Weighted Poincaré Inequalities and Applications in Domain Decomposition

Clemens Pechstein[1] and Robert Scheichl[2]

[1] Institute of Computational Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria, Clemens.Pechstein@jku.at,
supported by the Austrian Sciences Fund (FWF) under grant P19255
[2] Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK, R.Scheichl@bath.ac.uk

Summary. Poincaré type inequalities play a central role in the analysis of domain decomposition and multigrid methods for second-order elliptic problems. However, when the coefficient varies within a subdomain or within a coarse grid element, then standard condition number bounds for these methods may be overly pessimistic. In this short note we present new weighted Poincaré type inequalities for a class of piecewise constant coefficients that lead to sharper bounds independent of any possible large contrasts in the coefficients.

1 Introduction

Poincaré type inequalities play a central role in the analysis of domain decomposition (DD) methods for finite element discretisations of elliptic PDEs of the type

$$-\nabla \cdot (\alpha \, \nabla u) = f \,. \tag{1}$$

In many applications the coefficient α in (1) is discontinuous and varies over several orders of magnitude throughout the domain in a possibly very complicated way. Standard analyses of DD methods for (1) that use classical Poincaré type inequalities will often lead to pessimistic bounds. Two examples are the popular two-level overlapping Schwarz and FETI. If the subdomain partition can be chosen such that α is constant (or almost constant) on each subdomain as well as in each element of the coarse mesh (for two-level methods), then it is possible to prove bounds that are independent of the coefficient variation (cf. [2, 7, 14]). However, if this is not possible and the coefficient varies strongly within a subdomain, then the classical bounds depend on the local variation of the coefficient, which may be overly pessimistic in many cases. To obtain sharper bounds in some of these cases, it is possible to refine the standard analyses and use Poincaré inequalities on annulus type boundary layers of each subdomain [5, 8, 10, 12, 15], or weighted Poincaré type inequalities [4, 9, 13]. See also [1, 3, 6, 11, 16].

In this short note we want to collect and expand on the results in [4, 9] and present a new class of weighted Poincaré-type inequalities for a rather general class of piecewise constant coefficients. Due to space restrictions we have to refer the interested reader to [5, 8, 9, 13], to see where exactly these new inequalities can be used in the analysis of FETI and two-level Schwarz methods.

2 Weighted Poincaré Inequalities

Let D be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. For simplicity we only consider piecewise constant coefficient functions α with respect to a non-overlapping partitioning $\{Y_\ell : \ell = 1, \ldots, n\}$ of D into open, connected Lipschitz polygons (polyhedra), i.e. $\alpha|_{Y_\ell} \equiv \alpha_\ell \equiv$ const. The results generalise straightforwardly to more general coefficients that vary mildly within each of the regions Y_ℓ.

Definition 1. *The region $P_{\ell_1, \ell_s} := (\overline{Y}_{\ell_1} \cup \overline{Y}_{\ell_2} \cup \cdots \cup \overline{Y}_{\ell_s})^\circ$ is called a type-m quasi-monotone path from Y_{ℓ_1} to Y_{ℓ_s}, if*

(i) for $i = 1, \ldots, s-1$ the subregions $\overline{Y}_{\ell_i}$ and $\overline{Y}_{\ell_{i+1}}$ share a common m-dimensional manifold X_i

(ii) $\alpha_{\ell_1} \leq \alpha_{\ell_2} \leq \cdots \leq \alpha_{\ell_s}$.

Definition 2. *Let $X^* \subset \overline{D}$ be a manifold of dimension m, with $0 \leq m < d$. The coefficient distribution α is called type-m X^*-quasi-monotone on D, if for all $\ell = 1, \ldots, n$ there exists an index k such that $X^* \subset \overline{Y}_k$ and such that there is a type-m quasi-monotone path $P_{\ell,k}$ from Y_ℓ to Y_k.*

Definition 3. *Let $\Gamma \subset \partial D$. The coefficient distribution α is called type-m Γ-quasi-monotone on D, if for all $\ell = 1, \ldots, n$ there exists a manifold $X_\ell^* \subset \Gamma$ of dimension m and an index k such that $X_\ell^* \subset \partial Y_k$ and such that there is a type-m quasi-monotone path $P_{\ell,k}$ from Y_ℓ to Y_k.*

Note that the above definitions generalize the notion of quasi-monotone coefficients introduced in [2]. Definition 2 will be used to formulate weighted (discrete) Poincaré type inequalities, whereas Definition 3 will be used in weighted (discrete) Friedrichs inequalities. In Fig. 1 we give some examples of coefficient distributions that satisfy Definition 2.

