Space-Time Nonconforming Optimized Schwarz Waveform Relaxation for Heterogeneous Problems and General Geometries

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1 Introduction

In many fields of applications it is necessary to couple models with very different spatial and time scales and complex geometries. Amongst them are ocean-atmosphere coupling and far field simulations of underground nuclear waste disposal. For such problems with long time computations, a splitting of the time interval into windows is essential. This allows for robust and fast solvers in each time window, with the possibility of nonconforming space-time grids, general geometries, and ultimately adaptive solvers.

Optimized Schwarz Waveform Relaxation (OSWR) methods were introduced and analyzed for linear advection-reaction-diffusion problems with constant coefficients in [1, 3, 9]. All these methods rely on an algorithm that computes independently in each subdomain over the whole time interval, exchanging space-time boundary data through optimized transmission operators. They can apply to different space-time discretization in subdomains, possibly nonconforming and need a very small number of iterations to converge. Numerical evidences of the performance of the method with variable smooth coefficients were given in [9].

An extension to discontinuous coefficients was introduced in [4], with asymptotically optimized Robin transmission conditions in some particular cases. In [2, 6], semi-discretization in time in one dimension was performed using discontinuous Galerkin, see [8, 10]. In [7], we extended the analysis to the two dimensional case. We obtained convergence results and error estimates for rectangular or strip-subdomains.

For the space discretization, we extended numerically the nonconforming approach in [5] to advection-diffusion problems and optimized order 2 transmission conditions, to allow for non-matching grids in time and space on the boundary. The space-time projections between subdomains were computed with an optimal projection algorithm without any additional grid, as in [5]. In [7], two dimensional simulations with continuous coefficients were presented.

We present here new results in two directions: we extend the proof of convergence of the OSWR algorithm to nonoverlapping subdomains with curved interfaces. We also present simulations for two subdomains, with piecewise smooth coefficients and a curved interface, for which no error estimates are available. We finally present an application to the porous media equation.

We consider the advection-diffusion-reaction equation,

$$\partial_t u + \nabla \cdot (\boldsymbol{b}u - \nu \nabla u) + cu = f \text{ in } \mathbb{R}^N \times (0,T), \tag{1}$$

with initial condition u_0, and $N = 2$. The advection, diffusion and reaction coefficients $\boldsymbol{b}$, ν and c, are piecewise smooth, we suppose $\nu \geq \nu_0 > 0$ *a.e.*.

2 The Continuous OSWR Algorithm

We consider a decomposition into nonoverlapping subdomains $\Omega_i, i \in \{1, ..., I\}$, organized as depicted in Fig. 1. The interfaces between the subdomains are supposed to be flat at infinity. For any $i \in \{1, ..., I\}$, $\partial\Omega_i$ is the boundary of Ω_i, $\boldsymbol{n}_i$ the unit exterior normal vector to $\partial\Omega_i$, $\mathcal{N}_i$ is the set of indices of the neighbors of Ω_i. For $j \in \mathcal{N}_i$, $\Gamma_{i,j}$ is the common interface.

Fig. 1. Decomposition in subdomains. *Left*: Robin transmission conditions, *right*: second order transmission conditions.

Following [1, 2, 3, 4], we introduce the boundary operators $\mathcal{S}_{i,j}$ acting on functions defined on $\Gamma_{i,j}$:

$$\mathcal{S}_{i,j}\varphi = p_{i,j}\varphi + q_{i,j}(\partial_t \varphi + \nabla_{\Gamma_{i,j}} \cdot (\boldsymbol{r}_{i,j}\varphi - s_{i,j}\nabla_{\Gamma_{i,j}}\varphi)),$$

with respectively ∇_Γ and $\nabla_\Gamma \cdot$ the gradient and divergence operators on Γ. $p_{i,j}$, $q_{i,j}$, $\boldsymbol{r}_{i,j}$, $s_{i,j}$ are real parameters. $q_{i,j} = 0$, will be referred to as a Robin operator. We introduce the coupled problems

$$\begin{aligned} &\partial_t u_i + \nabla \cdot (\boldsymbol{b}_i u_i - \nu_i \nabla u_i) + c_i u_i = f \text{ in } \Omega_i \times (0,T) \\ &\left(\nu_i \partial_{\boldsymbol{n}_i} - \boldsymbol{b}_i \cdot \boldsymbol{n}_i\right) u_i + \mathcal{S}_{i,j} u_i = \\ &\qquad \left(\nu_j \partial_{\boldsymbol{n}_i} - \boldsymbol{b}_j \cdot \boldsymbol{n}_i\right) u_j + \mathcal{S}_{i,j} u_j \text{ on } \Gamma_{i,j} \times (0,T), j \in \mathcal{N}_i. \end{aligned} \tag{2}$$

