# Robust Preconditioner for $\mathrm{H}(\mathrm{curl})$ Interface Problems

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**Summary.** In this paper, we construct an auxiliary space preconditioner for Maxwell's equations with interface, and generalize the HX preconditioner developed in [9] to the problem with strongly discontinuous coefficients. For the $\mathbf{H}(\mathbf{curl})$ interface problem, we show that the condition number of the HX preconditioned system is uniformly bounded with respect to the coefficients and meshsize.

**Key words:** HX preconditioner, AMG, $\mathbf{H}(\mathbf{curl})$ systems, Nédélec, interface

## 1 Introduction

The space $\mathbf{H}_0(\mathbf{curl})$ consists of square integrable vector fields with square integrable $\mathbf{curl}$ whose tangential component vanishes on $\partial\Omega$. In this paper, we try to develop robust and efficient preconditioners for the $\mathbf{H}(\mathbf{curl})$ interface problem:

$$\text{find } \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : (\mu \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + (\sigma \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \ \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}). \tag{1}$$

Here, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is a vector field and the coefficients $\mu(x)$ and $\sigma(x)$ are assumed to be uniformly positive but may have large variations in a simply connected open polyhedral domain $\Omega \subset \mathbb{R}^3$.

This equation arises naturally from many engineering and physical applications based on Maxwell's equations. In some applications (see [12, 16] for example), the coefficients in (1) satisfy that $\mu(x)/\sigma(x) = c$ is the speed of light. In this case, Eq. (1) can be reduced to (2) by a simple scaling:

$$\text{find } \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : (\omega \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + \tau(\omega \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}), \tag{2}$$

where $\tau \in (0,1)$ is a constant, and $\omega > 0$ is piecewise constant but may possibly have large jump across the interfaces.

The finite element discretization of (2) reads:

$$\text{find } \mathbf{u}_h \in \mathbf{V}_h : (\omega \mathbf{curl}\ \mathbf{u}_h, \mathbf{curl}\ \mathbf{v}_h) + \tau(\omega \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (3)$$

where $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$ is a conforming finite element space, e.g. Nedéléc elements. It gives rise to the following linear system:

$$\mathbf{A}x = F, \quad (4)$$

where $\mathbf{A} = (a_{ij})$ is defined by $a_{ij} = \int_\Omega \omega \mathbf{curl}\ \mathbf{b}_j \cdot \mathbf{curl}\ \mathbf{b}_i + \tau\omega \mathbf{b}_j \cdot \mathbf{b}_i dx$ for any basis functions $\mathbf{b}_i, \mathbf{b}_j \in \mathbf{V}_h$. It is well-known that the operator $\mathrm{curl}$ has a large kernel, which should be taken into account in the development of efficient solvers. This kernel causes most existing AMG solvers for Poisson equations to fail; see [23] for a theoretical explanation. In order to deal with this issue, most work has been done for developing efficient solvers for (4) with constant coefficients; see [2, 8, 11, 15, 18, 19].

Recently, Hiptmair and Xu [9] proposed an innovative approach for solving $\mathbf{H}(\mathrm{curl})$ systems, known as the HX-preconditioner. It relies on a *regular decomposition* of $\mathbf{H}(\mathrm{curl})$ vector fields (see Sect. 2) and the framework of the auxiliary space method (cf. [22]). A related method, which is based on the compatible discretization framework, was introduced in [4]. Although the analysis in [9] is only for constant coefficients case, extensive numerical experiments (cf. [13, 14]) demonstrate that this preconditioner is also efficient and robust for general coefficients. It is the purpose of this paper to give an theoretical justification of the robustness of the HX-preconditioner for (3).

The remainder of this paper is organized as follows. In Sect. 2, we discuss the regular decompositions at the continuous level. In particular, we prove the regular decomposition in a weighted norm. Then in Sect. 3, we adapt the decomposition into a discrete form, develop the HX preconditioner, and prove its robustness.

## 2 Regular Decomposition

The theoretical foundation in the development of the HX preconditioner is the following theorem, which originates from [3, 7] for Maxwell's equations.

