# N–N Solvers for a DG Discretization for Geometrically Nonconforming Substructures and Discontinuous Coefficients

Maksymilian Dryja[1], Juan Galvis[2], and Marcus Sarkis[3,4]

[1] Department of Mathematics, Warsaw University, Warsaw 02-097, Poland. This work was supported in part by The Polish Sciences Foundation under grant NN201006933.
[2] Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
[3] Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro 22460-320, Brazil
[4] Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA

## 1 Summary

A discontinuous Galerkin discretization for second order elliptic equations with *discontinuous coefficients* in 2-D is considered. The domain of interest $\Omega$ is assumed to be a union of polygonal substructures $\Omega_i$ of size $O(H_i)$. We allow this substructure decomposition to be geometrically nonconforming. Inside each substructure $\Omega_i$, a conforming finite element space associated to a triangulation $\mathcal{T}_{h_i}(\Omega_i)$ is introduced. To handle the nonmatching meshes across $\partial\Omega_i$, a discontinuous Galerkin discretization is considered. In this paper additive Neumann–Neumann Schwarz methods are designed and analyzed. Under natural assumptions on the coefficients and on the mesh sizes across $\partial\Omega_i$, a condition number estimate $C(1+\max_i \log \frac{H_i}{h_i})^2$ is established with $C$ independent of $h_i$, $H_i$, $h_i/h_j$, and the jumps of the coefficients. The method is well suited for parallel computations and can be straightforwardly extended to three dimensional problems. Numerical results, which are not included in this paper, confirm the theoretical results.

## 2 Introduction

In this paper a *discontinuous Galerkin* (DG) approximation of elliptic problems with *discontinuous coefficients* is considered [3]. See [1, 9] and references therein for an overview on local DG discretizations. The problem is considered in a two-dimensional polygonal region $\Omega$ which is a *geometrically nonconforming* union of disjoint polygonal substructures $\Omega_i$, $i = 1, \ldots, N$. For simplicity of presentation we assume that inside each substructure $\Omega_i$ the coefficient $\rho_i$ is constant. The extension of the results to mildly variation of $\rho_i$ inside $\Omega_i$ is straightforward. Large discontinuities of the coefficients are assumed to occur only across the interfaces of the substructures $\partial\Omega_i$. Inside each substructure $\Omega_i$ a conforming finite element method is introduced to discretize the local problem, and *nonmatching triangulations* are allowed to occur across the $\partial\Omega_i$. This kind of composite discretization is motivated by the location of the discontinuities of the coefficients and by the regularity of the solution of the problem. The discrete problem is formulated using a symmetric DG method with interior penalty (IPDG) terms on $\partial\Omega_i$. To deal with the discontinuities of the coefficients across the substructure interfaces, *harmonic averages* of the coefficients are considered on these interfaces; see [3].

The main goal of this paper is to design and analyze additive Neumann–Neumann algorithms for the resulting DG-discrete problem. This type of algorithms is well established for standard conforming and nonconforming discretizations; see [10] and references therein. We note that other two-level and multilevel preconditioners have been considered for solving discrete IPDG problems; see [2, 6, 8] and references therein. These papers focus on the scalability of the preconditioners with respect to the mesh parameters, however, little has been said about the robustness with respect to jumps of the coefficients and nonmatching grids across the substructuring interfaces. The notion of discrete harmonic extension in the DG sense was also introduced in [4] to achieve these desirable robustness for geometrically conforming substructures. In this paper we consider both the *geometrically nonconforming case* and *discontinuous coefficients*.

The problem is reduced to the Schur complement form with respect to unknowns on $\partial\Omega_i$, for $i = 1, \ldots, N$. Discrete harmonic functions defined in a special way, see Sect. 3.3, are used in this step. The methods are designed and analyzed for the Schur complement problem using the general theory of N–N methods; see [10]. The local problems are defined on $\Omega_i$ and edges or part of the edge of $\partial\Omega_j$ which are common to $\Omega_i$. The coarse space is defined by using a special partitioning of unity with respect to the subdomains $\Omega_i$ and by introducing master and slave sides of the local interfaces between the substructures. Recall that we work with a geometrically nonconforming partition of $\Omega$ into substructures $\Omega_i$, $i = 1, \ldots, N$. A (part of the) edge $E_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a master side when $\rho_i \geq C\rho_j$, otherwise it is a slave side. Hence, if $E_{ij} \subset \partial\Omega_i$ is a master side then $E_{ji} \subset \partial\Omega_j$, $E_{ij} = E_{ji}$, is a slave. The $h_i$-triangulation on $E_{ij}$ and $h_j$-triangulation on $E_{ji}$ are built in such a way that $h_i \geq Ch_j$ if $\rho_i \geq C\rho_j$. Here $h_i$ and $h_j$ are the parameters of the triangulation in $\Omega_i$ and $\Omega_j$, respectively, and $C$ is a generic $O(1)$ constant. We prove that the algorithms are almost optimal and their rates of convergence are independent of the mesh parameters, the number of subdomains $\Omega_i$ and the jumps of the coefficients. The algorithms are well suited for parallel computations and they can be straightforwardly extended to three-dimensional problems.