As coefficients ν and $\boldsymbol{b}$ are possibly discontinuous on the interface, we note, for $s \in \Gamma_{i,j}$, $\nu_i(s) = \lim_{\varepsilon\to 0} \nu(s-\varepsilon \boldsymbol{n}_i)$. The same notation holds for $\boldsymbol{b}$. Under regularity assumptions, solving (1) is equivalent to solving (2) for $i \in \{1, ..., I\}$ with u_i the restriction of u to Ω_i. We now introduce an algorithm to solve (2). An initial guess $(g_{i,j})$ is given in $L^2((0,T)\times\Gamma_{i,j})$ for $i \in \{1, ..., I\}$, $j \in \mathcal{N}_i$. We solve iteratively

$$\begin{aligned} \partial_t u_i^k + \nabla\cdot(\boldsymbol{b}_i u_i^k - \nu_i \nabla u_i^k) + c_i u_i^k = f \text{ in } \Omega_i \times (0,T), \\ \left(\nu_i \partial_{\boldsymbol{n}_i} - \boldsymbol{b}_i\cdot\boldsymbol{n}_i\right) u_i^k + \mathcal{S}_{i,j} u_i^k = \\ \left(\nu_j \partial_{\boldsymbol{n}_i} - \boldsymbol{b}_j\cdot\boldsymbol{n}_i\right) u_j^{k-1} + \mathcal{S}_{i,j} u_j^{k-1} \text{ on } \Gamma_{i,j}\times(0,T), j \in \mathcal{N}_i. \end{aligned} \tag{3}$$

with the convention $\left(\nu_i \partial_{\boldsymbol{n}_i} - \boldsymbol{b}_i\cdot\boldsymbol{n}_i\right) u_i^1 + \mathcal{S}_{i,j} u_i^1 = g_{i,j},\ j \in \mathcal{N}_i$.

Theorem 1. *Assume* $\boldsymbol{b}_i \in (W^{1,\infty}(\Omega_i))^N$, $\nu_i \in W^{1,\infty}(\Omega_i)$, $p_{i,j} \in W^{1,\infty}(\Gamma_{i,j})$ *with* $p_{i,j} > 0$ *a.e.. If* $q_{i,j} = 0$, *or if* $q_{i,j} = q > 0$ *with* $\boldsymbol{r}_{i,j} \in (W^{1,\infty}(\Gamma_{i,j}))^{N-1}$, $\boldsymbol{r}_{i,j} = \boldsymbol{r}_{j,i}$ *on* $\Gamma_{i,j}$, $s_{i,j} \in W^{1,\infty}(\Gamma_{i,j})$, $s_{i,j} > 0$, $s_{i,j} = s_{j,i}$ *on* $\Gamma_{i,j}$, *the algorithm* (3) *converges in each subdomain to the solution of problem* (2).

Proof. We first need some results in differential geometry. For every $j \in \mathcal{N}_i$, the normal vector $\boldsymbol{n}_i$ can be extended in a neighbourhood of $\Gamma_{i,j}$ as a smooth function $\tilde{\boldsymbol{n}}_i$ with length one. Let $\psi_{i,j} \in \mathcal{C}^\infty(\overline{\Omega_i})$, such that $\psi_{i,j} \equiv 1$ in a neighbourhood of $\Gamma_{i,j}$, $\psi_{i,j} \equiv 0$ in a neighbourhood of $\Gamma_{i,k}$ for $k \in \mathcal{N}_i$, $k \neq j$ and $\sum_{j\in\mathcal{N}_i} \psi_{i,j} > 0$ on Ω_i. Let $\tilde{\boldsymbol{n}}_i$ be defined on a neighbourhood of the support of $\psi_{i,j}$. We can extend the tangential gradient and divergence operators in the support of $\psi_{i,j}$ by:

$$\widetilde{\nabla}_{\Gamma_{i,j}} \varphi := \nabla\varphi - (\partial_{\tilde{\boldsymbol{n}}_i}\varphi)\tilde{\boldsymbol{n}}_i, \quad \widetilde{\nabla}_{\Gamma_{i,j}}\cdot\boldsymbol{\varphi} := \nabla\cdot(\boldsymbol{\varphi} - (\boldsymbol{\varphi}\cdot\tilde{\boldsymbol{n}}_i)\tilde{\boldsymbol{n}}_i).$$