To formulate our results we define for any $u \in H^1(D)$ the average

$$\overline{u}^{X^*} := \frac{1}{|X^*|} \int_{X^*} u \, ds \quad \text{if } m > 0, \qquad \overline{u}^{X^*} := u(X^*) \quad \text{if } m = 0,$$

as well as the weighted norm and seminorm

$$\|u\|_{L^2(D),\alpha} := \Big(\int_D \alpha \, |u|^2 \, dx \Big)^{1/2} \quad \text{and} \quad |u|_{H^1(D),\alpha} := \Big(\int_D \alpha \, |\nabla u|^2 \, dx \Big)^{1/2}.$$

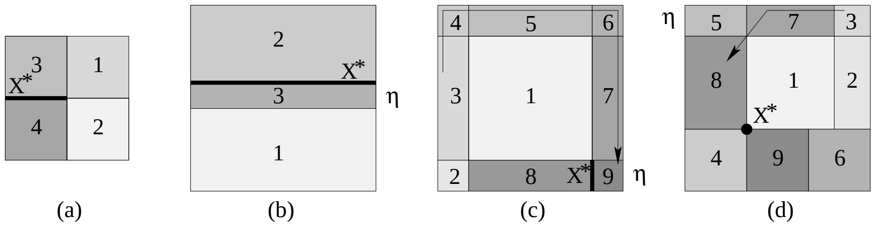

Fig. 1. Examples of quasi-monotone coefficients. The numbering of the regions is according to the relative size of the coefficients on these regions with the smallest coefficient in region Y_1. Note that the first case is quasi-monotone in the sense of [2], but the other three are not. The first three examples are type-1. The last example is type-0. The manifold X^* is shown in each case, together with a typical path in some of the cases.

Lemma 1 (weighted Poincaré inequality). *Let the coefficient* α *be type-*$(d-1)$ X^**-quasi-monotone on* D *with the* $(d-1)$*-dimensional manifold* X^**. For each index* $\ell = 1, \ldots, n$*, let* $P_{\ell,k}$ *be the path in Definition 2 with* $X^* \subset \overline{Y}_k$*, and let* $C^P_{\ell,k} > 0$ *be the best constant in the inequality*

$$\|u - \overline{u}^{X^*}\|^2_{L^2(Y_\ell)} \leq C^P_{\ell,k} \operatorname{diam}(D)^2 \, |u|^2_{H^1(P_{\ell,k})} \quad \text{for all } u \in H^1(P_{\ell,k}). \tag{2}$$

Then there exists a constant $C^P \leq \sum_{\ell=1}^n C^P_{\ell,k}$ *independent of* α *and* $\operatorname{diam}(D)$ *such that*

$$\|u - \overline{u}^{X^*}\|^2_{L^2(D),\alpha} \leq C^P \operatorname{diam}(D)^2 \, |u|^2_{H^1(D),\alpha} \quad \text{for all } u \in H^1(D).$$

Proof. Let us fix one of the subregions Y_ℓ and suppose without loss of generality that $\int_{X^*} u \, ds = 0$ and that $\operatorname{diam}(D) = 1$. Due to the assumption on α, we have $\|u\|^2_{L^2(Y_\ell),\alpha} = \alpha_\ell \|u\|^2_{L^2(Y_\ell)}$. Combining this identity with inequality (2) and using that the coefficients are monotonically increasing in the path from Y_ℓ to Y_k, we obtain

$$\|u\|^2_{L^2(Y_\ell),\alpha} \leq C^P_{\ell,k} \, \alpha_\ell \, |u|^2_{H^1(P_{\ell,k})} \leq C^P_{\ell,k} \, |u|^2_{H^1(P_{\ell,k}),\alpha} \leq C^P_{\ell,k} \, |u|^2_{H^1(D),\alpha} .$$

The proof is completed by adding up the above estimates for $\ell = 1, \ldots, n$.

