Coupled FE/BE Formulations for the Fluid–Structure Interaction

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Summary. We present several coupled finite and boundary element formulations for the vibro-acoustic simulation of completely immersed bodies such as submarines. All formulations are based on the different use of standard boundary integral equations. In addition to the well known symmetric coupling we discuss two different approaches which are based on the weakly singular boundary integral equation only.

1 Introduction

The simulation of the sound radiation of time-harmonic vibrating elastic structures is of main interest in many applications with the acoustic fluid being air or water. Relevant applications are the sound radiation of passenger car bodies, where the acoustic region is bounded, of partially immersed bodies such as ships, where the acoustic region is a half space, or of completely immersed bodies such as submarines with a full space acoustic region.

In this paper, we consider coupled finite and boundary element formulations for a direct simulation of a three-dimensional time-harmonic vibrating structure in a surrounding fluid [3, 7]. In particular, the time-harmonic vibrating structure in Ω_S is modeled by the Navier equations in the frequency domain,

$$-\varrho_S \omega^2 u(x) - \mu \Delta u(x) - (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) = f(x) \quad \text{for } x \in \Omega_S, \qquad (1)$$

where λ and μ are the Lamé parameters, ϱ_S is the density of the structure, ω is the frequency, and u is the unknown displacement field. Note that Ω_S is in general a bounded, multiple connected domain with an interior boundary Γ_N where Neumann boundary conditions

$$t(x) = \lambda \operatorname{div} u(x)\, n_x + 2\mu \frac{\partial}{\partial n_x} u(x) + \mu\, n_x \times \operatorname{curl} u(x) \;=\; g(x) \quad \text{for } x \in \Gamma_N \quad (2)$$

are considered, and with an exterior boundary Γ where transmission conditions are formulated for the coupling with the surrounding fluid. In particular, in the low frequency regime we use the Laplace equation

$$-\Delta p(x) = 0 \quad \text{for } x \in \Omega_F \tag{3}$$

to describe the acoustic pressure p in the unbounded domain Ω_F surrounding the structure in Ω_S. Note that p has to satisfy a radiation condition at infinity,

$$p(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to \infty.$$

In addition to the partial differential equations (1) and (3) and the Neumann boundary conditions (2) we consider the transmission conditions on the interface $\Gamma = \overline{\Omega}_F \cap \overline{\Omega}_S$,

$$q(x) = \frac{\partial}{\partial n_x} p(x) = \varrho_F \omega^2 [u(x) \cdot n_x], \quad t(x) = -p(x) n_x \quad \text{for } x \in \Gamma, \tag{4}$$

where ϱ_F is the density of the fluid, and n_x is the exterior normal vector with respect to Ω_S.

The aim of this paper is to derive and to discuss different coupled finite and boundary element formulations for the solution of the transmission boundary value problem (1), (2), (3) and (4). Besides an efficient solution of the direct problem a main interest in applications is the determination of critical frequencies ω which correspond to eigenvalues of the coupled problem with homogeneous data, see, e.g., [1, 2] and the references given therein.

2 Integral Equations and Variational Formulations

The solution of the Laplace equation (3) in the unbounded exterior domain Ω_F is given by the representation formula for $x \in \Omega_F$, see, e.g., [5],

$$p(x) = -\frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} q(y) ds_y + \frac{1}{4\pi} \int_\Gamma \frac{(x-y, n_y)}{|x-y|^3} p(y) ds_y. \tag{5}$$

From (5) we obtain a system of boundary integral equations given as

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{6}$$

For the structural part we introduce the bilinear forms

$$a_S(u,v) := \int_{\Omega_S} \sum_{i,j=1}^{3} \sigma_{ij}(u(x)) \overline{e_{ij}(v(x))}\, dx, \quad \langle u, v \rangle_{\Omega_S} := \int_{\Omega_S} u(x) \cdot \overline{v(x)}\, dx,$$

for $u, v \in [H^1(\Omega_S)]^3$ as well as the duality pairing, for $t \in [H^{-1/2}(\Gamma)]^3$,

$$\langle t, v \rangle_\Gamma := \int_\Gamma t(x) \cdot \overline{v(x)}_{|\Gamma}\, ds_x.$$

The variational formulation of the structural problem (1) and (2) is to find the displacement field $u \in [H^1(\Omega_S)]^3$ such that

$$a_S(u,v) - \varrho_S\omega^2\langle u,v\rangle_{\Omega_S} - \langle t,v\rangle_\Gamma = F(v) \tag{7}$$

is satisfied for all $v \in [H^1(\Omega_S)]^3$, where the linear form of the right hand side is given by

$$F(v) := \int_{\Omega_S} f(x)\cdot\overline{v(x)}\,dx + \int_{\Gamma_N} g(x)\cdot\overline{v(x)}_{|\Gamma}\,ds_x\,.$$

By using the second transmission boundary condition in (4), we can rewrite the variational formulation (7) as

$$a_S(u,v) - \varrho_S\omega^2\langle u,v\rangle_{\Omega_S} + \langle p, v\cdot n\rangle_\Gamma = F(v) \quad \text{for all } v \in [H^1(\Omega_S)]^3, \tag{8}$$

where in addition to $u \in [H^1(\Omega_S)]^3$ also $p \in H^{1/2}(\Gamma)$ is unknown. By using the boundary integral equations as given in (6), and by using the first transmission condition in (4), we will derive a second variational equation to link the two unknowns u and p. Since such an approach is not unique, we will discuss several possible methodologies.