The paper is organized as follows. In Sects. 3.1 and 3.2 the differential problem and its DG discretization are formulated. In Sect. 3.3 the Schur complement problem is derived using discrete harmonic functions in a special way. Section 4 is dedicated to introducing notation and the *interface condition* on the coefficients and the mesh parameters. Two additive Neumann–Neumann Schwarz preconditioners, one based on a small coarse space and the other based on a larger coarse space, are defined and analyzed in Sect. 5.

## 3 Differential and Discrete Problems

In this section we formulate the discrete problem and its Schur complement problem.

### 3.1 Differential Problem

Consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \qquad \text{for all } v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u, v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \ \text{ and } \ f(v) := \int_{\Omega} f v dx.$$

Here, $\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i$ where the substructures $\Omega_i$ are disjoint regular polygonal subregions of diameter $O(H_i)$. We assume that the substructures $\Omega_i$ form a geometrically nonconforming partition of $\Omega$, therefore, for all $i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty, a vertex of $\Omega_i$ and/or $\Omega_j$, or a common edge or part of an edge of $\partial\Omega_i$ and $\partial\Omega_j$. If the decomposition is geometrically conforming, then the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty or a common vertex of $\Omega_i$ and $\Omega_j$, or a common edge of $\Omega_i$ and $\Omega_j$. For simplicity of presentation we assume that the right-hand side $f \in L^2(\Omega)$ and the coefficients $\rho_i$ are all positive constants.

### 3.2 Discrete Problem

In each $\Omega_i$ presentation, we introduce a shape regular triangulation $\mathcal{T}_{h_i}(\Omega_i)$ with triangular elements and the mesh parameter $h_i$. The resulting triangulation of $\Omega$ is in general nonmatching across $\partial\Omega_i$. We let $X_i(\Omega_i)$ be the regular finite element (FE) space of piecewise linear and continuous functions in $\mathcal{T}_{h_i}(\Omega_i)$. We do not assume that the functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. We define

$$X_h(\Omega) := X_1(\Omega_1) \times \cdots \times X_N(\Omega_N)$$

and represent functions $v$ of $X_h(\Omega)$ as $v = \{v_i\}_{i=1}^N$ with $v_i \in X_i(\Omega_i)$.

The discrete problem obtained by the DG method, see [1, 3, 9], is of the form: Find $u_h^* \in X_h(\Omega)$ such that

$$\hat{a}_h(u_h^*, v_h) = f(v_h) \quad \text{for all } v_h \in X_h(\Omega) \tag{2}$$

where

$$\hat{a}_h(u,v) = \sum_{i=1}^{N} \hat{a}_i(u,v) \quad \text{and} \quad f(v) = \sum_{i=1}^{N} \int_{\Omega_i} f v_i dx. \tag{3}$$

Each bilinear form $\hat{a}_i$ is given as a sum of three bilinear forms:

$$\hat{a}_i(u,v) := a_i(u,v) + s_i(u,v) + p_i(u,v), \tag{4}$$

where

$$a_i(u,v) := \int_{\Omega_i} \rho_i \nabla u_i \cdot \nabla v_i dx, \tag{5}$$

$$s_i(u,v) := \sum_{E_{ij} \subset \partial\Omega_i} \int_{E_{ij}} \frac{\rho_{ij}}{l_{ij}} \left( \frac{\partial u_i}{\partial n_i}(v_j - v_i) + \frac{\partial v_i}{\partial n_i}(u_j - u_i) \right) ds$$

and

$$p_i(u,v) := \sum_{E_{ij} \subset \partial\Omega_i} \int_{E_{ij}} \frac{\rho_{ij}}{l_{ij}} \frac{\delta}{h_{ij}} (u_j - u_i)(v_j - v_i) ds. \tag{6}$$