**Theorem 1 ([10, 17]).** *For any $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$ there exist $\mathbf{\Phi} \in \mathbf{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ such that $\mathbf{u} = \mathbf{\Phi} + \nabla p$, which satisfy the following stability estimates:*

$$\|\mathbf{\Phi}\|_{1,\Omega} \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega}, \text{ and } \|\nabla p\|_{0,\Omega} \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}.$$

This theorem states that roughly speaking, the gap between $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}_0(\mathbf{curl})$ can be bridged by contributions from the kernel of $\mathbf{curl}$.

In some circumstances, the $\mathbf{H}(\mathbf{curl})$ systems are imposed with mixed boundary conditions. To deal with this situation, we consider the regular decomposition for the vector fields in the Hilbert space

$$\mathbf{H}_\Gamma(\mathbf{curl}) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}) : \mathbf{u} \times \mathbf{n}|_\Gamma = 0, \text{ for } \Gamma \subset \partial\Omega\},$$

where $\Gamma \neq \emptyset$ is the Dirichlet boundary. We have a similar regular decomposition for $\mathbf{u} \in \mathbf{H}_\Gamma(\mathbf{curl})$ as follows:

**Theorem 2.** *For any* $\mathbf{u} \in \mathbf{H}_\Gamma(\mathbf{curl})$ *there exist* $\mathbf{\Phi} \in \mathbf{H}^1_\Gamma(\Omega)$ *and* $p \in H^1_\Gamma(\Omega)$ *such that*

$$\mathbf{u} = \mathbf{\Phi} + \nabla p.$$

*This decomposition satisfies*

$$\|\mathbf{\Phi}\|_{1,\Omega} \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega}, \text{ and } \|\nabla p\|_{0,\Omega} \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}.$$

*Proof.* We need to take special care of the boundary conditions. Without loss of generality, we assume that $\Gamma$ is simply connected (otherwise, we just treat different connected components similarly). Let $\widetilde{\Omega}$ be a ball such that $\Omega \subset\subset \widetilde{\Omega}$, and $\widetilde{\Omega} = \Omega \cup O_\Gamma \cup O$ where $O_\Gamma$ is the subdomain with $\partial O_\Gamma \cap \partial\Omega = \Gamma$, and $O = \widetilde{\Omega} \backslash (\Omega \cup O_\Gamma)$ (see Fig. 1). We extend $\mathbf{u}$ to $\bar{\mathbf{u}} \in \mathbf{H}_0(\mathbf{curl}, \widetilde{\Omega})$ defined by $\bar{\mathbf{u}}|_\Omega := \mathbf{u}$, $\bar{\mathbf{u}}|_{O_\Gamma} := 0$. On the subdomain $O$, we define $\bar{\mathbf{u}}$ as the $\mathbf{H}(\mathbf{curl})$ extension of $\mathbf{u}$ such that $\bar{\mathbf{u}}|_{\partial\Omega\backslash\Gamma} = \mathbf{u}|_{\partial\Omega\backslash\Gamma}$ and 0 on the remaining boundary of $O$. We refer to [1] for the existence of such an extension. The remainder of the proof is almost identical to that of Theorem

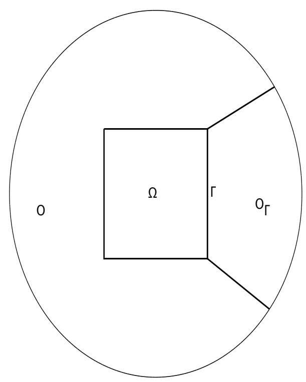


**Fig. 1.** Extension of $\mathbf{u} \in \mathbf{H}_\Gamma(\mathbf{curl}, \Omega)$ to $\bar{\mathbf{u}} \in \mathbf{H}_0(\mathbf{curl}, \widetilde{\Omega})$.

1 (see [17] for example). We omit the details.

*Remark 1.* For some other geometric structure of $\Gamma$, Theorem 2 still holds, for example if $\Gamma$ is a closed surface, or a “screen” (see [6, 16]).