3 Symmetric Coupling of Finite and Boundary Elements

When inserting the first boundary integral equation as given in (6) into the variational problem (8), and by using the first transmission condition in (4), i.e.,

$$p(x) = \frac{1}{2}p(x) + (Kp)(x) - (Vq)(x), \quad q(x) = \varrho_F\omega^2[u(x)\cdot n_x] \quad \text{for } x \in \Gamma,$$

we have to find $(u,p) \in [H^1(\Omega_S)]^3 \times H^{1/2}(\Gamma)$ satisfying

$$a_S(u,v) - \varrho_S\omega^2\langle u,v\rangle_{\Omega_S} - \varrho_F\omega^2\langle V[u\cdot n], v\cdot n\rangle_\Gamma + \langle(\frac{1}{2}I+K)p, v\cdot n\rangle_\Gamma = F(v) \tag{9}$$

for all $v \in [H^1(\Omega_S)]^3$. In addition we consider the weak formulation of the second, hypersingular, boundary integal equation in (6). Together with the first transmission condition in (4), this gives

$$\langle Dp,\pi\rangle_\Gamma + \varrho_F\omega^2(\frac{1}{2}I + K')[u\cdot n],\pi\rangle_\Gamma \;=\; 0 \quad \text{for all } \pi \in H^{1/2}(\Gamma). \tag{10}$$

From the hypersingular boundary integral equation (10) as well as from the coupled variational form (9) we conclude that the acoustic pressure p is only unique up to constants. Hence, to fix the constants we may introduce the modified hypersingular boundary integral operator via the bilinear form

$$\langle \widetilde{D}p,\pi\rangle_\Gamma \;:=\; \langle Dp,\pi\rangle_\Gamma + \langle p,1\rangle_\Gamma\langle\pi,1\rangle_\Gamma \quad \text{for all } p,\pi \in H^{1/2}(\Gamma).$$

Instead of (10) we now consider the modified variational problem

$$\langle \widetilde{D}p, \pi\rangle_\Gamma + \varrho_F \omega^2 \langle (\frac{1}{2}I + K')[u\cdot n], \pi\rangle_\Gamma \,=\, 0 \quad \text{for all } \pi \in H^{1/2}(\Gamma), \tag{11}$$

which implies the related scaling of the pressure by $\langle p, 1\rangle_\Gamma \,=\, 0$. To summarize, we have to find $(u,p) \in [H^1(\Omega_S)]^3 \times H^{1/2}(\Gamma)$ from the coupled variational problem (9) and (11). Since the modified hypersingular boundary integral operator $\widetilde{D}$ is $H^{1/2}(\Gamma)$–elliptic, we obtain from (11) the representation

$$p \,=\, -\varrho_F \omega^2 \widetilde{D}^{-1}(\frac{1}{2}I + K')[u\cdot n],$$

and therefore the continuous Schur complement problem to find $u \in [H^1(\Omega_S)]^3$ such that

$$a_S(u,v) - \omega^2 \Big[\varrho_S \langle u, v\rangle_{\Omega_S} + \varrho_F \langle T[u\cdot n], v\cdot n\rangle_\Gamma \Big] = F(v) \tag{12}$$

for all $v \in [H^1(\Omega_S)]^3$. Note that

$$T \,:=\, V + (\frac{1}{2}I + K)\widetilde{D}^{-1}(\frac{1}{2}I + K') \,:\, H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{13}$$

is the symmetric and $H^{-1/2}(\Gamma)$-elliptic representation of the Poincaré–Steklov operator realizing the Neumann to Dirichlet map which is related to the Neumann boundary value problem of the Laplace equation in the unbounded exterior domain Ω_F. As a direct consequence of the mapping properties of all involved operators, we can formulate the following result.

Lemma 1. *If ω^2 is not an eigenvalue of the eigenvalue problem*

$$a_S(u,v) \,=\, \lambda \Big[\varrho_S \langle u, v\rangle_{\Omega_S} + \varrho_F \langle T[u\cdot n], v\cdot n\rangle_\Gamma \Big] \quad \textit{for all } v \in [H^1(\Omega_S)]^3,$$

then there exists a unique solution of the variational problem (12), and therefore of the coupled variational problem (9) and (11).

