Coarse Spaces over the Ages

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1 Introduction

The objective of this paper is to explain the principles of the design of a coarse space in a simplified way and by pictures. The focus is on ideas rather than on a more historically complete presentation. That can be found, e.g., in [28]. Also, space limitation does not allow us even to mention many important methods and papers that should be rightfully included.

The coarse space facilitates a global exchange of information in multigrid and domain decomposition methods for elliptic problems. This exchange is necessary, because the solution is non-local: its value at any point depends on the right-hand-side at any other point. Both multigrid and domain decomposition combine a global correction in the coarse space with local corrections, called smoothing in multigrid and subdomain solves in domain decomposition. In multigrid the coarse space is large (typically, the mesh ratio is 2 or 3 at most) and the local solvers are not very powerful (usually, relaxation). In domain decomposition, the coarse space is small (just one or a few degrees of freedom per subdomain), and the local solvers are powerful (direct solvers on subdomain). But the mathematics is more or less the same.

2 Local Nullspace and Bounded Energy Conditions

Consider the variational problem

$$u \in V : \qquad a(u,v) = f(v) \qquad \forall v \in V, \tag{1}$$

where a is symmetric positive definite and V is a finite dimensional space. Most, if not all, multigrid, domain decomposition, and substructuring methods for (1) can be

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cast as variants of the additive Schwarz method (ASM), which is the preconditioning by the approximate solver

$$M : r \mapsto \sum_{i=0}^{N} u_i \tag{2}$$

where u_i are solutions of the subproblems

$$u_i \in V_i : \qquad a(u_i, v_i) = r(v_i) \qquad \forall v_i \in V_i \tag{3}$$

where

$$V = V_0 + V_1 + \cdots + V_N \tag{4}$$

The resulting condition number of the preconditioned problem is then bounded by nC_0, where $n \leq N + 1$ is the maximal number of the subspaces $V_0, V_1, \ldots, V_N$ that have nontrivial intersections, and C_0 is the constant from the bounded energy decomposition property

$$\forall v \in V \exists v_i \in V_i : v = \sum_{i=0}^{N} v_i, \quad \sum_{i=0}^{N} a\left(v_i, v_i\right) \leq C_0 a\left(v, v\right), \tag{5}$$

cf., [1, 9, 29].

Variants of ASM include the multiplicative use of the subspace correction in [16, 18], and the use of other forms in place of a in subproblems (3), cf., [9, 26].

Now consider V to be a space of functions on a domain Ω. The subspaces V_i range from the span of one basis vector in multigrid (for the simplest case, Jacobi iteration) to spaces of functions on large overlapping subdomains Ω_i. When the domain Ω is the union of non-overlapping subdomains Ω_j, $j = 1, \ldots, M$, the spaces V_i are defined as certain subspaces of the space $W = W_1 \times \cdots \times W_M$, where W_j is a space of functions on Ω_j. The natural splitting of the bilinear form $a\left(\cdot, \cdot\right)$ into integrals over Ω_j is then $a\left(u, v\right) = \int_\Omega \nabla u \cdot \nabla v = \sum_{j=1}^{N} a_j\left(u, v\right)$, where the local bilinear forms

$$a_j\left(u, v\right) = \int_{\Omega_j} \nabla u \cdot \nabla v \tag{6}$$

are used on W_j instead of the bilinear form $a\left(\cdot, \cdot\right)$.

The space V_0 is the coarse space, and the rest of this paper deals with its construction. It had been long understood and then formulated explicitly in [14] that for condition numbers to be independent of the number of subdomains, the coarse space needs to contain the nullspace of the local bilinear forms $a_j\left(\cdot, \cdot\right)$. For the scalar problem as in (6), this means constant functions, while for elasticity, the coarse space needs to contain the rigid body modes of every substructure. Much of the development of the coarse space in domain decomposition has been driven by the need for the coarse space to satisfy this *local nullspace condition* at the same time as the *bounded energy condition* $a\left(v_0, v_0\right) \leq C_0 a\left(v, v\right)$, required as a part of (5).

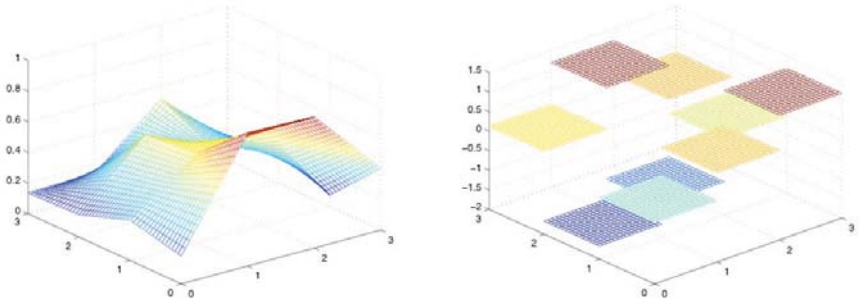

Fig. 1. *Left*: Piecewise bilinear coarse space function. *Right*: piecewise constant functions.