In order to deal with the interface problem (2), we consider the regular decomposition for $\mathbf{H}(\mathbf{curl})$ in the setting of the weighted norms, which are the natural norm to deal with the interface problems. More precisely, we denote

$$\|v\|^2_{0,\omega} = \int_\Omega \omega |v|^2 dx, \quad |v|^2_{1,\omega} = \int_\Omega \omega |\nabla v|^2 dx \text{ and } \|v\|^2_{1,\omega} = \|v\|^2_{0,\omega} + |v|^2_{1,\omega}.$$

For simplicity, let $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where in $\Omega_1$ and $\Omega_2$ the equation has different constant coefficients $\omega_1, \omega_2$, respectively (see Fig. 2), with $\omega_1 \geq \omega_2 > 0$. The main

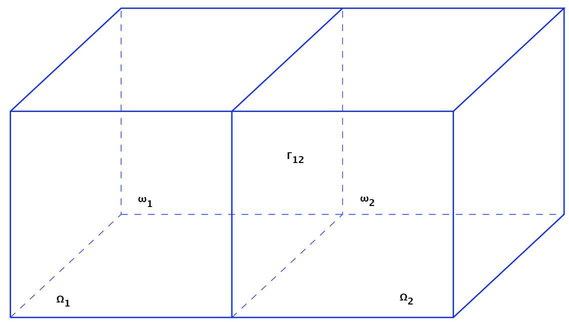


**Fig. 2.** Two domains with $\omega_1 \geq \omega_2 > 0$.

result of this section is the following decomposition. The idea of the proof is similar to the one used in [12] for proving a weighted Helmholtz decomposition.

**Theorem 3.** *For any* $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$, *we have* $\mathbf{u} = \mathbf{\Phi} + \nabla p$, *where* $\mathbf{\Phi} \in \mathbf{H}_0^1(\Omega)$ *and* $p \in H_0^1(\Omega)$ *such that*

$$\|\mathbf{\Phi}\|_{1,\omega}^2 \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\omega}^2 \text{ and } \|\nabla p\|_{0,\omega}^2 \lesssim \|\mathbf{u}\|_{0,\omega}^2 + \|\mathbf{curl}\ \mathbf{u}\|_{0,\omega}^2.$$

*Proof.* First we apply Theorem 2 on $\Omega_1$ with the Dirichlet boundary $\Gamma_1 = \partial\Omega \cap \partial\Omega_1$. For given $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl})$, we have, $\mathbf{u}|_{\Omega_1} = \mathbf{\Phi}_1 + \nabla p_1$ with $\mathbf{\Phi}_1 \in \mathbf{H}_{\Gamma_1}^1(\Omega_1)$ and $p_1 \in H_{\Gamma_1}^1(\Omega_1)$ such that

$$\|\mathbf{\Phi}_1\|_{1,\Omega_1} \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1} \quad \text{and} \quad \|\nabla p_1\|_{0,\Omega_1} \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)}. \tag{5}$$

Let $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ be the interface. We then extend $\mathbf{\Phi}_1$ and $p_1$ to harmonic functions on $\Omega_2$, and denote these extensions by $\widetilde{\mathbf{\Phi}}_1$ and $\widetilde{p}_1$. By the properties of harmonic extension (cf. [20]), the trace theorem and (5), we obtain

$$\|\widetilde{\mathbf{\Phi}}_1\|_{1,\Omega_2} \lesssim \|\mathbf{\Phi}_1\|_{\frac{1}{2},\Gamma_{12}} \lesssim \|\mathbf{\Phi}_1\|_{1,\Omega_1} \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1},$$
$$\|\widetilde{p}_1\|_{1,\Omega_2} \lesssim \|p_1\|_{\frac{1}{2},\Gamma_{12}} \lesssim \|p_1\|_{1,\Omega_1} \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)}.$$