It is easy to see that $(\widetilde{\nabla}_{\Gamma_{i,j}}\varphi)_{|\Gamma_{i,j}} = \nabla_{\Gamma_{i,j}}\varphi$, $(\widetilde{\nabla}_{\Gamma_{i,j}}\cdot\boldsymbol{\varphi})_{|\Gamma_{i,j}} = \nabla_{\Gamma_{i,j}}\cdot\boldsymbol{\varphi}$ and for $\boldsymbol{\varphi}$ and χ with support in $supp(\psi_{i,j})$, we have

$$\int_{\Omega_i} (\widetilde{\nabla}_{\Gamma_{i,j}}\cdot\boldsymbol{\varphi})\,\chi\,dx = -\int_{\Omega_i} \boldsymbol{\varphi}\cdot\widetilde{\nabla}_{\Gamma_{i,j}}\chi\,dx. \tag{4}$$

Now we prove Theorem 1. The key point is to obtain energy estimates for the homogeneous problem (2), i.e. for $f = u_0 = 0$. We sketch the proof in the most difficult case $q_{i,j} = q > 0$. For the geometry, we consider the case depicted in the right part of Fig. 1. In that case Ω_i has at most two neighbours with

We set $\|\varphi\|_i = \|\varphi\|_{L^2(\Omega_i)}$, $|||\varphi|||_i^2 = \|\sqrt{\nu_i}\,\nabla\varphi\|^2_{L^2(\Omega_i)}$, $\|\varphi\|_{i,\infty} = \|\varphi\|_{L^\infty(\Omega_i)}$, $\|\varphi\|_{i,1,\infty} = \|\varphi\|_{W^{1,\infty}(\Omega_i)}$ and $\beta_i = \sum_{j\in\mathcal{N}_i}\psi_{i,j}\beta_{i,j}$ with $\beta_{i,j} = \sqrt{\frac{p_{i,j}+p_{j,i}}{2}}$.

1. We multiply the first equation of (3) by $\beta_i^2\, u_i^k$, integrate on $\Omega_i\times(0,t)$ then integrate by parts in space,

$$\frac{1}{2}\|\beta_i u_i^k(t)\|_i^2 + \int_0^t |||\beta_i u_i^k(\tau,\cdot)\,|||_i^2\ d\tau - \int_0^t \int_{\Omega_i} \beta_i(\boldsymbol{b}_i \cdot \nabla\beta_i)(u_i^k)^2\,dx\,d\tau$$
$$+ \int_0^t \int_{\Omega_i} (c_i + \frac{1}{2}\nabla\cdot\boldsymbol{b}_i)\beta_i^2(u_i^k)^2\,dx\,d\tau - \int_0^t \int_{\Omega_i} \nu_i|\nabla\beta_i|^2(u_i^k)^2\,dx\,d\tau$$
$$- \int_0^t \int_{\Gamma_{i,j}} \beta_{i,j}^2(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \frac{\boldsymbol{b}_i\cdot\boldsymbol{n}_i}{2}u_i^k)\,u_i^k\,d\sigma\,d\tau = 0. \quad (5)$$

2. We multiply the first equation of (3) by $\partial_t u_i^k$, integrate on $\Omega_i \times (0,t)$ and integrate by parts in space,

$$\int_0^t \|\partial_t u_i^k\|_i^2\,d\tau + \frac{1}{2}\,|||\,u_i^k(t)\,|||_i^2 + \int_0^t \int_{\Omega_i} (c_i u_i^k + \nabla\cdot(\boldsymbol{b}_i u_i^k))\,\partial_t u_i^k\,dx\,d\tau$$
$$- \int_0^t \int_{\Gamma_{i,j}} \nu_i\partial_{\boldsymbol{n}_i}u_i^k\,\partial_t u_i^k\,d\sigma\,d\tau = 0. \quad (6)$$

3. We multiply the first equation of (3) by $\widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2\boldsymbol{r}_{i,j}u_i^k)$ integrate on $\Omega_i\times(0,t)$ integrate by parts in space to obtain

$$\int_0^t \int_{\Omega_i} \partial_t u_i^k \widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2\boldsymbol{r}_{i,j}u_i^k)\,dx\,d\tau + \int_0^t \int_{\Omega_i} \nabla\cdot(\boldsymbol{b}_i u_i^k)\,\widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2\boldsymbol{r}_{i,j}u_i^k)\,dx\,d\tau$$
$$+ \int_0^t \int_{\Omega_i} c_i u_i^k \widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2\boldsymbol{r}_{i,j}u_i^k)\,dx\,d\tau - \int_0^t \int_{\Gamma_{i,j}} \nu_i\partial_{\boldsymbol{n}_i}u_i^k\,\nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}u_i^k)\,d\sigma\,d\tau$$
$$- \frac{1}{4}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\,s_{i,j}}\,\nabla\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k\|_i^2\,d\tau \le C\int_0^t (\|\sqrt{\nu_i}\nabla u_i^k\|_i^2 + \|\beta_i u_i^k\|_i^2)\,d\tau. \quad (7)$$

