Mixed Multiscale Finite Element Analysis for Wave Equations Using Global Information

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Summary. A mixed multiscale finite element method (MsFEM) for wave equations is presented. Global information is used in the mixed MsFEM to construct multiscale basis functions. The solution of the wave equation smoothly depends on the global information. We investigate the relation between the smoothness of the global information and convergence rate of the mixed MsFEM.

1 Introduction

Over the past few decades, there has been growing interest in wave propagation in heterogeneous media. Many important problems such as earthquake motions, oceanography, medical and material sciences, and the morphology of oil and gas deposits can be understood through some use of mathematical and numerical modelings of wave propagation in heterogeneous media. In addition to heterogeneity, wave propagation is also a challenging multiscale problem. Among typical length scales present in wave propagation are wave length, propagation distance, and correlation length. In some problems such as in reflection seismology, the wave can propagate over a distance significantly larger than the wave length.

Consideration for accuracy suggests that the heterogeneity of media has to be sufficiently resolved when numerically simulating wave propagation, which can easily results in very expensive computations. While much more efficient and inexpensive in practice, standard upscaling techniques and multiscale methods employing some local information often fail to accurately transfer the fine scale information in media to the coarse formulation. Previous investigations (see e.g., [8]) indicate that appropriately taking into account some type of global information can potentially improve the accuracy of the multiscale methods. The importance of global information has been illustrated in porous media flow within the context of upscaling procedures [2] and also in multiscale finite (volume) elements [1]. The information is determined by some global fields that the solution of equations smoothly depends on. In the context of weak formulation, this global field is imbedded in the (multiscale) basis functions which in turn is used to represent the solution. Our objective is to develop a mixed MsFEM using global information that can capture the solution of wave equations in multiscale heterogeneous media and to make a priori error estimates for the mixed MsFEM.

The rest of the paper is organized as follows. In Sect. 2, we present some preliminaries. In Sect. 3, we present a mixed MsFEM for a model wave equation using global information and derive the error estimates. Some conclusions are drawn finally.

2 Preliminaries

In this section, we describe a model wave equation and some notations of function spaces.

Define D_{tt} to be the second order partial derivative operator with respect to t. Let $a(x)$ and $u(t,x)$ represent the density of the material and the unknown pressure, respectively. We define a time-space domain $\Omega_T := (0,T] \times \Omega$. Then a model wave equation reads as following:

$$\begin{aligned} D_{tt}u(t,x) - \nabla \cdot a(x)\nabla u(t,x) &= f(t,x) \quad \text{in} \quad \Omega_T \\ u(t,x) &= 0 \quad \text{on} \quad [0,T] \times \partial\Omega \\ u(x,0) &= g_0(x) \quad \text{in } \Omega \\ D_t u(x,0) &= g_1(x) \quad \text{in } \Omega. \end{aligned} \tag{1}$$

Here we assume that $a(x)$ is uniformly positive, symmetric and bounded in Ω. We assume that $f(t,x)$, $g_0(x)$ and $g_1(x)$ are smooth and do not have multiscale structures. This equation arises from geophysics and seismology. It is frequently observed that the spatial scales inherent in $a(x)$ cannot be clearly separated.

We introduce some notation which are used in the following sections. The usual Lebesgue and Sobolev spaces are denoted by $L^p(D)$, $W^{k,p}(D)$. In particular, $H^k(D) := W^{k,2}(D)$. Define $H(div, D) := \{f | f \in [L^2(D)]^d \quad \text{and} \quad \nabla \cdot f \in L^2(D)\}$. The vector-valued Sobolev space is equipped with the norm

$$\|u\|_{W^{m,p}(0,T;X)} := \left(\int_0^T \sum_{0 \le k \le m} \|D_t^k u\|_X^p dt \right)^{\frac{1}{p}},$$

when X is a normed space. If $p = 2$, we use $H^m(0,T;X)$ instead. When no ambiguity occurs, we use $W^{m,p}(X)$ to denote $W^{m,p}(0,T;X)$.

Without loss of generality, our discussion is concentrated on problems in $\Omega \subset \mathbb{R}^2$. We denote by K a generic coarse element with $h = \text{diam}(K)$, and τ_h a quasi-uniform family of coarse elements K. We shall not write the variables x and t for simplicity of presentation.

3 Mixed MsFEM Analysis

In this section, we first present a mixed MsFEM for the wave equation (1) using multiple global information, and then derive a priori error estimates in pressure and velocity.