Here, the bilinear form $p_i$ is called the penalty term with a positive penalty parameter $\delta$. In the above equations, we set $l_{ij} = 2$ when $E_{ij} = \partial\Omega_i \cap \partial\Omega_j$ is a common edge (or part of an edge) of $\partial\Omega_i$ and $\partial\Omega_j$, and define $\rho_{ij} := 2\rho_i\rho_j/(\rho_i + \rho_j)$ as the harmonic average of $\rho_i$ and $\rho_j$, and $h_{ij} := 2h_ih_j/(h_i + h_j)$. In order to simplify notation we include the index $j = \partial$ when $E_{i\partial} := \partial\Omega_i \cap \partial\Omega$ is an edge of $\partial\Omega_i$ and set $l_{i\partial} := 1$ and let $v_\partial = 0$ for all $v \in X_h(\Omega)$, and define $\rho_{i\partial} := \rho_i$ and $h_{i\partial} := h_i$. The outward normal derivative on $\partial\Omega_i$ is denoted by $\frac{\partial}{\partial n_i}$. We note that when $\rho_{ij}$ is given by the harmonic average then $\min\{\rho_i, \rho_j\} \le \rho_{ij} \le 2\min\{\rho_i, \rho_j\}$.

A priori error estimates for the method are optimal for constant coefficients, and also for the case where $h_i$ and $h_j$ are of the same order; see [1, 9]. For discontinuous coefficients $\rho_i$ and/or for mesh sizes $h_i$ and $h_j$ are not on the same order, see Theorem 4.2 of [3] and Lemma 2.2 of [5].

### 3.3 Schur Complement Problem

In this subsection we derive the Schur complement bilinear form for the problem (2). We first introduce auxiliary notation.

Define $X_i^\circ(\Omega_i)$ as the subspace of $X_i(\Omega_i)$ of functions that vanish on $\partial\Omega_i$. A function $u_i \in X_i(\Omega)$ can be represented as

$$u_i = \mathcal{H}_i u_i + \mathcal{P}_i u_i \tag{7}$$

where $\mathcal{H}_i u_i$ is the discrete harmonic part of $u_i$ in the sense of $a_i(.,.)$, see (5), i.e.,

$$\begin{cases} a_i(\mathcal{H}_i u_i, v_i) = 0 \quad \text{for all } v_i \in X_i^\circ(\Omega_i) \\ \mathcal{H}_i u_i = u_i \quad \text{on} \quad \partial\Omega_i, \end{cases} \tag{8}$$

while $\mathcal{P}_i u_i$ is the projection of $u_i$ into $X_i^\circ(\Omega_i)$ in the sense of $a_i(.,.)$, i.e.,

$$a_i(\mathcal{P}_i u_i, v_i) = a_i(u_i, v_i) \quad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{9}$$

Note that $\mathcal{H}_i u_i$ is the classical discrete harmonic part of $u_i$. Let us denote by $X_h^\circ(\Omega)$ the subspace of $X_h(\Omega)$ defined by $X_h^\circ(\Omega) := X_1^\circ(\Omega_1) \times \cdots \times X_N^\circ(\Omega_N)$ and consider the global projections $\mathcal{H}u := \{\mathcal{H}_i u_i\}_{i=1}^N$ and $\mathcal{P}u := \{\mathcal{P}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow X_h^\circ(\Omega)$ in the sense of $\sum_{i=1}^N a_i(\cdot,\cdot)$. Hence, a function $u \in X_h(\Omega)$ can then be decomposed as

$$u = \mathcal{H}u + \mathcal{P}u. \tag{10}$$

Alternatively to (10), a function $u \in X_h(\Omega)$ can be represented as

$$u = \hat{\mathcal{H}}u + \hat{\mathcal{P}}u, \tag{11}$$

where $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N : X_h(\Omega) \rightarrow X_h^\circ(\Omega)$ is the projection in the sense of the original bilinear for $\hat{a}_h(\cdot,\cdot)$, see (3), and $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N \in X_h(\Omega)$ where $\hat{\mathcal{H}}_i u$ is the discrete harmonic part of $u$ in the sense of $\hat{a}_i(.,.)$ defined in (4), i.e., $\hat{\mathcal{H}}_i u \in X_i(\Omega_i)$ is the solution of

$$\begin{cases} \hat{a}_i(\hat{\mathcal{H}}_i u, v_i) = 0 \quad \text{for all } v_i \in X_i^\circ(\Omega_i), \\ \hat{\mathcal{H}}_i u = u_i \text{ on } \partial\Omega_i \\ \hat{\mathcal{H}}_i u = u_j \text{ on every (part of) edge } E_{ji} \subset \partial\Omega_j. \end{cases} \tag{12}$$

Here the index $j$ in the last equation of (12) runs over all $\Omega_j$ and $j = \partial$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j$ and $\overline{\Omega}_i \cap \partial\Omega$ has one-dimensional nonzero measure, respectively. In the latter case, recall that $u_\partial = 0$.