4. We multiply the first equation of (3) by $-\widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2 s_{i,j}\,\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k)$ integrate on $\Omega_i\times(0,t)$, integrate by parts in space using (4). Using that

$$- \int_0^t \int_{\Omega_i} \nu_i\nabla u_i^k\cdot\nabla(\widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2 s_{i,j}\,\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k))\,dx\,d\tau$$
$$\ge \frac{1}{2}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\,s_{i,j}}\,\nabla\widetilde{\nabla}_{\Gamma_{i,j}},u_i^k)\|_i^2\,d\tau - C\int_0^t \|\sqrt{\nu_i}\nabla u_i^k\|_i^2\,d\tau,$$

we obtain

$$\frac{1}{2}\|\psi_{i,j}\sqrt{s_{i,j}}\,\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k(t)\|_i^2 + \frac{1}{2}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\,s_{i,j}}\,\nabla\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k)\|_i^2\,d\tau$$
$$+ \int_0^t \int_{\Omega_i} \psi_{i,j}^2 s_{i,j}\,c_i|\widetilde{\nabla}_{\Gamma_{i,j}}u_i^k|^2\,dx\,d\tau + \int_0^t \int_{\Gamma_{i,j}} \nu_i\partial_{\boldsymbol{n}_i}u_i^k\,\nabla_{\Gamma_{i,j}}\cdot(s_{i,j}\,\nabla_{\Gamma_{i,j}}\,u_i^k)\,d\sigma\,d\tau$$
$$\le \int_0^t \int_{\Omega_i} \nabla\cdot(\boldsymbol{b}_i u_i^k)\,\widetilde{\nabla}_{\Gamma_{i,j}}\cdot(\psi_{i,j}^2 s_{i,j}\,\widetilde{\nabla}_{\Gamma_{i,j}}\,u_i^k)\,dx\,d\tau + C\int_0^t \|\sqrt{\nu_i}\nabla u_i^k\|_i^2\,d\tau. \quad (8)$$

We add (6), (7) and (8), multiply the result by q, and add it to (5). We use $ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2$ in the integral terms in the right-hand side, simplify with the left-hand side, and obtain

$$\frac{1}{2}\left(\|\beta_i u_i^k(t)\|_i^2 + q \,|||\, u_i^k(t)\, |||_i^2 + q\|\psi_{i,j}\sqrt{s_{i,j}}\,\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k(t)\|_i^2\right) + \int_0^t |||\beta_i u_i^k(\tau,\cdot)|||_i^2\, d\tau$$
$$+ \frac{q}{2}\int_0^t \|\partial_t u_i^k\|_i^2\, d\tau + \frac{q}{8}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\, s_{i,j}}\,\nabla\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k\|_i^2\, d\tau$$
$$- q\int_0^t\int_{\Gamma_{i,j}} \nu_i\partial_{\boldsymbol{n}_i} u_i^k\, \left(\partial_t u_i^k + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}u_i^k) - \nabla_{\Gamma_{i,j}}\cdot(s_{i,j}\,\nabla_{\Gamma_{i,j}}\, u_i^k)\right)\, d\sigma\, d\tau$$
$$- \int_0^t\int_{\Gamma_{i,j}} \beta_{i,j}^2(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \frac{\boldsymbol{b}_i\cdot\boldsymbol{n}_i}{2}u_i^k)\, u_i^k\, d\sigma\, d\tau \le \frac{q}{2}(\|\boldsymbol{b}_i\|_{i,1,\infty} + \|c_i\|_{i,\infty})\|u_i^k(t)\|_i^2$$
$$+ C\left(\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau + q\int_0^t \|\sqrt{\nu_i}\nabla u_i^k\|_i^2\, d\tau\right). \quad (9)$$

Recalling that $s_{i,j} = s_{j,i}$ on $\Gamma_{i,j}$ and $\boldsymbol{r}_{i,j} = \boldsymbol{r}_{j,i}$ on $\Gamma_{i,j}$, we use now the identity:

$$\left(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \boldsymbol{b}_i\cdot\boldsymbol{n}_i u_i^k + \mathcal{S}_{i,j}u_i^k\right)^2 - \left(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \boldsymbol{b}_i\cdot\boldsymbol{n}_i u_i^k - \mathcal{S}_{j,i}u_i^k\right)^2$$
$$= 4\big(\beta_{i,j}^2(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \frac{\boldsymbol{b}_i\cdot\boldsymbol{n}_i}{2}u_i^k)u_i^k + q\nu_i\partial_{\boldsymbol{n}_i}u_i^k(\partial_t u_i^k + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}u_i^k))\big)$$
$$- 4\nabla_{\Gamma_{i,j}}\cdot(s_{i,j}\,\nabla_{\Gamma_{i,j}}\, u_i^k) + 2q(p_{i,j} - p_{j,i} - 2\boldsymbol{b}_i\cdot\boldsymbol{n}_i)(\partial_t u_i^k + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}u_i^k)$$
$$- \nabla_{\Gamma_{i,j}}\cdot(s_{i,j}\,\nabla_{\Gamma_{i,j}}\, u_i^k))u_i^k + (p_{i,j}+p_{j,i})(p_{i,j}-p_{j,i}-\boldsymbol{b}_i\cdot\boldsymbol{n}_i)(u_i^k)^2. \quad (10)$$

Replacing (10) into (9), we obtain

$$\frac{1}{2}\left(\|\beta_i u_i^k(t)\|_i^2 + q \,|||\, u_i^k(t)\, |||_i^2 + q\|\psi_{i,j}\sqrt{s_{i,j}}\,\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k(t)\|_i^2\right) + \int_0^t |||\beta_i u_i^k(\tau,\cdot)|||_i^2\, d\tau$$
$$+ \frac{q}{2}\int_0^t \|\partial_t u_i^k\|_i^2\, d\tau + \frac{1}{4}\int_0^t\int_{\Gamma_{i,j}} \left(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \boldsymbol{b}_i\cdot\boldsymbol{n}_i\, u_i^k - \mathcal{S}_{j,i}u_i^k\right)^2\, d\sigma\, d\tau$$
$$+\frac{q}{8}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\, s_{i,j}}\,\nabla\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k\|_i^2\, d\tau \le \frac{1}{4}\int_0^t\int_{\Gamma_{i,j}} \left(\nu_i\partial_{\boldsymbol{n}_i}u_i^k - \boldsymbol{b}_i\cdot\boldsymbol{n}_i\, u_i^k + \mathcal{S}_{i,j}u_i^k\right)^2\, d\sigma\, d\tau$$
$$+\int_0^t\int_{\Gamma_{i,j}} (p_{i,j}+p_{j,i})(-p_{i,j}+p_{j,i}+\boldsymbol{b}_i\cdot\boldsymbol{n}_i)(u_i^k)^2\, d\sigma\, d\tau + \frac{q}{2}(\|\boldsymbol{b}_i\|_{i,1,\infty}+\|c_i\|_{i,\infty})\|u_i^k(t)\|_i^2$$
$$+\frac{q}{2}\int_0^t\int_{\Gamma_{i,j}} (-p_{i,j}+p_{j,i}+2\boldsymbol{b}_i\cdot\boldsymbol{n}_i)(\partial_t u_i^k + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}u_i^k) - \nabla_{\Gamma_{i,j}}\cdot(s_{i,j}\,\nabla_{\Gamma_{i,j}}\, u_i^k))\, u_i^k\, d\sigma\, d\tau$$
$$+ C\left(\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau + q\int_0^t \|\sqrt{\nu_i}\nabla u_i^k\|_i^2\, d\tau\right). \quad (11)$$

In order to estimate the fourth term in the right-hand side of (11), we observe that

$$\int_0^t\int_{\Gamma_{i,j}} (-p_{i,j}+p_{j,i}+2\boldsymbol{b}_i\cdot\boldsymbol{n}_i)u_i^k\,\partial_t u_i^k\, d\sigma\, d\tau = \frac{1}{2}\int_{\Gamma_{i,j}} (-p_{i,j}+p_{j,i}+2\boldsymbol{b}_i\cdot\boldsymbol{n}_i)u_i^k(t)^2\, d\sigma.$$

By the trace theorem in the right-hand side, we write:

$$\int_0^t\int_{\Gamma_{i,j}} (-p_{i,j} + p_{j,i} + 2\boldsymbol{b}_i\cdot\boldsymbol{n}_i)u_i^k\,\partial_t u_i^k\, d\sigma\, d\tau \le C\|u_i^k(t)\|_i\|\sqrt{\nu_i}\nabla u_i^k(t)\|_i,$$

and

$$\|u_i^k(t)\|_i^2 = 2\int_0^t \int_{\Omega_i} (\partial_t u_i^k) u_i^k \le 2\left(\int_0^t \|\partial_t u_i^k\|_i^2\right)^{\frac{1}{2}} \left(\int_0^t \|u_i^k\|_i^2\right)^{\frac{1}{2}}, \quad (12)$$

we obtain

$$\frac{q}{2}\int_0^t \int_{\Gamma_{i,j}} (-p_{i,j} + p_{j,i} + 2\boldsymbol{b}_i \cdot \boldsymbol{n}_i) u_i^k\, \partial_t u_i^k\, d\sigma\, d\tau \\ \le \frac{q}{8}\int_0^t \|\partial_t u_i^k\|_i^2\, d\tau + \frac{q}{4} \interleave u_i^k(t) \interleave_i^2 + C\left(\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau\right). \quad (13)$$