Now notice that on $\Omega_2$, we have $\mathbf{u}_2^0 = \mathbf{u}|_{\Omega_2} - (\widetilde{\mathbf{\Phi}}_1 + \nabla\widetilde{p}_1)|_{\Omega_2} \in \mathbf{H}_0(\mathbf{curl}, \Omega_2)$. Then by Theorem 1 we get the decomposition $\mathbf{u}_2^0 = \mathbf{\Phi}_2^0 + \nabla p_2^0$ with $\mathbf{\Phi}_2^0 \in \mathbf{H}_0^1(\Omega_2)$ and $p_2^0 \in H_0^1(\Omega_2)$. This decomposition of $\mathbf{u}_2^0$ satisfies:

$$\begin{aligned}\|\mathbf{\Phi}_2^0\|_{1,\Omega_2} &\lesssim \|\mathbf{curl}\ \mathbf{u}_2^0\|_{0,\Omega_2} \leq \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_2} + \|\mathbf{curl}\ \widetilde{\mathbf{\Phi}}_1\|_{0,\Omega_2} \\ &\leq \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_2} + \|\widetilde{\mathbf{\Phi}}_1\|_{1,\Omega_2} \lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_2} + \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1},\end{aligned}$$

and similarly $\|\nabla p_2^0\|_{0,\Omega_2} \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. Let the decomposition of $\mathbf{u}$ in the whole domain be $\mathbf{u} = \mathbf{\Phi} + \nabla p$ where

$$\mathbf{\Phi} = \begin{cases} \mathbf{\Phi}_1 & \text{in } \Omega_1 \\ \mathbf{\Phi}_2^0 + \widetilde{\mathbf{\Phi}}_1 & \text{in } \Omega_2 \end{cases} \quad \text{and} \quad p = \begin{cases} p_1 & \text{in } \Omega_1 \\ p_2^0 + \widetilde{p}_1 & \text{in } \Omega_2 \end{cases}.$$

Recalling that $\omega_1 \geq \omega_2 > 0$, this decomposition satisfies

$$\begin{aligned} \|\mathbf{\Phi}\|_{1,\omega}^2 &\leq \omega_1 \|\mathbf{\Phi}_1\|_{1,\Omega_1}^2 + \omega_2 \|\mathbf{\Phi}_2^0\|_{1,\Omega_2}^2 + \omega_2 \|\widetilde{\mathbf{\Phi}}_1\|_{1,\Omega_2}^2 \\ &\lesssim \omega_1 \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1}^2 + \omega_2 \left( \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_2}^2 + \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1}^2 \right) + \omega_2 \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1}^2 \\ &= \left( 1 + \frac{2\omega_2}{\omega_1} \right) \omega_1 \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_1}^2 + \omega_2 \|\mathbf{curl}\ \mathbf{u}\|_{0,\Omega_2}^2 \\ &\lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\omega}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|\nabla p\|_{0,\omega}^2 &\leq \omega_1 \|\nabla p_1\|_{0,\Omega_1}^2 + \omega_2 \|\nabla p_2^0\|_{0,\Omega_2}^2 + \omega_2 \|\nabla \widetilde{p}_1\|_{0,\Omega_2}^2 \\ &\lesssim \omega_1 \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)}^2 + \omega_2 \|u\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 \\ &\lesssim \|\mathbf{curl}\ \mathbf{u}\|_{0,\omega}^2 + \|\mathbf{u}\|_{0,\omega}^2. \end{aligned}$$

This completes the proof.

*Remark 2.* The above result can be generalized to more general interface problems. For example, to cases where the subdomains have no "cross edge", that is, there is no edge which belongs to more than two subdomains. In these cases, the same conclusion holds because the coefficients satisfy a certain monotonicity.

## 3 Auxiliary Space Preconditioners

To realize the preconditioners for the finite element discretization of the model equations (1), the decomposition discussed in the previous section should be adapted to the discrete setting.