Remark 1. Obviously, inequality (2) follows from the standard Poincaré type inequality $\|u - \overline{u}^{X^*}\|^2_{L^2(P_{\ell,k})} \leq C \, |u|_{H^1(P_{\ell,k})}$ for all $u \in H^1(P_{\ell,k})$, with some constant C depending on $P_{\ell,k}$ and on X^*. However, this may lead to a sub-optimal constant. In general, the constants $C^P_{\ell,k}$ depend on the choice of the manifold X^*, as well as on the number, shape, and size of the subregions Y_ℓ. In Sect. 3, we give a bound of $C^P_{\ell,k}$ in terms of local Poincaré constants on the individual subregions Y_ℓ to make this dependency more explicit.

On the other hand, if X^* is a manifold of dimension less than $d-1$ (i.e. an edge or a point), inequality (2) does not hold for all functions $u \in H^1(D)$. However, there is a discrete analogue for finite element functions which holds under some geometric assumptions on the subregions Y_ℓ, cf. [14, Sect. 4.6].

Let $\{\mathcal{T}_h(D)\}$ be a family of quasi-uniform, simplicial triangulations of D with mesh width h. By $V^h(D)$ we denote the space of continuous piecewise linear functions with respect to the elements of $\mathcal{T}_h(D)$. Note that we do not prescribe any boundary conditions. We further assume that the fine mesh $\mathcal{T}_h(D)$ resolves the interfaces between the subregions Y_ℓ.

Assumption 1 (cf. [14, Assumption 4.3]) There exists a parameter η with $h \le \eta \le \operatorname{diam}(D)$ such that each subregion Y_ℓ is the union of a few simplices of diameter η, and the resulting coarse mesh is globally conforming on all of D.

Before stating the next lemma, we define the function

$$\sigma^\delta(x) := \begin{cases} (1+\log(x)) & \text{for } \delta = 2, \\ x & \text{for } \delta = 3. \end{cases} \tag{3}$$

Lemma 2 (weighted discrete Poincaré inequality). *Let Assumption 1 hold and let α be type-m X^*-quasi-monotone on D with the manifold X^* having dimension $m < d-1$. If $m = 1$, assume furthermore that X^* is an edge of the coarse triangulation in Assumption 1. For each $\ell = 1, \ldots, n$, let $P_{\ell,k}$ be the path in Definition 2 with $X^* \subset \overline{Y}_k$ and let $C^{P,m}_{\ell,k} > 0$ be the best constant independent of h such that*

$$\|u - \overline{u}^{X^*}\|^2_{L^2(Y_\ell)} \le C^{P,m}_{\ell,k}\, \sigma^{d-m}\big(\tfrac{\eta}{h}\big)\, \operatorname{diam}(D)^2\, |u|^2_{H^1(P_{\ell,k})} \quad \text{for all } u \in V^h(P_{\ell,k}). \tag{4}$$

Then, there exists a constant $C^{P,m} \le \sum_{\ell=1}^n C^{P,m}_{\ell,k}$, independent of h, of α, and of $\operatorname{diam}(D)$ such that

$$\|u - \overline{u}^{X^*}\|^2_{L^2(D),\alpha} \le C^{P,m}\, \sigma^{d-m}\big(\tfrac{\eta}{h}\big)\, \operatorname{diam}(D)^2\, |u|^2_{H^1(D),\alpha} \quad \text{for all } u \in V^h(D).$$

Proof. The proof is analogous to that of Lemma 1, but uses (4) instead of (2).

We remark that the existence of the constants $C^{P,m}_{\ell,k}$ fulfilling inequality (4) will follow from the results summarized in [14, Sect. 4.6] and from our investigation in Sect. 3. For simplicity, let us also define $\sigma^1 \equiv 1$ and $C^{P,d-1} := C^P$.