Next we consider a Galerkin discretization of the coupled variational formulation (9) and (11). Let $S_h^1(\Omega_S) \subset H^1(\Omega_S)$ be a conformal finite element space of, e.g., piecewise linear and continuous basis functions with respect to some admissible finite element mesh $\Omega_{S,h}$, and let $S_h^1(\Gamma)$ be some boundary element ansatz space of, e.g., piecewise linear and continuous basis functions which can be defined independently of $S_h^1(\Omega_S)$. The Galerkin discretization of the coupled variational problem (9) and (11) results in the linear system

$$\begin{pmatrix} K_S - \varrho_S\omega^2 M_S - \varrho_F\omega^2 C^\top V_h C & C^\top(\frac{1}{2}M_h + K_h) \\ (\frac{1}{2}M_h^\top + K_h^\top)C & \frac{1}{\varrho_F\omega^2}\widetilde{D}_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix} \tag{14}$$

where K_S and M_S are the finite element stiffness and mass matrices, respectively. $\widetilde{D}_h$ is the Galerkin matrix of the modified hypersingular boundary integral operator

$\widetilde{D}$. The matrix C describes the basis transformation of a piecewise linear and continuous vector function u_h to a scalar piecewise linear but discontinuous function $u_h \cdot n$ when considering a polygonal boundary mesh Γ_h. Note that V_h is the Galerkin discretization of the single layer potential V when using piecewise linear but discontinuous basis functions, while K_h and M_h are the Galerkin boundary element matrices of the double layer potential K and of the identity.

Since the Galerkin discretization $\widetilde{D}_h$ of the modified hypersingular boundary integral operator $\widetilde{D}$ is invertible, the Schur complement system of (14) is given by

$$\Big(K_S - \omega^2 \Big[\varrho_S M_S + \varrho_F C^\top [V_h + (\frac{1}{2}M_h + K_h)\widetilde{D}_h^{-1}(\frac{1}{2}M_h^\top + K_h^\top)]C\Big]\Big)\underline{u} = \underline{f}. \quad (15)$$

As in the continuous case, see (12), we conclude unique solvability of the Schur complement system (14), if ω^2 is not an eigenvalue of the algebraic eigenvalue problem

$$K_S \underline{u} = \lambda \Big(\varrho_S M_S + \varrho_F C^\top [V_h + (\frac{1}{2}M_h + K_h)\widetilde{D}_h^{-1}(\frac{1}{2}M_h^\top + K_h^\top)]C\Big)\underline{u} \quad (16)$$

which is the discrete counterpart of the eigenvalue problem as considered in Lemma 1. Note that

$$T_h \,=\, V_h + (\frac{1}{2}M_h + K_h)\widetilde{D}_h^{-1}(\frac{1}{2}M_h^\top + K_h^\top)$$

is a symmetric boundary element approximation of the Poincaré–Steklov operator as defined in (13).

4 Nonsymmetric Finite and Boundary Element Coupling

Instead of the symmetric coupling of finite and boundary elements, the use of the weakly singular boundary integral equation is very popular in applications in engineering and in industry. This is due to the use of the single layer potential V and the double layer potential K only. Hence we will discuss related formulations which also allow the use of simpler collocation methods for the boundary element discretization.

For the non-sysmmetric coupling we consider two different combinations of the first boundary integral equation as given in (6), of the first transmission condition as given in (4), and of the variational formulation (8).

4.1 A Second Kind Boundary Integral Equation Approach

Inserting the first transmission condition of (4) into the first boundary integral equation in (6) gives the second kind boundary integral equation

$$(\frac{1}{2}I - K)p = -Vq = -\varrho_F \omega^2 V[u \cdot n] \quad \text{on } \Gamma. \quad (17)$$

Since $\frac{1}{2}I - K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible, see, e.g. [6], we obtain

$$p = -\varrho_F \omega^2 (\frac{1}{2}I - K)^{-1} V[u \cdot n] = -\varrho_F \omega^2 T[u \cdot n],$$

where

$$T := (\frac{1}{2}I - K)^{-1} V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is a second representation of the Poincaré–Steklov operator as introduced in (13). From (8) we obtain the variational formulation: find $u \in [H^1(\Omega_S)]^3$ such that

$$a_S(u,v) - \omega^2 \Big[\varrho_S \langle u, v \rangle_{\Omega_S} + \varrho_F \langle T[u \cdot n], v \cdot n \rangle_\Gamma \Big] = F(v)$$

for all $v \in [H^1(\Omega_S)]^3$, which corresponds to the variational problem (12). However, the Galerkin discretization of the variational formulation (8) and of the boundary integral equation (17) now results in the different linear system

$$\begin{pmatrix} K_S - \varrho_S \omega^2 & C^\top \\ -\overline{V}C & \frac{1}{\varrho_F \omega^2}[\frac{1}{2}\overline{M}_h - \overline{K}_h] \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}. \tag{18}$$

