{10}.

Table 1. Convergence rate q resulting from the FA ($\varepsilon = 10^{-5}$).

$\log_{10}\sigma$	-10	-8	-6	-4	-2	0	2	4	6	8	10
$h = 1/60$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	0.0005	0.046	0.17	0.17	0.021	0.0002	$< \varepsilon$
$h = 1/120$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	0.0005	0.046	0.17	0.17	0.072	0.0009	$< \varepsilon$
$h = 1/1{,}200$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	0.0005	0.046	0.17	0.17	0.17	0.072	0.0009
$h = 1/12{,}000$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	0.0005	0.046	0.17	0.17	0.17	0.17	0.072
$h = 1/120{,}000$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	$< \varepsilon$	0.0005	0.046	0.17	0.17	0.17	0.17	0.17

Table 2 provides the MinRes iteration numbers which are needed for reducing the initial error by the factor $\varepsilon = 10^{-5}$ for different h and σ. The second line contains the preconditioned GMRes iterations for the constellation $h = 1/60$ and $H = 1/10$, where we use the preconditioner (11) with $\mathbf{B_h} = \mathbf{A_h}$. Both the FA (Table 1) and the numerical experiments (Table 2) were performed for the one-dimensional linear problem resulting in the stiffness matrix $K_h = h^{-1}\text{tridiag}(-1, 2, -1)$ and in the mass matrix $M_h = (h/6)\text{tridiag}(1, 4, 1)$ for the case $\nu = 1$. However, due to the estimates (14) and (15) the numerical behavior observed in our one-dimensional example is characteristic for the three-dimensional linear problem as well.

Table 2. Number of GMRes and MinRes iterations for $\varepsilon = 10^{-5}$.

$\log_{10}\sigma$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
GMRes	1	1	1	1	1	1	2	2	2	3	4	5	9	18	36	52	52	52	52	52	52
$h = 1/60$	1	1	1	1	1	1	3	3	3	5	7	11	13	13	14	10	6	4	4	2	2
$h = 1/120$	1	1	1	1	1	1	3	3	3	5	7	11	13	13	14	12	8	4	4	2	2
$h = 1/1{,}200$	1	1	1	1	1	1	3	3	3	5	7	11	15	13	14	13	12	10	6	4	4
$h = 1/12{,}000$	1	1	1	1	1	1	3	3	3	5	7	11	15	13	14	13	12	12	11	10	6
$h = 1/120{,}000$	1	1	1	1	1	1	3	3	3	5	7	11	15	13	14	13	12	12	11	10	10

5 Conclusions, Outlook, and Acknowledgments

In this paper we have considered the harmonic and multiharmonic approach to the solution of linear and nonlinear parabolic initial-boundary value problems with harmonic excitation. We have proposed two solution strategies based on a preconditioned GMRes method for the positive definite and non-symmetric problem formulation and a preconditioned MinRes iteration method for the symmetric and indefinite reformulation of the problem. The preconditioner for the GMRes method is a two-level Schwarz preconditioner consisting of a coarse grid solver for the original non-symmetric problem and a wire-basket-based domain decomposition preconditioner for the SPD part. This iterative solver works well for both the linear system (10) arizing from the linear time-harmonic problem and the Jacobi-systems (9) arizing in every step of the Newton iteration (8) for solving the nonlinear equations (7). This preconditioner is highly parallel, but not robust with respect to the bad parameter σ. A robust preconditioner can be constructed for the linear case where ν is independent of $|\nabla u|$. The preconditioner used in the MinRes method has a block-diagonal structure and is robust with respect to both the discretization parameter h and the bad parameter σ. Of course, other iterative methods are possible like the symmetric Uzawa CG method considered in [10] or the QMR method used in [3]. Furthermore, the robust all-at-once multigrid solvers developed by [9] for solving saddle point problems can be an alternative to the preconditioned Krylov-subspace methods considered in this paper. The preconditioned GMRes and MinRes solvers presented in this paper can be generalized to nonlinear eddy current problems studied in [2] and [3].

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