Moreover, integrating by parts and using the trace theorem, we have:

$$-\frac{q}{2}\int_0^t \int_{\Gamma_{i,j}} \nabla_{\Gamma_{i,j}} \cdot (s_{i,j}\, \nabla_{\Gamma_{i,j}}\, u_i^k)(-p_{i,j} + p_{j,i} + 2\boldsymbol{b}_i \cdot \boldsymbol{n}_i) u_i^k\, d\sigma\, d\tau \\ \le \frac{q}{16}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\, s_{i,j}}\, \nabla\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k\|_i^2\, d\tau \\ + C(\int_0^t \|\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k\|_i^2\, d\tau + \int_0^t \|\beta_i u_i^k\|_i^2\, d\tau). \quad (14)$$

Using (12), we estimate the third term in the right-hand side of (11) by:

$$\frac{q}{2}(\|\boldsymbol{b}_i\|_{i,1,\infty} + \|c_i\|_{i,\infty})\|u_i^k(t)\|_i^2 \le \frac{q}{8}\int_0^t \|\partial_t u_i^k\|_i^2\, d\tau + C\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau. \quad (15)$$

Replacing (14), (13) and (15) in (11), then using the transmission conditions, we have:

$$\frac{1}{2}\left(\|\beta_i u_i^k(t)\|_i^2 + \frac{q}{2} \interleave u_i^k(t) \interleave_i^2 + q\|\psi_{i,j}\sqrt{s_{i,j}}\,\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k(t)\|_i^2\right) \\ + \int_0^t \interleave \beta_i u_i^k(\tau,\cdot)\interleave_i^2\, d\tau + \frac{q}{4}\int_0^t \|\partial_t u_i^k\|_i^2\, d\tau + \frac{q}{16}\int_0^t \|\psi_{i,j}\sqrt{\nu_i\, s_{i,j}}\,\nabla\,\widetilde{\nabla}_{\Gamma_{i,j}}\, u_i^k\|_i^2\, d\tau \\ + \frac{1}{4}\int_0^t \int_{\Gamma_{i,j}} \left(\nu_i \partial_{\boldsymbol{n}_i} u_i^k - \boldsymbol{b}_i \cdot \boldsymbol{n}_i\, u_i^k - \mathcal{S}_{j,i} u_i^k\right)^2\, d\sigma\, d\tau \\ \le \frac{1}{4}\int_0^t \int_{\Gamma_{i,j}} \left(\nu_j \partial_{\boldsymbol{n}_i} u_j^{k-1} - \boldsymbol{b}_j \cdot \boldsymbol{n}_i\, u_j^{k-1} + \mathcal{S}_{i,j} u_j^{k-1}\right)^2\, d\sigma\, d\tau \\ + C\left(\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau + \frac{q}{2}\int_0^t \|\sqrt{\nu_i}\,\nabla u_i^k\|_i^2\, d\tau\right).$$

We now sum up over the interfaces $j \in \mathcal{N}_i$, then over the subdomains for $1 \le i \le I$, and over the iterations for $1 \le k \le K$, the boundary terms cancel out, and with $\alpha(t) = \frac{1}{4}\sum_{i\in\{1,\ldots,I\}}\sum_{j\in\mathcal{N}_i}\int_0^t \int_{\Gamma_{i,j}} \left(\nu_j \partial_{\boldsymbol{n}_i} u_i^0 - \boldsymbol{b}_j \cdot \boldsymbol{n}_i\, u_j^0 + \mathcal{S}_{i,j} u_j^0\right)^2\, d\sigma\, d\tau$, we obtain for any $t \in (0,T)$,

$$\sum_{k\in\{1,\ldots,K\}}\sum_{i\in\{1,\ldots,I\}} \left(\|\beta_i u_i^k(t)\|_i^2 + \frac{q}{2}\|\sqrt{\nu_i}\,\nabla u_i^k(t)\|_i^2 + \nu_0\int_0^t \|\nabla(\beta_i u_i^k)\|_i^2\, d\tau\right) \\ \le \alpha(t) + C\sum_{k\in\{1,\ldots,K\}}\sum_{i\in\{1,\ldots,I\}} \left(\int_0^t \|\beta_i u_i^k\|_i^2\, d\tau + \frac{q}{2}\int_0^t \|\sqrt{\nu_i}\,\nabla u_i^k\|_i^2\, d\tau\right).$$

We conclude with Gronwall's lemma that the sequence converges in $L^2(0,T;H^1(\Omega_i))$.