3 Some Early Domain Decomposition Methods

By taking v_0 in (5) first, we see that the design objective of the coarse space is that there should exist a mapping $v \in V \longmapsto v_0 \in V_0$ such that (i) the energy of v_0 is not too large, and (ii) the remainder $v - v_0$ can be decomposed in the spaces V_i, $i = 1, \ldots, N$, without increasing the energy too much. Definition of v_0 by linear or bilinear interpolation is the natural first choice (Fig. 1 left). Because of the discrete Sobolev inequality, this works fine in 2D: values of v at interpolation nodes are bounded by the energy of v up to a logarithmic factor in the mesh size h. The remainder $v - v_0$ is tied to zero by its zero values at the interpolation nodes, and it turns out it can be decomposed into v_i's with bounded energy (up to a logarithmic factor). In 3D, however, the pointwise values of v for constant energy of v can grow quickly as $h \to 0$, so interpolation can no longer be used. Overlapping methods ([8]) use decomposition into v_i's by a partition of unity on overlapping subdomains Ω_i, and they carry over to 3D; only the interpolation from the values of v needs to be replaced by a method that is energy stable in 3D, such as interpolation from averages or L^2 projection. In some non-overlapping methods, however, the functions v_i are defined in such way that they are zero at the nodes that define the values of v_0, e.g., [2]. Then a straightforward extension of the method to 3D forces v_0 to be linear interpolant from pointwise values of v. [3] resolved this problem by redefining the coarse bilinear form a_0 so that $a_0(u, u) = \sum_{i=1}^{N} \min_{c_i} \int_\Omega |\nabla u - c_i|^2$; cf., [15] for a generalization to elasticity and an algebraic explanation. The coarse space degrees of freedom are one number c_i per substructure, thus the coarse space can be thought of as piecewise constant (Fig. 1 right). Piecewise constant coarse space used with the original bilinear form $a(\cdot, \cdot)$ results in aggregation methods [27]. Reference [7] defined the interpolant by discrete harmonic functions, which have lower energy than piecewise linear functions.

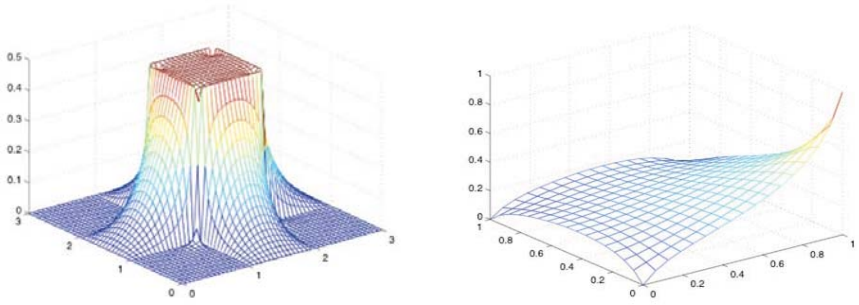

Fig. 2. *Left*: BDD coarse basis function, with support on one substructure and adjacent ones. *Right*: Coarse function on one substructure of BDD for plates, and BDDC (reproduced from [20]).

4 Balancing Domain Decomposition (BDD) and FETI

The BDD method was created by [16] by adding a special coarse space to the Neumann–Neumann (NN) method from [5]. The NN method uses the additive preconditioner with the local forms a_i from (6) and no coarse space. In the NN method, the local forms are generally singular and the local problems (3) are not consistent. The BDD preconditioner applies multiplicatively a coarse correction based on a known superspace Z_i of the local nullspace and designed so that the right-hand side in (3) is orthogonal to Z_i. Since the nullspace of a_i is contained in Z_i, (3) is now guaranteed to be consistent. The coarse space is obtained by averaging between adjacent substructures and extending the functions from the substructure boundaries in the interior with minimal energy (i.e., as discrete harmonic). A basis function of the resulting coarse space is in Fig. 2 right. Of course, for elasticity, rigid body modes are used rather than constants, giving 6 coarse degrees of freedom per substructure in 3D.

BDD is completely *algebraic*. It can be implemented only by calls to subdomain matrix-vector multiplication and by access to a basis of the local space Z_i (such as the rigid body modes written in terms of the degrees of freedom). This made possible a black-box type application of BDD to mixed finite elements in [4]: the substructure matrix-vector multiply becomes the mapping of pressure on substructure faces to the velocity in the normal direction. (Some components of other methods can be generated algebraically also; e.g., overlapping Schwarz methods are used as smoothers in adaptive algebraic multigrid in [24].)