We would like to mention that similar inequalities than those in Lemmas 1 and 2 can also be proved, if u vanishes on part of the boundary of D. Here, we just display the case $m = d-1$. The generalisation to $m < d-1$ is straightforward and follows Lemma 2.

Lemma 3 (weighted Friedrichs inequality). *Let $\Gamma \subset \partial D$ and let α be type-$(d-1)$ Γ-quasi-monotone on D (according to Definition 3). Then there exists a constant $C^F = C^{F,d-1}$ independent of α and of $\operatorname{diam}(D)$ such that*

$$\|u\|^2_{L^2(D),\alpha} \le C^F \operatorname{diam}(D)^2\, |u|^2_{H^1(D),\alpha} \quad \text{for all } u \in H^1(D),\ u|_\Gamma = 0.$$

3 Explicit Dependence on Geometrical Parameters

In this section we will study the dependence of the constants $C^{P,m}_{\ell,k}$ (and consequently $C^{P,m}$) in the above lemmas on the choice of X^* and on the number, size and shape of the regions Y_ℓ (in particular the ratio $\mathrm{diam}(D)/\eta$). In [9, §3] the dependence on the geometry of the subregions is made more explicit. The lemmas presented there are in fact special cases of Lemmas 1 and 2 here.

First, we show that bounds for the constants $C^{P,m}_{\ell,k}$ can be obtained from inequalities on the individual subregions Y_ℓ. Secondly, we will look at a series of examples in two dimensions. For further 3D examples see [9].

Lemma 4. *Let α be type-m X^*-quasi-monotone on D with $0 \le m \le d-1$, and let P_{ℓ_1,ℓ_s} be any of the paths in Definition 2. If $m < d-1$, let Assumption 1 hold. If $m = 1$ and $d = 3$, assume additionally that X^* is an edge of the coarse triangulation. For each $i = 1, \dots, s$, let $C^{P,m}_{\ell_i}$ be the best constant, such that*

$$\|u - \overline{u}^X\|^2_{L^2(Y_{\ell_i})} \;\le\; C^{P,m}_{\ell_i}\, \sigma^{d-m}\big(\tfrac{\eta}{h}\big)\; \mathrm{diam}(Y_{\ell_i})^2\, |u|^2_{H^1(Y_{\ell_i})} \quad \text{for all } u \in V^h(Y_{\ell_i}), \tag{5}$$

where $X \subset \overline{Y}_{\ell_i}$ is any of the manifolds X_{i-1}, X_i or X^* in Definition 2 (as appropriate), cf. [14, Sect. 4.6]. Then

$$C^{P,m}_{\ell_1,\ell_s} \;\le\; 4\left\{\sum_{i=1}^{s} \frac{\mathrm{meas}(Y_{\ell_1})}{\mathrm{meas}(Y_{\ell_i})}\,\frac{\mathrm{diam}(Y_{\ell_i})^2}{\mathrm{diam}(D)^2}\, C^{P,m}_{\ell_i}\right\}.$$

If $m = d-1$ we can extend the result to the whole of H^1.

Proof. We give the proof for the case $m = d-1$. The other cases are analogous. For convenience let $X_s := X^*$. Then, telescoping yields

$$\|u - \overline{u}^{X^*}\|_{L^2(Y_{\ell_1})} \;\le\; \|u - \overline{u}^{X_1}\|_{L^2(Y_{\ell_1})} + \sum_{i=2}^{s} \sqrt{\mathrm{meas}(Y_{\ell_1})}\,\big|\overline{u}^{X_{i-1}} - \overline{u}^{X_i}\big|.$$

Due to (5), $\|u - \overline{u}^{X_1}\|_{L^2(Y_{\ell_1})} \le \sqrt{C^{P,m}_{\ell_1}}\,\mathrm{diam}(Y_{\ell_1})\,|u|_{H^1(Y_{\ell_1})}$, and for each i,

$$\begin{aligned}\big|\overline{u}^{X_{i-1}} - \overline{u}^{X_i}\big|^2 &\le \frac{2}{\mathrm{meas}(Y_{\ell_i})}\Big(\|\overline{u}^{X_{i-1}} - u\|^2_{L^2(Y_{\ell_i})} + \|u - \overline{u}^{X_i}\|^2_{L^2(Y_{\ell_i})}\Big)\\ &\le \frac{4}{\mathrm{meas}(Y_{\ell_i})}\, C^{P,m}_{\ell_i}\,\mathrm{diam}(Y_{\ell_i})^2\,|u|^2_{H^1(Y_{\ell_i})}.\end{aligned}$$

An application of Cauchy's inequality (in $\mathbb{R}^s$) yields the final result.