Observe that since $\hat{\mathcal{P}}_i u_i \in X_i^\circ(\Omega_i)$ we have that for all $v_i \in X_i^\circ(\Omega_i)$,

$$a_i(\hat{\mathcal{P}}_i u, v_i) = \hat{a}_h(u, R_i^T v_i),$$

where $R_i^T$ is the standard discrete zero extension operator, i.e., $R_i^T v_i := \{v_j\}_{j=1}^N$, where $v_j$ vanishes for $j \neq i$; see also Sect. 4 for the definition of other discrete zero extension operators $I_i$ and $\tilde{I}_i$.

The discrete solution of (2) can be decomposed as $u_h^* = \hat{\mathcal{H}}u_h^* + \hat{\mathcal{P}}u_h^*$. To compute the projection $\hat{\mathcal{P}}u_h^*$ we need to solve the following set of standard discrete Dirichlet problems:

$$a_i(\hat{\mathcal{P}}_i u_h^*, v_i) = f(R_i^T v_i) \qquad \text{for all } v_i \in X_i^\circ(\Omega_i). \tag{13}$$

Note that these problems, for $i = 1, \ldots N$, are local and independent, and so, they can be solved in parallel. This is a precomputational step.

We next formulate the problem for $\hat{\mathcal{H}}u_h^*$. We first point out that for $v_i \in X_i^\circ(\Omega_i)$ we have

$$\hat{a}_i(u_i, v_i) = (\rho_i \nabla u_i, \nabla v_i)_{L^2(\Omega_i)} + \sum_{E_{ij} \subset \partial\Omega_i} \frac{\rho_{ij}}{l_{ij}} (\frac{\partial v_i}{\partial n}, u_j - u_i)_{L^2(E_{ij})}. \tag{14}$$

For $u \in X_h(\Omega)$ observe that (12) is obtained from

$$\hat{a}_h(\hat{\mathcal{H}}u, v) = 0 \tag{15}$$

by taking $v = \{v_i\}_{i=1}^N \in X_h^\circ(\Omega)$. It is easy to see that $\hat{\mathcal{H}}u = \{\hat{\mathcal{H}}_i u\}_{i=1}^N$ and $\hat{\mathcal{P}}u = \{\hat{\mathcal{P}}_i u_i\}_{i=1}^N$ are orthogonal in the sense of $\hat{a}_h(.,.)$, i.e.,

$$\hat{a}_h(\hat{\mathcal{H}}u, \hat{\mathcal{P}}v) = 0, \qquad u, v \in X^h(\Omega). \tag{16}$$

In addition,

$$\mathcal{H}\hat{\mathcal{H}}u = \mathcal{H}u \quad \text{and} \quad \hat{\mathcal{H}}\mathcal{H}u = \hat{\mathcal{H}}u \tag{17}$$

since neither $\hat{\mathcal{H}}u$ nor $\mathcal{H}u$ changes the values of $u$ at the nodes on the boundaries of the subdomains $\Omega_i$; see (8) and (12).

Define

$$\Gamma_h := (\cup_i \partial\Omega_{ih_i}), \tag{18}$$

where $\partial\Omega_{ih_i}$ is the set of nodal points of $\partial\Omega_i$. *We note that the definition of $\Gamma_h$ includes the nodes on both triangulations of $\cup_i\partial\Omega_i$.*

We are now in a position to derive the Schur complement problem for (2). Applying the decomposition (11) in (2) we obtain

$$\hat{a}_h(\hat{\mathcal{H}}u_h^* + \hat{\mathcal{P}}u_h^*, \hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h) = f(\hat{\mathcal{H}}v_h + \hat{\mathcal{P}}v_h).$$

Using (13) and (15) we have

$$\hat{a}_h(\hat{\mathcal{H}}u_h^*, \hat{\mathcal{H}}v_h) = f(\hat{\mathcal{H}}v_h) \qquad \text{for all } v_h \in X_h(\Omega). \tag{19}$$

