Optimized Schwarz Methods for Domains with an Arbitrary Interface

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1 Introduction

Optimized Schwarz methods form a class of domain decomposition methods for the solution of partial differential equations. Optimized Schwarz methods employ a first or higher order boundary condition along the artificial interface to accelerate its convergence. In the literature, analysis of optimized Schwarz methods rely on Fourier analysis and so the domains are restricted to be regular (rectangle or disk). By expressing the interface operator in terms of Poincare–Steklov operators, we are able to derive upper bounds of the spectral radius of the operator for Poisson-like problems for two essentially arbitrary subdomains. For a first order (Robin) boundary operator, an optimal choice of the parameter in the boundary operator leads to an upper bound of $1 - O(h^{1/2})$ of the spectral radius, where h is the discretization parameter. For a certain higher order boundary operator, a clever choice of the two parameters in the boundary operator leads to an upper bound of $1 - O(h^{1/4})$ of the spectral radius. These agree with the predicted rates for rectangular subdomains available in the literature and are also the observed rates in numerical simulations. This contribution summarizes the author's work in [11, 12].

Let Ω be a bounded domain in $\mathbf{R}^N$ with a smooth boundary. Suppose Ω is composed of two nonoverlapping open subdomains, that is, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Assume that the artificial boundary $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$ is non-trivial (non-zero measure in R^{N-1}) and is a smooth curve. We shall always assume that $\partial\Omega_i \setminus \Gamma$ is non-trivial for both $i = 1, 2$.

Recall the trace space

$$H_{00}^{1/2}(\Gamma) = \{v|_\Gamma,\ v \in H_0^1(\Omega)\}$$

with dual $H^{-1/2}(\Gamma)$. For $i = 1, 2$, let

$$V_i = \{v_i \in H^1(\Omega_i),\ v_i = 0 \text{ on } \partial\Omega_i \cap \partial\Omega\}.$$

Define the trace operators $T_i : V_i \to H_{00}^{1/2}(\Gamma)$ by

$$T_i v_i = v_i|_\Gamma, \qquad v_i \in V_i.$$

For simplicity, consider the model problem

$$-\triangle u = f \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

One candidate for the subdomain problem is

$$\begin{aligned} -\triangle u_i &= f \ \text{ on } \Omega_i, \\ u_i &= p \ \text{ on } \Gamma \end{aligned}$$

with $u_i \in V_i$ for some function $p \in H_{00}^{1/2}(\Gamma)$. Note that p is the correct function ($p = T_i u$) if

$$\frac{\partial u_1}{\partial \nu_1} + \frac{\partial u_2}{\partial \nu_2} = 0 \text{ on } \Gamma.$$

This is known as the transmission condition. Define $u_i = u_i^e + z_i$ where $u_i^e = \mathcal{H}_i p \in V_i$ is the harmonic extension of p:

$$\begin{aligned} -\triangle u_i^e &= 0 \ \text{ on } \Omega_i, \\ u_i^e &= p \ \text{ on } \Gamma \end{aligned}$$

and $z_i = \triangle_i^{-1} f$ where $\triangle_i$ is the Laplacian operator with domain $H_0^1(\Omega_i)$. Define the Poincare–Steklov operators $S_i : H_{00}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$S_i p = \frac{\partial \mathcal{H}_i p}{\partial \nu_i}$$

or by

$$\langle S_i p, q \rangle = \int_{\Omega_i} \nabla p^e \cdot \nabla q^e, \qquad \forall p, q \in H_{00}^{1/2}(\Gamma)$$

with $p^e = \mathcal{H}_i p,\ q^e = \mathcal{H}_i q$. In the above inner product, S_i is self-adjoint and positive definite. Hence the transmission condition can also be expressed as

$$(S_1 + S_2) u|_\Gamma = w \tag{1}$$

for some w.

2 First-Order Boundary Condition

In [10], the author defined the Schwarz sequence $\{u_i^{(n)} \in V_i,\ n \geq 0\}$ by

$$\begin{aligned} -\triangle u_i^{(n)} &= f \ \text{ on } \Omega_i, \\ \frac{\partial u_i^{(n)}}{\partial \nu_i} + \lambda u_i^{(n)} &= g_i^{(n)} \ \text{ on } \Gamma. \end{aligned} \tag{2}$$

Here λ is a positive constant. Noting that $\nu_1 = -\nu_2$ on Γ, the Robin data can be updated as

$$g_{3-i}^{(n+1)} = -\frac{\partial u_i^{(n)}}{\partial \nu_i} + \lambda u_i^{(n)} \text{ on } \Gamma, \qquad i = 1, 2.$$

The iteration can be started for any initial $g_i^{(0)} \in L^2(\Gamma)$. In practice, the choice $g_i^{(0)} = 0$ is convenient.