Let us look at some examples now. Firstly, if Assumption 1 holds with constant $\eta \gtrsim \mathrm{diam}(D)$ (e. g. in Fig. 1a), then $n = \mathcal{O}(1)$ and each path $P_{\ell,k}$ in Definition 2 contains $\mathcal{O}(1)$ subregions. If we choose X^* to be a vertex, edge or face of the coarse triangulation in Assumption 1, then by standard arguments $C^{P,m}_{\ell} = \mathcal{O}(1)$ in (5) for all $\ell = 1, \dots, n$. Hence, it follows from Lemma 4 that the constants $C^{P,m}$ in Lemmas 1–2 are all $\mathcal{O}(1)$.

Before we look to more complicated examples, which involve in particular long, thin regions, let us first derive two auxiliary results.

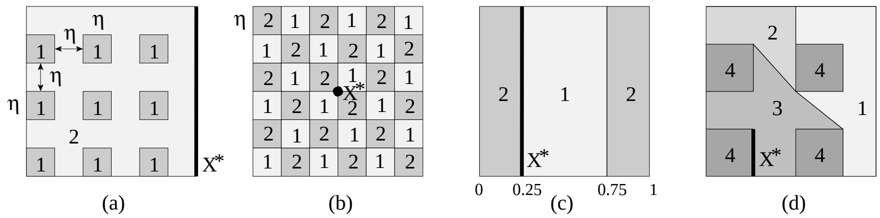

Fig. 2. More examples (with $\alpha_1 \ll \alpha_2$): The first two examples are quasi-monotone of type-1 and type-0, respectively. X^* is shown in each case. The examples in (**c**) and (**d**) are not quasi-monotone.

(i) The middle region Y_3 in Fig. 1b is long and thin if $\eta \ll \mathrm{diam}(Y_3)$. With X^* as given in the figure, one can show that (5) holds with $C_3^{P,1} = \mathcal{O}(1)$, independent of η and $\mathrm{diam}(Y_3)$. Note that $\mathrm{diam}(X^*) \simeq \mathrm{diam}(Y_3)$.

(ii) The region Y_8 in Fig. 1c has essentially the same shape, but here X^* has diameter $\eta \ll \mathrm{diam}(Y_8)$. Nevertheless, one can show that (5) holds with $X = X^*$ and $C_8^{P,1} = \mathcal{O}(1)$, independent of η and $\mathrm{diam}(Y_8)$. (This result can be obtained by sub-dividing Y_8 into small quadrilaterals of sidelength η and applying Lemma 4).

In Figs. 1 and 2, H denotes the sidelength of D (thus, $H \simeq \mathrm{diam}(D)$). We view η (if displayed) as a varying parameter $\le H$, with the other parameters fixed.

Fig. 1b. As just discussed, $C_{3,3}^{P,1} = C_3^{P,1} = \mathcal{O}(1)$. Similarly, $C_{2,2}^{P,1} = C_2^{P,1} = \mathcal{O}(1)$. To obtain $C_{1,3}^{P,1} = \mathcal{O}(1)$ we use $\|u - \overline{u}^{X^*}\|^2_{L^2(Y_1)} \le \|u - \overline{u}^{X^*}\|^2_{L^2(P_{1,3})}$ and apply a standard Poincaré inequality (rather than resorting to Lemma 4 which would yield a pessimistic bound). Hence, Lemma 1 holds with $C^{P,1} = \mathcal{O}(1)$.