Note that the test functions to be used in the Galerkin discretization of the second kind boundary integral equation (17) are the piecewise linear and continous basis functions of $S_h^1(\Gamma)$ as used for the approximation of the pressure p. Although, to the best of our knowledge, there is still no rigorous stability analysis available for general Lipschitz boundaries Γ, the elimination of $\underline{p}$ results in the Schur complement system

$$\Big(K_S - \omega^2 \Big[\varrho_S M_S + \varrho_F C^\top (\frac{1}{2}\overline{M}_h - \overline{K}_h)^{-1} \overline{V}_h C \Big] \Big) \underline{u} = \underline{f}, \tag{19}$$

which is uniquely solvable if ω^2 is not an eigenvalue of the related discrete eigenvalue problem

$$K_S \underline{u} = \lambda \Big(\varrho_S M_S + \varrho_F C^\top (\frac{1}{2}\overline{M}_h - \overline{K}_h)^{-1} \overline{V}_h C \Big) \underline{u}.$$

Note that

$$T_h = (\frac{1}{2}\overline{M}_h - \overline{K}_h)^{-1} \overline{V}_h$$

is a non-symmetric boundary element approximation of the Poincaré–Steklov operator T which is based on an approximate solution of the second kind boundary integral equation (17).

4.2 A First Kind Boundary Integral Equation Approach

Since the single layer potential $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible, we obtain from the first boundary integral equation of (6), and by using the first transmission boundary condition of (4), the relation

$$q = V^{-1}(-\frac{1}{2}I + K)p = -Sp = \varrho_F \omega^2 [u \cdot n] \quad \text{on } \Gamma,$$

where

$$S = V^{-1}(\frac{1}{2}I - K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is the Steklov–Poincaré operator describing the Dirichlet to Neumann map which is related to the Laplace equation in the exterior domain. We therefore obtain

$$p = -\varrho_F \omega^2 S^{-1}[u \cdot n] = -\varrho_F \omega^2 T[u \cdot n], \quad T = S^{-1} = (\frac{1}{2}I - K)^{-1}V,$$

which obviously corresponds to the nonsymmetric approach which is based on the solution of the second kind boundary integral equation (17). Hence, unique solvability of the continuous problem follows as above. However, for a finite and boundary element discretization we consider the coupled system based on the variational formulation (8), the first boundary integral equation in (6), and the first transmission condition in (4). The Galerkin discretization of the coupled system then results in the linear system

$$\begin{pmatrix} K_S - \varrho_S \omega^2 M_S & C^\top & \\ -\varrho_F \omega^2 C & & M_h^\top \\ & \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \\ \underline{q} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \\ \underline{0} \end{pmatrix}. \tag{20}$$

Since the discrete single layer potential V_h is invertible, after elimination of $\underline{q}$ we obtain the reduced system

$$\begin{pmatrix} K_S - \varrho_S \omega^2 M_S & C^\top \\ C & \frac{1}{\varrho_F \omega^2} M_h^\top V_h^{-1}(\frac{1}{2}M_h - K_h) \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}. \tag{21}$$

Note that

$$S_h := M_h^\top V_h^{-1}(\frac{1}{2}M_h - K_h)$$

is a non-symmetric representation of the Steklov–Poincaré operator. For stability we need to assume an appropriate choice of the boundary element spaces for an approximation of p and q, respectively, see, e.g. [4]. If S_h is invertible, the Schur complement system of (21),

$$\Big(K_S - \omega^2 \Big[\varrho_S M_S + \varrho_F C^\top S_h^{-1} C\Big]\Big)\underline{u} = \underline{f},$$

is uniquely solvable, if ω^2 is not an eigenvalue of the related eigenvalue problem

$$K_S \underline{u} = \lambda \Big(\varrho_S M_S + \varrho_F C^\top S_h^{-1} C\Big)\underline{u}.$$

5 Conclusions

The symmetric coupling of finite and boundary element methods as described in Sect. 3 admits a complete error and stability analysis, but requires the use of the hypersingular boundary integral operator D, and a Galerkin approach for the discretization of the boundary integral equations. In contrast, both nonsymmetric formulations as given in Sect. 4 are based on the single and double layer potential operators V and K only, and allow the use of a collocation scheme for a boundary element discretization.

Challenging problems appear in the construction of efficient and robust preconditioning strategies for the solution of the resulting linear systems, in particular when considering the Helmholtz equation instead of the Laplace equation when simulating the sound radiation in the mid frequency regime. The issue of appropriate eigensolvers for the determination of critical frequencies is also of interest. For preliminary and promising results, see [1, 2, 7].

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