The degrees of freedom specified for $\mathbf{V}_h$ determine the *nodal interpolation operator* $\mathbf{\Pi}_h$, defined by $\mathbf{\Pi}_h \mathbf{v} = \sum_{e \in \mathcal{E}_h} \left( \int_e \mathbf{v} \cdot dl \right) \mathbf{b}_e$, where $\mathcal{E}_h$ is the set of (interior) edges and $\mathbf{b}_e$ is the edge element basis function associated with the edge $e$. In the sequel, we let $S_h \subset H_0^1(\Omega)$ be the standard nodal finite element space and $\mathbf{S}_h \subset \mathbf{H}_0^1(\Omega)$ be the vector counterpart of $S_h$. Due to the local approximation property of $\mathbf{\Pi}_h$, we have the following standard estimate.

**Lemma 1.** *For any* $\mathbf{\Phi} \in \mathbf{H}_0^1(\Omega)$ *such that* $\mathbf{curl}\ \mathbf{\Phi} \in \mathbf{curl}\ \mathbf{V}_h$, *the interpolation operator* $\mathbf{\Pi}_h$ *satisfies*

$$\mathbf{curl}\ (\mathbf{\Pi}_h \mathbf{\Phi}) = \mathbf{curl}\ \mathbf{\Phi} \text{ and } \|h^{-1} (I - \mathbf{\Pi}_h) \mathbf{\Phi}\|_{0,\omega} \lesssim \|\mathbf{\Phi}\|_{1,\omega}.$$

Based on Theorem 3 and Lemma 1, we obtain the following main result.

**Theorem 4.** *For any* $\mathbf{v}_h \in \mathbf{V}_h$ *there exist* $\boldsymbol{\Phi}_h \in \mathbf{S}_h$, $p_h \in S_h$ *and* $\widetilde{\mathbf{v}}_h \in \mathbf{V}_h$ *such that* $v_h = \widetilde{\mathbf{v}}_h + \boldsymbol{\Pi}_h \boldsymbol{\Phi}_h + \nabla p_h$, *and for any constant* $\tau \in (0,1)$

$$\|(h^{-1} + \tau^{\frac{1}{2}})\widetilde{\mathbf{v}}_h\|_{0,\omega}^2 + \|\!|\boldsymbol{\Phi}_h|\!\|_\tau^2 + \tau\,|p_h|_{1,\omega}^2 \lesssim \|\mathbf{v}_h\|_A^2, \tag{6}$$

*where* $\|\mathbf{v}_h\|_A^2 = \int_\Omega \omega |\mathbf{curl}\ \mathbf{v}|^2 + \tau\omega|\mathbf{v}|^2 dx$ *and* $\|\!|\mathbf{w}|\!\|_\tau^2 = \int_\Omega \omega|\nabla \mathbf{w}|^2 + \tau\omega|\mathbf{w}|^2 dx$.

*Proof.* Notice that if $\mathbf{v}_h \in \mathbf{V}_h$, by Theorem 3 and Lemma 1 there exists a $\boldsymbol{\Phi} \in \mathbf{H}_0^1(\Omega)$ such that $\mathbf{curl}\ \mathbf{v}_h = \mathbf{curl}\ \boldsymbol{\Phi} = \mathbf{curl}\ \boldsymbol{\Pi}_h\boldsymbol{\Phi}$. That is, $\mathbf{v}_h - \boldsymbol{\Pi}_h\boldsymbol{\Phi}$ is in the kernel of $\mathbf{curl}$. Therefore, there exists a $p_h \in S_h$ such that $\nabla p_h = \mathbf{v}_h - \boldsymbol{\Pi}_h\boldsymbol{\Phi}$. It satisfies

$$\begin{aligned}
\|\nabla p_h\|_{0,\omega} &\le \|\mathbf{v}_h\|_{0,\omega} + \|\boldsymbol{\Pi}_h\boldsymbol{\Phi}\|_{0,\omega} \\
&\le \|\mathbf{v}_h\|_{0,\omega} + \|(I - \boldsymbol{\Pi}_h)\boldsymbol{\Phi}\|_{0,\omega} + \|\boldsymbol{\Phi}\|_{0,\omega} \\
&\lesssim \|\mathbf{v}_h\|_{0,\omega} + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\omega}.
\end{aligned}$$