Fig. 1c. Despite the fact that $C_1^{P,1} = \mathcal{O}(1)$ and $C_8^{P,1} = \mathcal{O}(1)$, the constant $C_{1,8}^{P,1}$ is not $\mathcal{O}(1)$: Since $\mathrm{diam}(Y_1) \sim H$, Lemma 4 yields

$$C_{1,8}^{P,1} \lesssim \frac{H^2}{H^2}\,\frac{H^2}{H^2} + \frac{H^2}{H\,\eta}\,\frac{H^2}{H^2} = \mathcal{O}\Big(\frac{H}{\eta}\Big).$$

We easily convince ourselves that this is the worst constant $C_{\ell,k}^{P,1}$, for all $\ell = 1,\dots,9$ (e. g., $C_{3,9}^{P,1} = \mathcal{O}(1)$), and so we obtain $C^{P,1} = \mathcal{O}(\frac{H}{\eta})$.

Fig. 1d. Here the coefficient is only type-0 quasi-monotone and so we cannot apply Lemma 1, but by applying Lemma 4 we find that $C_{7,8}^{P,0} = \mathcal{O}(1)$ and all the other constants are no worse. So in contrast to Case (c), we can show that the constant $C^{P,0}$ in Lemma 2 is $\mathcal{O}(1)$ in this case. The crucial difference is not that α is type-0 here, but that $\mathrm{diam}(Y_8) = \mathcal{O}(H)$ and $\mathrm{diam}(Y_9) = \mathcal{O}(H)$.

The examples in Fig. 2 are further, typical test cases used in the literature.

Fig. 2a. To obtain a sharp bound for $C^{P,1}$, it is better here to treat all the regions where $\alpha = \alpha_1$ as one single region Y_1, slightly modifying the proof of Lemma 1. Then $C_{1,2}^{P,1} = \mathcal{O}(1)$ (standard Poincaré on D). Due to a tricky overlapping argument that can be found in the Appendix of [8], $C_{2,2}^{P,1} = \mathcal{O}(1)$. Thus, $C^{P,1} = \mathcal{O}(1)$. Note that this is only possible if α takes the same values on all the inclusions. If there are

p distinct values in the inclusions, the constant $C^{P,1}$ depends (linearly) on p. This should be compared with one of the main results in [4], where a similar Poincaré inequality is proved with a constant depending on the number of inclusions.

Fig. 2b. For each region Y_ℓ we have $C_\ell^{P,0} = C_\square^{P,0} = \mathcal{O}(1)$. For a moment, let us restrict on the regions where the coefficient is α_1 and group them into $T := \frac{H}{2\eta}$ concentric layers starting from the two centre squares touching X^* where $\alpha = \alpha_1$. Obviously, for $t = 1, \dots, T$, layer t contains $2t - 2$ regions where $\alpha = \alpha_1$. Each region in layer t can be connected to one of the two centre squares by a type-0 quasi-monotone path of length t. By Lemma 4, $C_{\ell,k}^{P,0} \le 4 \sum_{j=1}^t \frac{\eta^2}{H^2} C_\square^{P,0} = 4\,t\,\frac{\eta^2}{H^2} C_\square^{P,0}$ for all the regions Y_ℓ in layer t where $\alpha = \alpha_1$. The same bound holds for the regions where $\alpha = \alpha_2$. Summing up these bounds over all regions and all layers, we obtain

$$C^{P,0} \;\le\; 2\sum_{t=1}^{T} (2t-2)\,4\,t\,\frac{\eta^2}{H^2}\,C_\square^{P,0} \;=\; 16\,\frac{\eta^2}{H^2}\,\frac{T^3 - T}{3} \;=\; \mathcal{O}\Big(\frac{H}{\eta}\Big).$$

Equivalently, as there are $n_\times = \mathcal{O}\big(\frac{H}{\eta}\big)^2$ crosspoints in this example, we have shown that $C^{P,0} = \mathcal{O}(\sqrt{n_\times})$. An enhanced bound of $\mathcal{O}\big((1+\log(H/\eta))^2\big)$ for $C^{P,0}$ in this example can be obtained using a multilevel argument, and will be proved in an upcoming paper.