3.1 Mixed MsFEM Formulation

Let velocity $\sigma = a\nabla u$. Then the mixed formulation for (1) is to find $\{u, \sigma\}$: $[0,T] \longrightarrow L^2(\Omega) \times H(div, \Omega)$ such that

$$\begin{aligned} (D_{tt}u, w) - (\nabla \cdot \sigma, w) &= (f, w) \quad \forall w \in L^2(\Omega) \\ (a^{-1}\sigma, \chi) + (u, \nabla \cdot \chi) &= 0 \quad \forall \chi \in H(div, \Omega) \\ (u(0), w) &= (g_0, w) \quad \forall w \in L^2(\Omega) \\ ((D_t u)(0), w) &= (g_1, w) \quad \forall w \in L^2(\Omega) \\ (a^{-1}\sigma(0), \chi) &= (\nabla g_0, \chi) \quad \forall \chi \in H(div, \Omega). \end{aligned} \tag{2}$$

We use global fields σ_i $(i = 1, \cdots, N)$ to build velocity basis function. We formulate an assumption for the global fields as following.

Assumption 1 *There exist functions $\sigma_1, \cdots, \sigma_N$ and $A_1(t,x), \cdots, A_N(t,x)$ such that*

$$\sigma = \sum_{i=1}^{N} A_i(t,x)\sigma_i,$$

where $A_i(t,x)$'s are smooth functions (we specify their smoothness later) and $\sigma_i = a(x)\nabla p_i$ $(i = 1, ..., N)$ solves an elliptic equation $\nabla \cdot a(x)\nabla p_i = 0$ with appropriate boundary conditions.

Remark 1. As an example in 2D, let u_1 and u_2 be the solution of the following equations

$$\begin{aligned} \nabla a \cdot \nabla u_i &= 0 \quad \text{in} \quad \Omega \\ u_i &= x_i \quad \text{on} \quad \partial\Omega, \quad i = 1, 2. \end{aligned} \tag{3}$$

set $u = u(t, u_1, u_2)$, then

$$\sigma = a\nabla u = \sum_{i=1}^{2} \frac{\partial u}{\partial u_i} a\nabla u_i := \sum_{i=1}^{2} A_i(t,x)\sigma_i,$$

where $A_i(t,x) = \frac{\partial u}{\partial u_i}$. Here $\sigma_i = a\nabla u_i$ are the global fields. Provided that $f \in L^\infty(L^p(\Omega)) \cap H^1(L^p(\Omega))$, $g_1 \in W^{1,p}(\Omega)$ and $D_{tt}u(0) \in L^p(\Omega)$, then the proof Theorem 1.1 in [8] implies that $A_i(t,x) = \frac{\partial u}{\partial u_i} \in L^\infty(W^{1,p}(\Omega))$. Consequently $A_i(t,x) \in L^2(C^{1-\frac{2}{p}}(\Omega))$ if $p > 2$ by using the Sobolev embedding theorem.

To numerically approximate the mixed problem (2), we construct the basis function for the velocity σ,

$$\begin{aligned} \nabla\cdot(a(x)\nabla\phi_{ij}^K) &= \frac{1}{|K|} \quad \text{in} \quad K \\ a(x)\nabla\phi_{ij}^K\cdot n_{e_l}^K &= \delta_{jl}\frac{\sigma_i\cdot n_{e_l}^K}{\int_{e_l}\sigma_i\cdot n_{e_l}ds} \quad \text{on} \quad \partial K, \end{aligned} \tag{4}$$

where $i=1,...,N$ and j is the index of the edges of the coarse block K (a triangle or rectangle), and

$$\delta_{jj}=1,\ \ \delta_{jl}=0\ \text{ if }\ j\neq l.$$

Here e_l denotes an edge of the coarse block. We shall omit the subscript e_l in n if the integral is taken along the edge. Note that for each edge, we have N basis functions and we assume that σ_1,..., σ_N are linearly independent in order to guarantee that the basis functions are linearly independent. To avoid the possibility that $\int_{e_l}\sigma_i\cdot nds$ is zero or unbounded, we make the following assumption for the convergence analysis. If $\int_{e_l}\sigma_i\cdot nds=0$ on some e_l, we can use the local mixed MsFEM basis function proposed in [3], i.e., replace $\frac{\sigma_i\cdot n_{e_l}^K}{\int_{e_l}\sigma_i\cdot n_{e_l}ds}$ with $\frac{1}{|e_l|}$ in (4).