The following is an equivalent update ([2]):

$$g_{3-i}^{(n+1)} = 2\lambda u_i^{(n)} - g_i^{(n)} \text{ on } \Gamma, \qquad i = 1, 2. \tag{3}$$

Note that the subdomain computations can be carried out concurrently. Many authors have studied the convergence of this method and the choice of the optimal parameter. See [1, 12, 15] which are most pertinent to this paper.

The function g_2 can be eliminated in (3) to obtain the following equation for g_1:

$$\begin{aligned} &\Big[I - (I - 2\lambda(S_2 + \lambda)^{-1})(I - 2\lambda(S_1 + \lambda)^{-1})\Big] g_1 \\ &= 2\lambda b \equiv 2\lambda(T_2 z_2 - (I - 2\lambda(S_2 + \lambda)^{-1})T_1 z_1). \end{aligned}$$

The operator for g_1 has alternative representations

$$\begin{aligned} &I - (S_2 + \lambda)^{-1}(S_2 - \lambda)(S_1 - \lambda)(S_1 + \lambda)^{-1} \\ &= (S_2 + \lambda)^{-1}\big((S_2 + \lambda)(S_1 + \lambda) - (S_2 - \lambda)(S_1 - \lambda)\big)(S_1 + \lambda)^{-1} \\ &= 2\lambda(S_2 + \lambda)^{-1}(S_1 + S_2)(S_1 + \lambda)^{-1}. \end{aligned}$$

Thus the above equation for g_1 is equivalent to

$$(S_2 + \lambda)^{-1}(S_1 + S_2)(S_1 + \lambda)^{-1} g_1 = b.$$

Recognizing that $(S_1 + \lambda)^{-1} g_1 = T_1(u - z_1)$ where u is the exact solution of the global Poisson equation and (g_1, g_2) is the solution of (3), we see that Lions' method is an iterative method which solves (1) using the left preconditioner $(S_2 + \lambda)^{-1}$.

Lions' method is equivalent to the following iterative method

$$g_1^{(n+1)} = \mathcal{G}_h g_1^{(n-1)} + b \tag{4}$$

to solve for the discrete counterpart of the boundary function g_1 where

$$\begin{aligned} \mathcal{G}_h &\equiv (I - 2\lambda(S_{2,h} + \lambda)^{-1})(I - 2\lambda(S_{1,h} + \lambda)^{-1}) \\ &= (S_{2,h} + \lambda)^{-1}(S_{2,h} - \lambda)(S_{1,h} + \lambda)^{-1}(S_{1,h} - \lambda). \end{aligned}$$

Here $S_{i,h}$ is a finite element discretization of S_i.

For a square matrix A, let the spectral radius of A be denoted by $\rho(A)$. The convergence of the iteration (4) depends on $\rho(\mathcal{G}_h)$ which will be analyzed below. In the following, $|\cdot|$ denotes the two-norm. We shall use c, c_1, c_2 to denote positive constants whose values may differ in different occurrences.

The analysis for the upper bound of $\rho(\mathcal{G}_h)$ is identical to that for the ADI method to solve PDEs. This is because $\mathcal{G}_h$ has the same form as the operator in the ADI method. Note

$$\rho(\mathcal{G}_h) \leq |\mathcal{G}_h| \leq |(S_{1,h}+\lambda)^{-1}(S_{1,h}-\lambda)|\,|(S_{2,h}+\lambda)^{-1}(S_{2,h}-\lambda)|,$$

Since $S_{1,h}$ and $S_{2,h}$ are symmetric and their eigenvalues have the same asymptotic behaviour, it is not difficult to show

Theorem 1.

$$\rho(\mathcal{G}_h) \leq \begin{cases} 1-c_1\lambda h, & \lambda \leq h^{-1/2}; \\ 1-c_2\lambda^{-1}, & \lambda \geq h^{-1/2}. \end{cases} \tag{5}$$

In case $\lambda = O(h^{-1/2})$*, then* $\rho(\mathcal{G}_h) \leq 1 - ch^{1/2}$.

A lower bound for $\rho(\mathcal{G}_h)$ is considerably more difficult to establish than an upper bound. In fact, we have only been able to obtain a lower bound for λ in special intervals. For $\lambda = h^s$ with $s \in (-\infty,-1)\cup(0,\infty)$, the upper bound established in the theorem is actually sharp. In the more interesting range $s \in [-1,0]$, the analysis is more complicated because $\mathcal{G}_h$ is a product of two symmetric indefinite matrices. We conjecture that the bounds in (5) are sharp for $s \in [-1,0]$ as well.

We conclude this section by mentioning that the analysis has been extended to the case of PDEs with discontinuous coefficients. See [3].