Fig. 2c. α is not quasi-monotone in this case, and indeed Lemmas 1–3 do not hold. For example, if we choose X^* as shown, then it suffices to choose u to be the continuous function that is equal to $2(x_1 - \frac{1}{4})$ for $\frac{1}{4} \le x_1 \le \frac{3}{4}$ and constant otherwise, to obtain a counter example in $V^h(D) \subset H^1(D)$ that satisfies $\overline{u}^{X^*} = 0$. We have $\|u\|^2_{L^2(D),\alpha} = \frac{\alpha_1}{6} + \frac{\alpha_2}{4}$ and $|u|^2_{H^1(D),\alpha} = 2\alpha_1$, and so the constant $C^{P,1}$ in Lemma 1 blows up with the contrast $\frac{\alpha_2}{\alpha_1}$. It is impossible to find X^* such that Lemma 2 holds.

Fig. 2d. Again α is not quasi-monotone and Lemmas 1–3 do not hold on all of the domain D. However, by choosing suitable (energy-minimising) coarse space basis functions in two-level Schwarz methods (cf. [5, 12, 15]), it often suffices to be able to apply Lemmas 1–3 on $D' := Y_1 \cup Y_2 \cup Y_3$. Since α is type-1 quasi-monotone on D', e.g. Lemma 1 holds for $u \in H^1(D')$ and it is easy to verify that $C^{P,1} = \mathcal{O}(1)$.

References

1. C.R. Dohrmann, A. Klawonn, and O.B. Widlund. Domain decomposition for less regular subdomains: Overlapping Schwarz in two dimensions. *SIAM J. Numer. Anal.*, 46:2153–2168, 2008.
2. M. Dryja, M.V. Sarkis, and O.B. Widlund. Multilevel Schwarz methods for elliptic problems with discontinuous coefficients in three dimensions. *Numer. Math.*, 72:313–348, 1996.
3. Y. Efendiev and J. Galvis. Domain decomposition preconditioners for multiscale problems. Preprint, Texas A&M University, 2009. Submitted.
4. J. Galvis and Y. Efendiev. Domain decomposition preconditioners for multiscale flows in high contrast media. Preprint, Texas A&M University, 2009. Submitted.

5. I.G. Graham, P. Lechner, and R. Scheichl. Domain decomposition for multiscale PDEs. *Numer. Math.*, 106:589–626, 2007.
6. A. Klawonn, O. Rheinbach, and O.B. Widlund. An analysis of a FETI-DP algorithm on irregular subdomains in the plane. *SIAM J. Numer. Anal.*, 46:2484–2504, 2008.
7. A. Klawonn and O.B. Widlund. FETI and Neumann–Neumann iterative substructuring methods: connections and new results. *Commun. Pure Appl. Math.*, 54:57–90, 2001.
8. C. Pechstein and R. Scheichl. Analysis of FETI methods for multiscale PDEs. *Numer. Math.*, 111:293–333, 2008.
9. C. Pechstein and R. Scheichl. Analysis of FETI methods for multiscale PDEs - Part II: Interface variation. BICS Preprint 7/09, University of Bath, 2009. Submitted.
10. C. Pechstein and R. Scheichl. Scaling up through domain decomposition. *Appl. Anal.*, 88 (10), 2009.
11. M. Sarkis. Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements. *Numer. Math.*, 77:383–406, 1997.
12. R. Scheichl and E. Vainikko. Additive Schwarz and aggregation-based coarsening for elliptic problems with highly variable coefficients. *Computing*, 80:319–343, 2007.
13. R. Scheichl, P.S. Vassilevski, and L. Zikatanov. Two-level Schwarz with non-matching coarse grids. Preprint, Lawrence Livermore National Labs, 2009.
14. A. Toselli and O. Widlund. *Domain Decomposition Methods – Algorithms and Theory*. Springer, Berlin, 2005.
15. J. Van lent, R. Scheichl, and I.G. Graham. Energy minimizing coarse spaces for two-level Schwarz methods for multiscale PDEs. *Numer. Linear Algebra Appl.*, 16:775–799, 2009.
16. Y. Zhu. Domain decomposition preconditioners for elliptic equations with jump coefficients. *Numer. Linear Algebra Appl.*, 15:271–289, 2008.