This is the Schur complement problem for (2). We denote by $V$ the set of all functions $v_h$ in $X_h(\Omega)$ such that $v_h \equiv \hat{\mathcal{H}}v_h$, i.e., the space of discrete harmonic functions in the sense of the $\hat{\mathcal{H}}$. We rewrite the Schur complement problem as follows: Find $u_h^* \in V$ such that

$$\mathcal{S}(u_h^*, v_h) = g(v_h) \qquad \text{for all } v_h \in V \tag{20}$$

where, here and below, $u_h^* \equiv \hat{\mathcal{H}}u_h^*$ and

$$\mathcal{S}(u_h, v_h) := \hat{a}_h(\hat{\mathcal{H}}u_h, \hat{\mathcal{H}}v_h) \text{ and } g(v_h) := f(\hat{\mathcal{H}}v_h). \tag{21}$$

The Schur complement problem (20) has a unique solution.

## 4 Notation and the Interface Condition

We first classify substructures according to their position with respect to the boundary $\partial\Omega$. We say that a substructure $\Omega_i$ is an *interior substructure* or *floating substructures* if $\Omega_i$ does not share an edge with the boundary of $\Omega$. Otherwise, we say it is a *boundary substructure* or *nonfloating substructure*. We denote by $\mathcal{N}_I$ and $\mathcal{N}_B$ the sets of indices of interior and boundary substructures, respectively.

Let $\overset{\circ}{\Omega}_{ih_i}$ and $\partial\Omega_{ih_i}$ be the interior and boundary nodes of $\mathcal{T}_{h_i}(\overline{\Omega}_i)$ in $\Omega_i$ and on $\partial\Omega_i$, respectively. Define $E_{ijh_i}$ as the set of nodes of $\partial\Omega_{ih_i}$ that are on $E_{ij}$. Recall that $E_{ij}$ is a *closed* interval. We also define $\partial E_{ijh_i}$ as the set of nodes on $E_{ijh_i}$ that are closest to the boundary $\partial E_{ij}$. Let $\overset{\circ}{E}_{ijh_i} := E_{ijh_i} \setminus \partial E_{ijh_i}$ be the set of interior nodes in $E_{ij}$. Additionally, we define the extended boundary nodes $\partial^e E_{ijh_i}$ as the union of $\partial E_{ijh_i}$ and the nodal points $y \in \partial\Omega_i \setminus E_{ij}$ closest to $x \in \partial E_{ij}$ when $x$ is not a nodal point. Note that when $E_{ij}$ is a full edge of $\partial\Omega_i$, then $\partial^e E_{ijh_i} = \partial E_{ij}$. Let $\overline{E}_{ijh_i} := \overset{\circ}{E}_{ijh_i} \cup \partial^e E_{ijh_i}$. We define

$$\Gamma_i := \partial\Omega_{ih_i} \cup \bigcup_{E_{ij}\subset\partial\Omega_i} \overline{E}_{jih_j}. \tag{22}$$

Note that $\Gamma_i$ is defined to include the nodes on $\Gamma_h$ necessary for computing $\hat{\mathcal{H}}_i$; see (12). Define $W_i$ as the space of piecewise linear functions or its vector representation defined by the nodal values on $\Gamma_i$ extended via $\hat{\mathcal{H}}_i$ (defined in (12)) inside $\Omega_i$, i.e.,

$$W_i := \left\{ \text{nodal values of } v \text{ defined on } \overset{\circ}{\Omega}_{ih_i} \cup \Gamma_i :\ v \equiv \hat{\mathcal{H}}_i v \text{ in } \Omega_i \right\}. \tag{23}$$

Observe that a function $u^{(i)} \in W_i$ can be represented as

$$u^{(i)} = \{u_l^{(i)}\}_{l\in\#(i)} \quad \text{where} \quad \#(i) = \{i\} \cup \{j : E_{ij} \subset \partial\Omega_i\}.$$

Here $u_i^{(i)}$ and $u_j^{(i)}$ stand for the nodal values of $u^{(i)}$ on $\overline{\Omega}_i$ and on $\overline{E}_{jih_j}$, respectively. Recall also that sometimes we write $u = \{u_i\}_{i=1}^N \in V$ to refer to a function defined on all of $\Gamma_h$ with each $u_i$ defined (only) on $\partial\Omega_i$; see Sect. 3.2. We point out that $E_{ij}$ and $E_{ji}$ are geometrically the same even though the mesh on the side $E_{ij}$ comes from the $\Omega_i$ triangulation while the mesh on the side $E_{ji}$ corresponds from the $\Omega_j$ triangulation. Note also that, according to our conventions, if $i \in \mathcal{N}_B$ and $u^{(i)} \in W_i$ then $u_\partial^{(i)} = 0$ on the fictitious edge $E_{\partial i}$.