3 Numerical Results

We recall the discrete time nonconforming Schwarz waveform relaxation method developed in [7].

Let $\mathcal{T}_i$ be the time partition in subdomain Ω_i, with N_i+1 intervals I_n^i, and time step k_n^i. We define interpolation operators $\mathcal{I}^i$ and projection operators $\mathcal{P}^i$ in each subdomain as in [7], and we solve

$$\begin{aligned}
&\partial_t(\mathcal{I}^i U_i^k) + \nabla\cdot(\boldsymbol{b}U_i^k - \nu_i\nabla U_i^k) + c_i\, U_i^k = \mathcal{P}^i f \text{ in } \Omega_i\times(0,T),\\
&\big(\nu_i\partial_{\boldsymbol{n}_i} - \frac{\boldsymbol{b}_i\cdot\boldsymbol{n}_i}{2}\big)\, U_i^k + S_{i,j}U_i^k = \\
&\qquad \mathcal{P}^i\big((\nu_j\partial_{\boldsymbol{n}_i} - \frac{\boldsymbol{b}_j\cdot\boldsymbol{n}_i}{2})\, U_j^{k-1} + \widetilde{S}_{i,j}U_j^{k-1}\big) \text{ on } \Gamma_{i,j}\times(0,T),
\end{aligned}$$

with $S_{i,j}\,U = p_{i,j}\,U + q_{i,j}\,(\partial_t(\mathcal{I}^i U) + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}U - s_{i,j}\nabla_{\Gamma_{i,j}}U))$, and $\widetilde{S}_{i,j}\,U = p_{i,j}\,U + q_{i,j}\,(\partial_t(\mathcal{I}^j U) + \nabla_{\Gamma_{i,j}}\cdot(\boldsymbol{r}_{i,j}U - s_{i,j}\nabla_{\Gamma_{i,j}}U))$.

The coefficients $p_{i,j}$ and $q_{i,j}$ are defined through an optimization procedure, see [1], restricted to values such that the subdomain problems are well-posed. The time semi-discrete analysis was performed in [7] in the case $\nabla\cdot\boldsymbol{b} = 0$. For the space discretization, we use the nonconforming approach in [5] extended to problem (1) and order 2 transmission conditions, to allow non-matching grids in time and space on the boundary. We have implemented the algorithm with $\mathbf{P}_1$ finite elements in space in each subdomain. Time windows are used in order to reduce the number of iterations of the algorithm. To reduce the number of parameters and following [1], we choose $\boldsymbol{r}_{i,j} = \boldsymbol{\Pi}_{\Gamma_{i,j}}\boldsymbol{b}_j$ with $\boldsymbol{\Pi}_{\Gamma_{i,j}}$ the tangential trace on $\Gamma_{i,j}$, and $s_{i,j} = \nu_j$ (even though the present analysis does not cover this case). The optimized parameters are constant along the interface. They correspond to a mean value of the parameters obtained by a numerical optimization of the convergence factor.

We first give an example of a multidomain solution with one time window. The physical domain is $\Omega = (0,1)\times(0,2)$, the final time is $T = 1$. The initial value is $u_0 = 0.25e^{-100((x-0.55)^2+(y-1.7)^2)}$ and the right-hand side is $f = 0$. The domain Ω is split into two subdomains $\Omega_1 = (0,0.5)\times(0,2)$ and $\Omega_2 = (0.5,1)\times(0,2)$. The reaction factor c is zero, the advection and diffusion coefficients are $\boldsymbol{b}_1 = (0,-1)$, $\nu_1 = 0.001\sqrt{y}$, and $\boldsymbol{b}_2 = (-0.1,0)$, $\nu_2 = 0.1\sin(xy)$. The mesh size over the interface and time step in Ω_1 are $h_1 = 1/32$ and $k_1 = 1/128$, while in Ω_2, $h_2 = 1/24$ and $k_2 = 1/94$. On Fig. 2, we observe, at final time $T = 1$, a very good behavior of the multidomain solution after 5 iterations. The relative error with the one domain solution is of the same order as the error of the scheme.