Assumption 2

$$\int_{e_l}|\sigma_i\cdot n|ds\leq Ch^{\beta_1}\ \text{ and }\ \|\frac{\sigma_i\cdot n}{\int_{e_l}\sigma_i\cdot nds}\|_{L^r(e_l)}\leq Ch^{-\beta_2+\frac{1}{r}-1}$$

uniformly for all edges e_l*, where* $\beta_1\leq 1$*,* $\beta_2\geq 0$ *and* $r\geq 1$*.*

We would like to note that *Assumption 2* is used to define the boundary data for the velocity basis equations well and to bound the velocity basis function ψ_{ij}^K. In fact, *Assumption 2* implies that $\|\psi_{ij}^K\|_{0,K}\leq Ch^{-\beta_2}$ (see [1]). If σ_i are bounded in $L^\infty(e_l)$ for all e_l and $|\int_{e_l}\sigma_i\cdot nds|$ remains positive uniformly for all e_l, then $\beta_1=1$ and $\beta_2=0$. The index r is only related to the L^r norm that appeared in Assumption 2 and has nothing to do with the convergence rate. We would like to note that local mixed MsFE basis function introduced in [3] is a special case defined in (4). To do this, one just needs to replace σ_1 in (4) by a constant vector.

We define $\psi_{ij}^K=a(x)\nabla\phi_{ij}^K$ and

$$\Sigma_h=\bigoplus_K\{\psi_{ij}^K\}\subset H(div,\Omega).$$

Let $Q_h=\bigoplus_K \mathrm{P}_0(K)\subset L^2(\Omega)$, i.e., piecewise constants, be the basis functions approximating u. For $t>0$, we define

$$\Pi_h|_K\sigma(t)=\sum_{i,j}(\int_{e_j}A_i(t,x)\sigma_i\cdot ndx)\psi_{ij}^K$$

The numerical mixed formulation is to find $\{u_h,\sigma_h\}:[0,T]\longrightarrow Q_h\times\Sigma_h$ such that

$$
\begin{aligned}
(D_{tt}u_h, w) - (\nabla \cdot \sigma_h, w) &= (f, w) \quad \forall w \in Q_h \\
(a^{-1}\sigma_h, \chi) + (u_h, \nabla \cdot \chi) &= 0 \quad \forall \chi \in \Sigma_h \\
(u_h(0), w) &= (g_0, w) \quad \forall w \in Q_h \\
((D_t u_h)(0), w) &= (g_1, w) \quad \forall w \in Q_h \\
(\sigma_h(0), \chi) &= (\sigma(0), \chi) \quad \forall \chi \in \Sigma_h.
\end{aligned} \tag{5}
$$

3.2 A Priori Error Estimates for Continuous Time

Before we proceed with the convergence analysis of the mixed MsFEM for the wave equation, we recall some properties for the basis defined in (4). By Lemma 3.1 in [1], it follows that

$$\sigma_i|_K = \sum_j \beta_{ij}^K \psi_{ij}^K, \tag{6}$$

where $\beta_{ij}^K = \int_{e_j} \delta_{ij}\sigma_i \cdot n dx$. For the interpolator Π_h, Lemma 3.2 in [1] claims

$$(\nabla \cdot (\sigma - \Pi_h \sigma), w) = 0 \quad w \in Q_h. \tag{7}$$

Let P_h be $L^2(\Omega)$ orthogonal projection onto Q_h. We define

$$\|\sigma\|^2_{L^2_a(\Omega)} = \int_\Omega \sigma^t \cdot a^{-1}(x)\sigma dx, \quad \|\sigma\|^2_{L^2(0,T;L^2_a(\Omega))} = \int_{\Omega_T} \sigma^t \cdot a^{-1}(x)\sigma dx ds.$$

By using (7) and standard estimate techniques (e.g., Schwarz inequality, Grownall's inequality, Jensen's inequality and triangle inequality), we can obtain the following lemma.

Lemma 1. *[6] Let $\{u, \sigma\}$ and $\{u_h, \sigma_h\}$ be respectively solution of (2) and (5). Then*

$$
\begin{aligned}
&\|u - u_h\|^2_{L^\infty(L^2(\Omega))} + \sup_t \|\int_0^t (\sigma(s) - \sigma_h(s)) ds\|^2_{L^2_a(\Omega)} \\
&\le C(\|P_h u - u\|^2_{L^\infty(L^2(\Omega))} + \|\Pi_h \sigma - \sigma\|^2_{L^2(L^2_a(\Omega))}).
\end{aligned} \tag{8}
$$

By Lemma 1, we get an priori error estimate for the scheme defined in (5).