Define the extension operator $\tilde{I}_i : W_i \to V$ as follows: Given $u^{(i)} \in W_i$, let $\tilde{I}_i u^{(i)}$ be equal to $u^{(i)}$ at the nodes of $\Gamma_i$ and $\overset{\circ}{\Omega}_{ih_i}$, equal to zero on $\Gamma_h \backslash \Gamma_i$, and extended by $\hat{\mathcal{H}}\tilde{I}_i u^{(i)}$ elsewhere and denoted also by $\tilde{I}_i$, i.e.,

$$\tilde{I}_i u(x) = \begin{cases} u(x) & \text{if } x \in \Gamma_i \\ 0 & \text{if } x \in \Gamma_h \backslash \Gamma_i \\ \hat{\mathcal{H}} \tilde{I}_i u & \text{elsewhere,} \end{cases} \tag{24}$$

where the last condition on (24) means that $\tilde{I}_i u$ is discrete harmonic in the sense of $\hat{\mathcal{H}}$.

To each pair $\{E_{ij}, E_{ji}\}$ we assign one master and one slave side. If $E_{ij}$ is chosen to be the slave side then $E_{ji}$ must be the master one. Note that since we are working with a geometrically nonconforming decomposition of $\Omega$, a part of an edge can be labeled as master side while other part of the same edge can be marked as the slave side. The choice of slave-master sides is such that the *interface condition*, stated next in Assumption 1, can be satisfied. Under this assumption, Theorems 1 below hold with constants $C_1$ and $C_2$ independent of the $\rho_i$, $h_i$ and $H_i$. This assumption says basically that the coarser meshes $h_i$ should be chosen where the coefficients $\rho_i$ are larger, and additionally, the master side should be chosen on the side where the coefficient is larger. In terms of accuracy, this condition is satisfied in practice since the solution $u^*$ in general varies less where the coefficient is larger. We note that this condition is similar to the ones adopted in mortar studies for geometrical nonconforming cases; [7].

**Assumption 1 (The interface condition)** *We say that the coefficients $\{\rho_i\}$ and the local mesh sizes $\{h_i\}$ satisfy the interface condition if there exist constants $\beta_1$ and $\beta_2$, of order $1$, such that for any (part of) edge $E_{ij}$, one of the following inequalities hold:*

$$\begin{cases} h_i \leq \beta_1 h_j \;\; \textit{and} \;\; \rho_i \leq \beta_2 \rho_j & \textit{if } E_{ij} \textit{ is a slave side, or} \\ h_j \leq \beta_1 h_i \;\; \textit{and} \;\; \rho_j \leq \beta_2 \rho_i & \textit{if } E_{ij} \textit{ is a master side.} \end{cases} \tag{25}$$

We associate to each $\Omega_i$ $1, \cdots, N$, a diagonal weighting matrix $D^{(i)} = \{D_l^{(i)}\}_{l \in \#(i)}$ on $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$. Let $x$ be a nodal point of $\Gamma_i \cup \overset{\circ}{\Omega}_{ih_i}$. Then, the diagonal element of $D^{(i)}$ associated to $x$ is defined by:

- On $\overset{\circ}{\Omega}_{ih_i} \cup \partial\Omega_{i,h_i} \quad (l = i)$

$$D_i^{(i)}(x) = \begin{cases} 0 \text{ if } x \in \overset{\circ}{E}_{ijh_i} \text{ and } E_{ij} \text{ is a slave side} \\ 1 \text{ otherwise,} \end{cases} \tag{26}$$

- On $\overline{E}_{jih_j} \quad (l = j)$

$$D_j^{(i)}(x) = \begin{cases} 0 \text{ if } x \in \partial^e E_{jih_j}, \\ 1 \text{ if } x \in \overset{\circ}{E}_{jih_j} \text{ and } E_{ij} \text{ is a master side} \\ 0 \text{ if } x \in \overset{\circ}{E}_{jih_j} \text{ and } E_{ij} \text{ is a slave side,} \end{cases} \tag{27}$$