In the last inequality, we used Lemma 1, the inverse inequality, and Theorem 3. We then define the other two terms in the decomposition in the theorem as

$$\widetilde{\mathbf{v}}_h := \boldsymbol{\Pi}_h\left(\boldsymbol{\Phi} - Q_h^\omega\boldsymbol{\Phi}\right) \in \mathbf{V}_h, \quad \boldsymbol{\Phi}_h := Q_h^\omega\boldsymbol{\Phi} \in \mathbf{S}_h,$$

where $Q_h^\omega$ is the weighted $L^2$ projection introduced in [5]. Note that in our setting of the interface problem, $Q_h^\omega$ satisfies

$$\|(I - Q_h^\omega)v\|_{0,\omega} \lesssim |v|_{1,\omega} \text{ and } |Q_h^\omega v|_{1,\omega} \lesssim |v|_{1,\omega}, \quad \forall v \in H_0^1(\Omega).$$

Hence, we have $\|\!|\boldsymbol{\Phi}_h|\!\|_\tau \lesssim \|\boldsymbol{\Phi}\|_{1,\omega} \lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\omega} \le \|\mathbf{v}_h\|_A$. Moreover, we have

$$\begin{aligned}
\|h^{-1}\widetilde{\mathbf{v}}_h\|_{0,\omega} &\le \|h^{-1}(I - \boldsymbol{\Pi}_h)(I - Q_h^\omega)\boldsymbol{\Phi}\|_{0,\omega} + \|h^{-1}(I - Q_h^\omega)\boldsymbol{\Phi}\|_{0,\omega} \\
&\lesssim \|\boldsymbol{\Phi}\|_{1,\omega} \lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\omega} \lesssim \|\mathbf{v}_h\|_A.
\end{aligned}$$

This completes the proof.

The resulting HX preconditioner for Eq. (2) reads

$$\mathbf{B} := \mathbf{D}_A^{-1} + \mathbf{P}_h(\mathbf{L}(\omega) + \tau\mathbf{M}(\omega))^{-1}{\mathbf{P}_h}^T + \tau^{-1}\mathbf{G}L(\omega)^{-1}\mathbf{G}^T, \tag{7}$$

where $\mathbf{D}_A$ is the diagonal of $\mathbf{A}$; $\mathbf{P}_h$ is the matrix representation of $\boldsymbol{\Pi}_h$; $\mathbf{L}(\omega) + \tau\mathbf{M}(\omega)$ is the matrix associated with the bilinear form $(\omega\nabla\boldsymbol{\Phi}, \nabla\boldsymbol{\Psi}) + \tau(\omega\boldsymbol{\Phi}, \boldsymbol{\Psi})$ on $\mathbf{S}_h$; $L(\omega)$ is the matrix associated with $(\omega\nabla\phi, \nabla\psi)$ on $S_h$; and $\mathbf{G}$ is the discrete gradient matrix. Standard multilevel preconditioners are robust for solving the $H^1$-interface problems $\mathbf{L}(\omega) + \tau\mathbf{M}(\omega)$ and $L(\omega)$ (see [21] for the theoretical justifications). In practical implementation, we can also replace $(\mathbf{L}(\omega) + \tau\mathbf{M}(\omega))^{-1}$ by an AMG solver for ${\mathbf{P}_h}^T\mathbf{A}\mathbf{P}_h$, and replace $L(\omega)^{-1}$ by an AMG solver for $\mathbf{G}^T\mathbf{A}\mathbf{G}$.

Based on Theorem 4 and the framework developed in [9], the HX preconditioner (7) is robust with respect to the coefficients and meshsize. More precisely, we have the following theorem:

**Theorem 5.** *The condition number* $\kappa\left(\mathbf{BA}\right) \le C$, *where the constant* $C$ *is independent of the coefficients and the mesh size.*

## 4 Conclusions

In this paper, we have developed HX-preconditioners for the $\mathbf{H}(\mathbf{curl})$ interface problems. We have shown the robustness of the preconditioner by showing that the condition number of the preconditioned system is uniformly bounded with respect to the coefficients and the meshsize.

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