BDD with the spaces Z_i given by constants or rigid body modes is not suitable for 4th order problems (such as plate bending), because the tearing at corners has high energy – the trace norm associated with 4th order problem is the Sobolev norm $H^{3/2}$. But empowering BDD by enriching the coarse space was envisioned already in [16], and all that was needed was to enlarge the spaces Z_i so that after the coarse

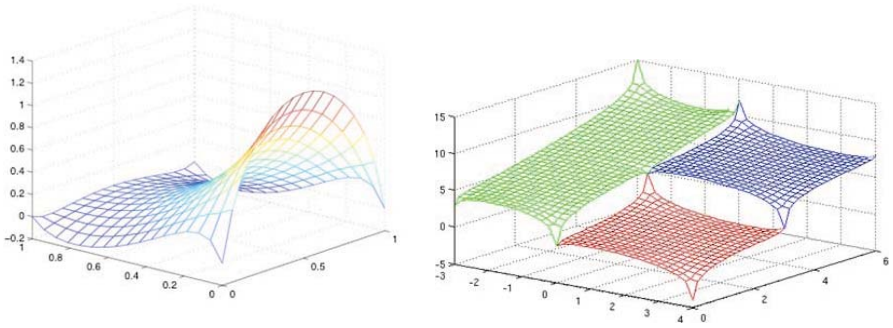

Fig. 3. *Left*: coarse function in BDDC for edge average degree of freedom on one substructure. *Right*: BDD for plates and BDDC coarse space with vertex degrees of freedom on several substructures (courtesy of Marta Čertíková and Jakub Šístek).

correction, the error is zero at corners, thus the tears across the corners do not matter. In [13], such Z_i consists of functions determined by their values at the corners of the substructure, and by having minimal energy (Fig. 2 right).

The FETI method by [11] runs in the dual space of Lagrange multipliers and it uses a coarse space constructed from the exact nullspace of the local problems (3). In the scalar case, this is the space of discontinuous piecewise constant functions (Fig. 1 right), and of piecewise rigid body modes for elasticity. Since the dual space (after elimination of the interior) is equipped with the $H^{-1/2}$ norm, jumps between subdomains do not cause a large energy increase. Like BDD, FETI is completely algebraic, which is why the two methods have become popular in practice. [23] generalized FETI to deal with 4th order problems analogously as in BDD, but the resulting method, called FETI-2, was quite complicated. Since the basic algebra of FETI relies on the exact nullspace of the local problems, the added coarse functions had to be in a new coarse space of their own, with the additional components of the coarse correction wrapped around the original FETI method. Eventually, FETI-2 was superseded by FETI-DP.

A Neumann–Neumann method, also called balancing but somewhat different from BDD, was developed in [9]. This method uses the same coarse space as BDD, but additively, and it takes care of the singularity in the local problems by adding small numbers to the diagonal. To guarantee optimal condition bounds, a modification of the form a_0 is needed. This method is not algebraic in the same sense as BDD or FETI, i.e., relying on the matrices only.

5 BDDC and FETI-DP

A satisfactory extension of FETI and BDD to 4th order problems came only with FETI-DP by [10] and BDDC by [6]. These methods are based on identical components and have the same spectrum. except possibly for the eigenvalues equal to zero

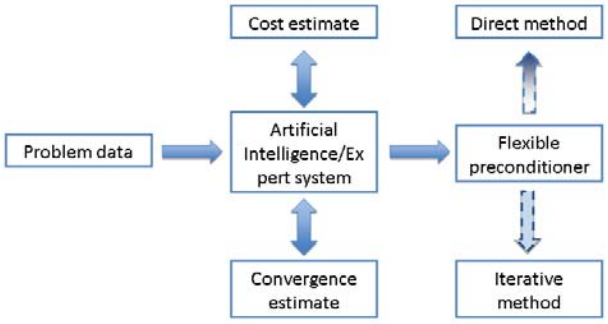

Fig. 4. Intelligent iterative method. Adapted from [17].

and one [21], so we can discuss BDDC only. The coarse space consists of functions given by their values of coarse degrees of freedom and energy minimal on every substructure independently. For coarse degrees of freedom given by values on substructure corners, this is the same coarse space as in BDD for plates in [13] (Fig. 2 right, Fig. 3 right), and the substructure spaces W_i are also the same. The new feature of BDDC is that the coarse correction is additive, not multiplicative, resulting in a sparser coarse matrix [20]. In 3D, FETI-DP and BDDC require additional degrees of freedom for optimal convergence, namely averages over faces or edges [10, 12], cf., Fig. 3 left for a visualization in 2D.

6 Adaptive Methods by Enriching the Coarse Space

Enlarging the coarse space is a powerful but expensive tool. When the coarse space is the whole space, domain decomposition turns into a direct solver. So, adding suitable functions to the coarse space adaptively can yield a robust method, which is fast on easy problems, but does not fail on hard ones (Fig. 4). In [19], the coarse space in the p-version finite element method consists of linear functions when all is good, quadratic functions when things get worse, all function in one direction in the case of strong anisotropy, up to all functions when the heuristic gives up. In [24], a similar methodology was applied in algebraic multigrid. In [22] and in the companion paper [25] in this volume, the coarse space in BDDC is enriched by adaptively selected linear combinations of basis functions on substructure faces.

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