Theorem 1. *Suppose that $f \in L^2(L^2(\Omega))$, $g_0 \in H^1(\Omega)$ and $g_1 \in L^2(\Omega)$. Let $\{u, \sigma\}$ and $\{u_h, \sigma_h\}$ be solution of (2) and (5), respectively. If Assumption 1 and Assumption 2 hold and $A_i(t, x) \in L^2(C^\alpha(\Omega))$ for $i = 1, \cdots, N$, then for $\alpha + \beta_1 - \beta_2 - 1 > 0$,*

$$\|u - u_h\|_{L^\infty(L^2(\Omega))} + \sup_t \|\int_0^t (\sigma(s) - \sigma_h(s)) ds\|_{L^2_a(\Omega)} \le Ch^{\min(1, \alpha+\beta_1-\beta_2-1)}.$$

Proof. If the source term $f \in L^2(L^2(\Omega))$, the initial conditions $g_0 \in H^1(\Omega)$ and $g_1 \in L^2(\Omega)$, then $u \in L^\infty(H^1(\Omega))$ (see [5]). Thanks to the fact that P_h is the $L^2(\Omega)$ projection onto Q_h,

$$\|u - P_h u\|_{L^\infty(L^2(\Omega))} \le Ch|u|_{L^\infty(H^1(\Omega))}, \tag{9}$$

which estimates the first term of right hand side in (8). Next we estimate the term $\|\sigma - \Pi_h \sigma\|^2_{L^2(L^2_a(\Omega))}$. Define

$$A_{ij}^K(t) = \int_{e_j} A_i(t,s)\sigma_i \cdot n ds$$

on each element K. With $\bar{A}_i^j$ the average $A_i(x)$ along e_j, then

$$\begin{aligned} |A_{ij}^K - \bar{A}_i^j \beta_{ij}^K| &= |\int_{e_j} A_i \sigma_i \cdot n ds - \bar{A}_i^j \int_{e_j} \sigma_i \cdot n ds| \\ &\le Ch^{\alpha+\beta_1} \|A_i(t)\|_{C^\alpha(\Omega)}, \end{aligned} \tag{10}$$

where we have used the *Assumption 2*.

Invoking *Assumption 1*, (6) and $\|\psi_{ij}^K\|_{0,K} \le Ch^{-\beta_2}$, see [1], we have in each element K

$$\begin{aligned} &\|\sigma - \Pi_h \sigma\|^2_{L^2(0,T;L^2_a(K))} = \\ &\int_0^T \int_K \sum_{i,j} (A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K \cdot a^{-1} \sum_{i,j} (A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K dxdt \\ &\le C \int_0^T \int_K (\sum_{i,j} (A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K)^2 dxdt \\ &= C\| \sum_{i,j} (A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K \|^2_{L^2(0,T;L^2(K))} \\ &\le C\| \sum_{i,j} (A_i(t,x) - \bar{A}_i^j(t))\beta_{ij}^K \psi_{ij}^K \|^2_{L^2(0,T;L^2(K))} \\ &+ C\| \sum_{i,j} (\bar{A}_i^j(t)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K \|^2_{L^2(0,T;L^2(K))} \\ &\le Ch^{2(\alpha+\beta_1)} (\sum_i \|A_i\|^2_{L^2(0,T;C^\alpha(K))}) \sum_{ij} \|\psi_{ij}^K\|^2_{0,K} \\ &\le Ch^{2(\alpha+\beta_1-\beta_2)} (\sum_i \|A_i\|^2_{L^2(0,T;C^\alpha(K))}), \end{aligned} \tag{11}$$

where we have used the facts that $A_i \in L^2(0,T;C^\alpha(\Omega))$ and (10). After making summation over all K in (11), we have

$$\|\sigma - \Pi_h \sigma\|_{L^2(0,T;L^2_a(\Omega))} \le Ch^{\alpha+\beta_1-\beta_2-1}. \tag{12}$$

Taking into account (9) , (12) and (8), the proof is complete.