3 Higher-Order Boundary Condition

One popular optimized Schwarz method using a second order boundary condition along the artificial interface is

$$-\frac{d^2u_i}{d\tau^2} + \eta\frac{du_i}{d\nu_i} + \lambda u_i = g_i \text{ on } \Gamma$$

where η and λ are positive parameters and τ is a unit tangent vector along Γ. In the literature, see [4, 5, 6, 7, 8, 9, 13, 14], for instance, Fourier analysis is used to analyze the convergence of the schemes, which means that the theory is applicable only to regular (rectangular) subdomains.

For $i = 1, 2$, the subdomain problems are

$$\begin{aligned} -\triangle u_i^{(n)} &= f \text{ on } \Omega_i, \\ -\frac{\partial^2 u_i^{(n)}}{\partial\tau^2} + \eta\frac{\partial u_i^{(n)}}{\partial\nu_i} + \lambda u_i^{(n)} &= g_i^{(n)} \text{ on } \Gamma \end{aligned} \tag{6}$$

where $g_i^{(n)}$ is some given function. Henceforth, we shall assume $f \equiv 0$. Unfortunately, we are also unable to prove a rate of convergence of $g_i^{(n)}$ to zero in non-rectangular geometry. Instead, we propose a different boundary condition for which a spectral radius estimate $1 - O(h^{1/4})$ can be proven for a general class of domains.

This is the same estimate as that for (6) for rectangular domains which is available in the literature.

We now give a heuristic derivation of our new boundary condition. Along Γ,

$$0 = f = \triangle u = \frac{\partial^2 u}{\partial \nu^2} + \frac{\partial^2 u}{\partial \tau^2} + Lu$$

where $L = \nabla \cdot \tau\, \partial_\tau + \nabla \cdot \nu\, \partial_\nu$ is a linear first order differential operator. We shall be taking $\eta = O(h^{-3/4})$ and $\lambda = O(h^{-1})$ where h is the discretization parameter and thus the term containing L will be insignificant. Ignoring it, (6) can be approximated as

$$\frac{\partial^2 u_i^{(n)}}{\partial \nu_i^2} + \eta \frac{\partial u_i^{(n)}}{\partial \nu_i} + \lambda u_i^{(n)} = g_i^{(n)} \text{ on } \Gamma. \tag{7}$$

A natural update for the boundary function $g_i^{(n)}$ is

$$g_{3-i}^{(n+1)} = g_i^{(n)} - 2\eta \frac{\partial u_i^{(n)}}{\partial \nu_i}. \tag{8}$$

To see this, note that $\nu_1 = -\nu_2$ and

$$\begin{aligned} g_{3-i}^{(n+1)} &= \frac{\partial^2 u_{3-i}^{(n+1)}}{\partial \nu_{3-i}^2} + \eta \frac{\partial u_{3-i}^{(n+1)}}{\partial \nu_{3-i}} + \lambda u_{3-i}^{(n+1)} \\ &\equiv \frac{\partial^2 u_i^{(n)}}{\partial \nu_i^2} - \eta \frac{\partial u_i^{(n)}}{\partial \nu_i} + \lambda u_i^{(n)} \\ &= g_i^{(n)} - 2\eta \frac{\partial u_i^{(n)}}{\partial \nu_i}. \end{aligned}$$

We next approximate the second normal derivative in (7) by S_i^2, leading to the new boundary condition

$$(S_i^2 + \eta S_i + \lambda) T_i u_i^{(n)} = g_i^{(n)} \text{ on } \Gamma.$$

We assume that

$$S_i T_i u_i^{(n)} \in H_{00}^{1/2}(\Gamma) \tag{9}$$

so that $S_i^2 T_i u_i^{(n)} \in H^{-1/2}(\Gamma)$. Two examples of Γ where the assumption (9) holds are one side of a rectangle and an arc of a circle, provided that $u_i^{(n)}$ is sufficiently smooth. For these two cases, S_i can be worked out analytically and it can be seen that S_i^2 and $\partial^2/\partial\nu_i^2$ differ when acting upon low order modes. Their difference goes to zero as the order of modes goes to infinity. It is in this sense that S_i^2 approximates the second normal derivative and is the reason why the algorithms employing the two boundary conditions have similar convergence rates. By writing an equivalent form

$$\frac{\partial^2}{\partial \nu_i^2} + \eta S_i + \lambda + L$$

of the boundary operator (6), we clearly see the two approximations involved in the proposed boundary operator $S_i^2 + \eta S_i + \lambda$: replacement of the second normal derivative by S_i^2 and the removal of L.