- On $\overline{E}_{i\partial h_i}$

$$D_i^{(i)}(x) = 1 \text{ for all } x \in \overline{E}_{i\partial h_i}.$$

The prolongation operators $I_i : W_i \to V$, $i = 1, \cdots, N$, are defined as

$$I_i = \tilde{I}_i D^{(i)}. \tag{28}$$

It is easy to see that the image of $I_i$ forms a decomposition of $V$ since

$$\sum_{i=1}^{N} I_i \tilde{I}_i^T u = u, \tag{29}$$

where the $\tilde{I}_i^T$ stand for the restriction of $V$ to $W_i$.

## 5 Additive Preconditioners

To design and analyze additive N–N type methods for solving (20) we use the general framework of ASMs; see Theorem 2.7 in [10]. We now consider an additive Schwarz method based on the coarse space $V_{0,I}$, i.e., a coarse space with one degree of freedom per interior substructure and no degrees of freedom for any boundary substructure; see (34). We now introduce the local and coarse problems to define the additive Schwarz method $T_{as,I}$.

### 5.1 Local Problems

Recall the definition of $\Gamma_i$ in (22), the space $W_i$ in (23) and the sets of $\mathcal{N}_B$ and $\mathcal{N}_I$ substructures, see Sect. 4. Define

$$\begin{cases} V_i = V_i(\Gamma_i) := \left\{ u^{(i)} \in W_i : \int_{\partial\Omega_i} u_i^{(i)} = 0 \right\}, & \text{if } i \in \mathcal{N}_I \\ V_i = V_i(\Gamma_i) := W_i, & \text{if } i \in \mathcal{N}_B \end{cases} \tag{30}$$

i.e., for interior substructures $\Omega_i$, $V_i$ is the subspace of $W_i$ consisting of functions with zero average value on $\partial\Omega_i$, while for boundary substructures, $V_i$ is the whole space $W_i$. We recall that for $v^{(i)} \in W_i$ (or $V_i$) then $v^{(i)} \equiv \hat{\mathcal{H}}_i v^{(i)}$ and $v \in V$ we have $v^{(i)} = \hat{\mathcal{H}}_i v^{(i)}$ and $v = \hat{\mathcal{H}} v$.

For $u^{(i)}, v^{(i)} \in V_i$, $i = 1, \ldots, N$, we define the local bilinear form $b_i$ as

$$b_i(u^{(i)}, v^{(i)}) := \hat{a}_i(u^{(i)}, v^{(i)}), \tag{31}$$

where the bilinear form $\hat{a}_i$ is defined in (4). We define the operators $T_i : V \to V$, $i = 1, \ldots, N$, by defining $\tilde{T}_i : V \to V_i$ as

$$b_i(\tilde{T}_i u, v^{(i)}) = \hat{a}_h(u, I_i v^{(i)}) \text{ for all } v^{(i)} \in V_i, \tag{32}$$

and then set $T_i = I_i \tilde{T}_i$. It is easy to see that these problems are well posed and that the $T_i$ are symmetric with respect to the $\hat{a}_h$-inner product.

### 5.2 Coarse Problems

Let $e^{(i)} \in W_i$ be the vector with value one at the nodes of $\Gamma_i$ and on $\overset{\circ}{\Omega}_{ih_i}$. Recall that the prolongation operators $\tilde{I}_i$ and $I_i$ are defined in (24) and (28), respectively. Define $\Theta_i \in V$, for $i = 1, \ldots, N$, as $\Theta_i := \tilde{I}_i \Theta^{(i)}$ where $\Theta^{(i)} = D^{(i)} e^{(i)}$, hence, $\Theta_i = I_i e^{(i)}$. Note from (26) and (27) we have that

$$\sum_{i=1}^{N} \Theta_i = 1 \text{ on } \Gamma_h. \tag{33}$$

We consider the following coarse space:

$$V_{0,I} = \text{Span}\,\{\Theta_i\}_{i \in \mathcal{N}_I} \subset V. \tag{34}$$

The coarse bilinear form is defined according to

$$b_0(u,v) = \left(1 + \log \frac{H}{h}\right)^{-2} \hat{a}_h(u,v), \quad u,v \in V_{0,I}. \tag{35}$$

Next we define the projection-like operator $T_0 : V \rightarrow V_{0,I}$ as

$$b_0(T_0 u, v^{(0)}) = \hat{a}_h(u, v^{(0)}) \text{ for all } v^{(0)} \in V_{0,I}. \tag{36}$$

This problem is well posed and symmetric with respect to the $\hat{a}_h$-inner product.