If the functions $A_i(t,x)$ ($i = 1, \cdots, N$) in *Assumption 1* have better regularity with respect to time t, we can obtain an convergence rate as follows:

Theorem 2. *Let* $\{u, \sigma\}$ *and* $\{u_h, \sigma_h\}$ *be the solution of (2), respectively and (5). If Assumption 1, 2 hold and* $A_i(t,x) \in L^\infty(C^\alpha(\Omega)) \cap H^1(C^\alpha(\Omega))$ *for* $i = 1, \cdots, N$, *then for* $\alpha + \beta_1 - \beta_2 - 1 > 0$,

$$\|u - u_h\|_{L^\infty(L^2(\Omega))} + \|\sigma - \sigma_h\|_{L^\infty(L_a^2(\Omega))} \leq C h^{\min(1, \alpha+\beta_1-\beta_2-1)}.$$

The proof can be found in [6].

Remark 2. If global fields $u_i (i = 1, 2)$ are defined in Remark 1, then $\sigma = \sum_{i=1}^2 A_i(t,x)\sigma_i$, where $A_i(t,x) = \frac{\partial u}{\partial u_i}$ and $\sigma_i = a\nabla u_i$. Provided that $f \in W^{1,\infty}(L^p(\Omega)) \cap W^{2,p}(L^p(\Omega))$, $D_{tt}u(0) \in W^{1,p}(\Omega)$ and $D_{ttt}u(0) \in L^p(\Omega)$, then the proof Lemma 2.6 in [8] implies that $A_i(t,x) = \frac{\partial u}{\partial u_i} \in W^{1,\infty}(W^{1,p}(\Omega))$. Consequently $A_i(t,x) \in H^1(C^{1-\frac{2}{p}}(\Omega))$ if $p > 2$ by using Sobolev embedding theorem.

3.3 A Priori Error Estimate for Discrete Time

We introduce the following notation for time-discetization,

$$D_t u^{\frac{1}{2}} = \frac{u^1 - u^0}{\Delta t}, \quad D_{tt}u^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}.$$

Because we assume that the media has only spatial multiscales, we use the mixed MsFEM for the space discretization and use conventional finite difference schemes to discretize the temporal variables. As an explicit-in-time scheme, the fully mixed formulation is to find $\{u_h^{n+1}, \sigma_h^{n+1}\} \in Q_h \times \Sigma_h$ such that

$$\begin{aligned}
(D_{tt}u_h^n, w) - (\nabla \cdot \sigma_h^n, w) &= (f^n, w) \quad \forall w \in Q_h \\
(a^{-1}\sigma_h^{n+1}, \chi) + (u_h^{n+1}, \nabla \cdot \chi) &= 0 \quad \forall \chi \in \Sigma_h \\
(u_h^0, w) &= (g_0, w) \quad \forall w \in Q_h \\
(\frac{2}{\Delta t} D_t u_h^{\frac{1}{2}}, w) - (\nabla \cdot \sigma_h^0, w) &= (f^0 + \frac{2}{\Delta t} g_1, w) \quad \forall w \in Q_h \\
(\sigma_h^0, \chi) &= (\sigma(0), \chi) \quad \forall \chi \in \Sigma_h.
\end{aligned} \tag{13}$$

It is known that the scheme in (13) is conditional stable (refer to [4]) and that the time consistence error is $O(\Delta t^2)$ if $u(t,x)$ is sufficiently smooth with respect to t. Consequently, we can use Theorem 2 and follow the proof of Theorem 5.2 in [4] to obtain the following estimate.

Theorem 3. *Let* $\{u, \sigma\}$ *and* $\{u_h, \sigma_h\}$ *be solution of (2) and (13), respectively. If* $u(t,x)$ *is sufficiently smooth with respect to* t *and the assumptions in Theorem 2 are satisfied, then*

$$\sup_{t_n} \|u - u_h^n\|_{L^2(\Omega)} + \sup_{t_n} \|\sigma - \sigma_h^n\|_{L_a^2(\Omega)} \leq C(h^{\min(1, \alpha+\beta_1-\beta_2-1)} + \Delta t^2).$$

We would like to note that an implicit-in-time scheme for the wave equation is presented in [4].

4 Conclusions

In the paper, we present a mixed MsFEM for a wave equation using global information. The global information is described by global fields (velocity fields). For construction of velocity basis functions, the global fields are employed. A prior error estimates are derived for the wave equation by the mixed MsFEM. The numerical results in some recent works (e.g., [6, 7, 8]) demonstrate that using global fields can capture non-local effects in simulations and significant improve accuracy and efficiency when the media are heterogenous and their scales are non-separable.

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