Define, for $i = 1, 2$,

$$\begin{aligned} -\triangle u_i^{(n)} &= 0 \text{ on } \Omega_i \\ (S_i^2 + \eta S_i + \lambda) T_i u_i^{(n)} &= g_i^{(n)}. \end{aligned} \tag{10}$$

The parameters η and λ are positive. The update (8) is still applicable here and can be written as

$$g_{3-i}^{(n+1)} = g_i^{(n)} - 2\eta S_i T_i u_i^{(n)}.$$

Since $T_i u_i^{(n)} = (S_i^2 + \eta S_i + \lambda)^{-1} g_i^{(n)}$, the update for the boundary function becomes

$$g_{3-i}^{(n+1)} = g_i^{(n)} - 2\eta S_i (S_i^2 + \eta S_i + \lambda)^{-1} g_i^{(n)}.$$

Eliminate $g_2^{(n)}$ from the above to obtain $g_1^{(n+1)} = \mathcal{K} g_1^{(n-1)}$ where

$$\mathcal{K} = \Big(I - 2\eta S_2 (S_2^2 + \eta S_2 + \lambda)^{-1}\Big)\Big(I - 2\eta S_1 (S_1^2 + \eta S_1 + \lambda)^{-1}\Big). \tag{11}$$

The discrete iteration is

$$g_1^{(n+1)} = \mathcal{K}_h g_1^{(n)} \tag{12}$$

where $\mathcal{K}_h$ denotes a finite element discretization of $\mathcal{K}$. Convergence of this iteration depends on $\rho(\mathcal{K}_h)$. If $\rho(\mathcal{K}_h) < 1$, then $g_1^{(n)} \to 0$, the exact solution. Since $\rho(\mathcal{K}_h) \leq |\mathcal{K}_h|$,

$$\rho(\mathcal{K}_h) \leq |I - 2\eta S_{2,h} (S_{2,h}^2 + \eta S_{2,h} + \lambda)^{-1}| \, |I - 2\eta S_{1,h} (S_{1,h}^2 + \eta S_{1,h} + \lambda)^{-1}|$$

with the matrices on the right-hand side symmetric. The proof of the following theorem appears in [11].

Theorem 2. *Let* $\lambda = O(h^{-1})$ *and* $\eta = O(h^{-3/4})$. *Then* $\rho(\mathcal{K}_h) \leq 1 - O(h^{1/4})$.

The above theorem gives an upper bound of the spectral radius. As before, a lower bound is much more difficult to establish. The following are some partial results. Suppose $\eta < O(1)$. Then

$$\rho(\mathcal{K}_h) = \begin{cases} 1 - O(\eta h), & \lambda \leq O(h^{-1}); \\ 1 - O(\eta \lambda^{-1}), & \lambda \geq O(h^{-1}). \end{cases}$$

Suppose $\eta > O(h^{-1})$ and $\lambda < O(\eta)$. Then

$$\rho(\mathcal{K}_h) = \begin{cases} 1 - O(\eta^{-1}), & \lambda < O(1); \\ 1 - O(\eta^{-1} \lambda h), & \lambda > O(h^{-2}). \end{cases}$$

We give one MATLAB numerical experiment. Let the domain be the rectangle $[0, 1.6] \times [0, 1]$ and the artificial interface be the line $y = x - 0.2$. Hence the two subdomains are trapezoids. Using a simple finite difference scheme, the result is shown

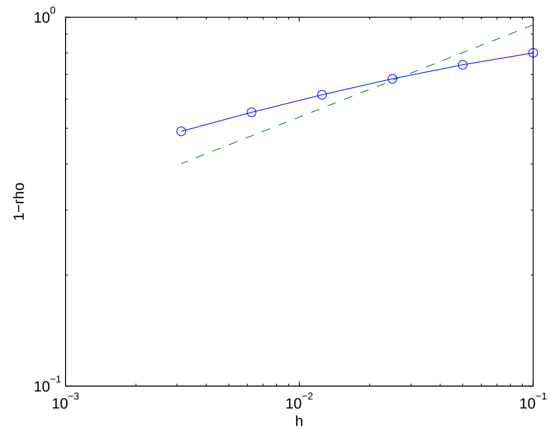

Fig. 1. *Solid line* is a plot of $1 - \rho(\mathcal{K}_h)$ versus h for two trapezoidal subdomains while the *dashed line* is a plot of $1 - O(h^{1/4})$.

in Fig. 1. Observe that for larger values of h, the spectral radius is actually better than the prediction $1 - O(h^{1/4})$. However, the spectral radius seems to approach the predicted rate for smaller values of h. For other numerical results, see [11].

There are a number of mathematical questions about the new boundary condition which have not been answered. Although the discrete iteration (12) is well defined and convergent, it remains to show well-posedness at the continuous level for the boundary condition (10). Also, the geometric meaning of the assumption (9) requires investigation. While we have not been able to establish a convergence rate for (6) on arbitrary domains, it is hoped that the present analysis gives some new insight to the convergence of (6).

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