The additive preconditioner is defined by

$$T_{as,I} = \sum_{i=0}^{N} T_i. \tag{37}$$

Note that $T_{as,I}$ is symmetric with respect to the inner product $\hat{a}_h(\cdot,\cdot)$.

### 5.3 Condition Number Estimate for $T_{as,I}$

In this section we state the main result concerning the preconditioner defined in (37) with $V_0 = V_{0,I}$.

**Theorem 1.** *Let Assumption 1 be satisfied. In addition, assume that for $i \in \mathcal{N}_B$, the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of $\Omega_i$. Then there exist positive constants $C_1$ and $C_2$ independent of $h_i$, $H_i$, $h_i/h_j$ and the jumps of $\rho_i$ such that*

$$C_1 \hat{a}_h(u,u) \leq \hat{a}_h(T_{as,I} u, u) \leq C_2 \left(1 + \log \frac{H}{h}\right)^2 \hat{a}_h(u,u) \quad \textit{for all } u \in V. \tag{38}$$

*Here* $\log(H/h) = \max_i \log(H_i/h_i)$.

*Proof.* By the general theory of ASMs we need to check three key assumptions; see Theorem 2.7 [10]. The proof can be found in [5].

## 6 Final Remarks

The ASM considered can be generalized replacing the coarse space $V_{0,I}$, see (34), by adding boundary coarse basis functions, i.e.,

$$V_{0,I\cup B} = \text{Span}\left\{\Theta_i\right\}_{i\in\mathcal{N}_I\cup\mathcal{N}_B} . \tag{39}$$

The additive preconditioner is then defined by

$$T_{as,I\cup B} = \sum_{i=0}^{N} T_i, \tag{40}$$

where the $T_0$ is defined as in (36) except that now we replace $V_{0,I}$ by $V_{0,I\cup B}$. For this preconditioner, the Theorem 1 is also valid, moreover, it is valid without the assumption that the size of $\partial\Omega_i \cap \partial\Omega$ is of the same order as the diameter of $\partial\Omega_i$ when $i \in \mathcal{N}_B$.

The tools of the discussed methods can be used to design and analyze hybrid (BDD) methods for (20). We can also consider hybrid versions of $T_{as,I\cup B}$, see [5].

The numerical tests carried out for the above methods confirm the theoretical results, see [5]. In particular, Assumption 1 is necessary and sufficient.

The discussed methods can be straightforwardly extended to 3-D cases.

## References


1. D.N. Arnold, F. Brezzi, B. Cockburn, and L. Donatella Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02. ISSN 0036-1429.
2. B.A. de Dios and L. Zikatanov. Uniformly convergent iterative methods for discontinuous Galerkin discretizations. *J. Sci. Comput.*, 40(1–3):4–36, 2009. ISSN 0885-7474.
3. M. Dryja. On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients. *Comput. Methods Appl. Math.*, 3(1):76–85 (electronic), 2003. ISSN 1609-4840. Dedicated to Raytcho Lazarov.
4. M. Dryja, J. Galvis, and M. Sarkis. BDDC methods for discontinuous Galerkin discretization of elliptic problems. *J. Complex.*, 23(4–6):715–739, 2007. ISSN 0885-064X.
5. M. Dryja, J. Galvis, and M. Sarkis. Neumann–Neumann methods for a DG discretization on geometrically nonconforming substructures. Technical Report 188, Department of Mathematics, Warsaw University, 2009.
6. X. Feng and O.A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4): 1343–1365 (electronic), 2001. ISSN 0036-1429.
7. H.H. Kim, M. Dryja, and O.B. Widlund. A BDDC method for mortar discretizations using a transformation of basis. *SIAM J. Numer. Anal.*, 47(1):136–157, 2008/09. ISSN 0036-1429. URL http://dx.doi.org/10.1137/070697859.

8. C. Lasser and A. Toselli. An overlapping domain decomposition preconditioner for a class of discontinuous Galerkin approximations of advection-diffusion problems. *Math. Comput.*, 72(243):1215–1238 (electronic), 2003. ISSN 0025-5718.
9. B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Applications*, volume 35 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. ISBN 978-0-898716-56-6.
10. A. Toselli and O. Widlund. *Domain Decomposition Methods – Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2005. ISBN 3-540-20696-5.