We analyze now the precision in time. The space mesh is conforming and the converged solution is such that the residual is smaller than 10^{-8}. We compute a variational reference solution on a time grid with 4,096 time steps. The nonconforming solutions are interpolated on the previous grid to compute the error. We start with a time grid with 128 time steps for the left domain and 94 time steps for the right domain. Thereafter the time steps are divided by 2 several times. Figure 3 (left) shows

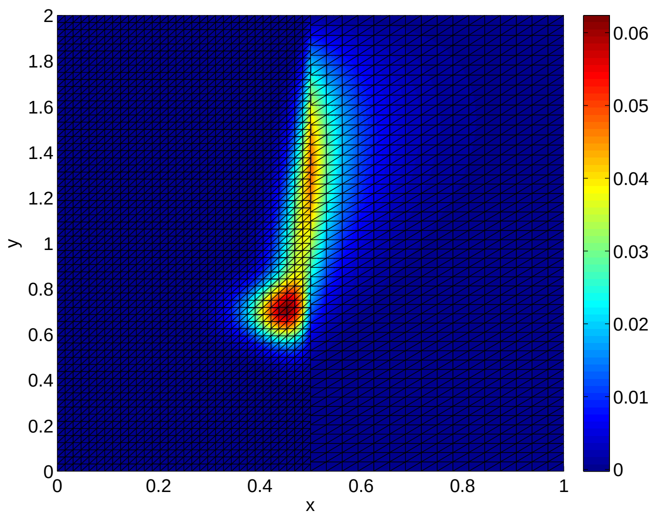

Fig. 2. Nonconforming DG-OSWR solution after 5 iterations.

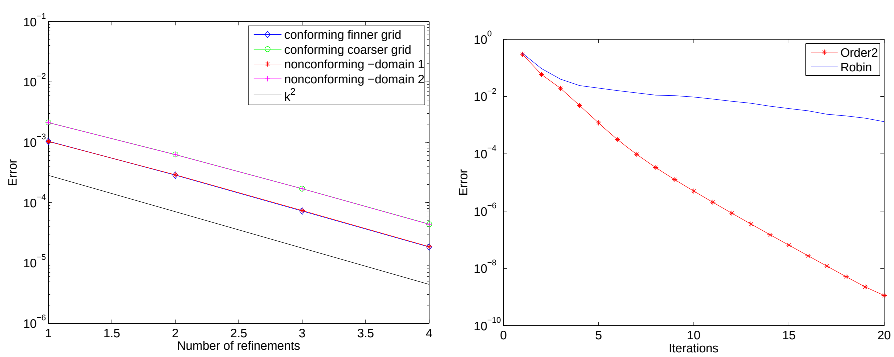

Fig. 3. Error between variational and DG-OSWR solutions versus the refinement in time (*left*), and versus the iterations (*right*).

the norms of the error in $L^\infty(I; L^2(\Omega_i))$ versus the number of refinements, for both subdomains. First we observe the order 2 in time for the nonconforming case. This fits the theoretical estimates, even though we have theoretical results only for Robin transmission conditions. Moreover, the error obtained in the nonconforming case, in the subdomain where the grid is finer, is nearly the same as the error obtained in the conforming finer case.

The computations are done using Order 2 transmissions. Indeed, the error between the multidomain and variational solutions decreases much faster with the Order 2 transmissions conditions than with the Robin transmissions conditions as we can see on Fig. 3 (right), in the conforming case.

We now consider advection-diffusion equations with discontinuous porosity:

$$\omega\partial_t u + \nabla\cdot(\boldsymbol{b}u - \nu\nabla u) = 0.$$

The physical domain is $\Omega = (0,1)\times(0,2)$, the final time is $T = 1.5$. The initial value is $u_0 = 0.5e^{-100((x-0.7)^2+(y-1.5)^2)}$. Domain Ω is split into two subdomains $\Omega_1\times(0,1.5)$ and $\Omega_2\times(0,1.5)$ with $(\frac{1}{2}-\frac{sin(2\pi s)}{8}, 2s)$, $0 < s < 1$ a parametrization of the interface, as in Fig. 4. The advection and diffusion coeffi-

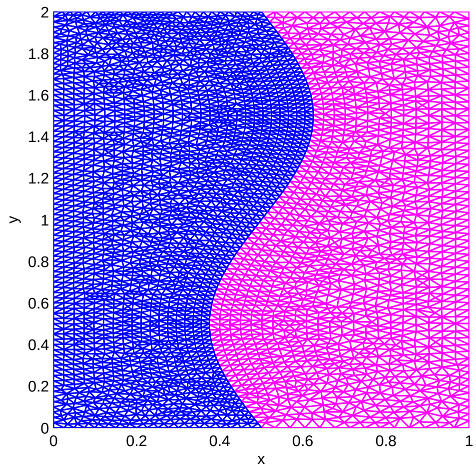

Fig. 4. Domain decomposition with Ω_1 (*left*) and Ω_2 (*right*).

cients are $\boldsymbol{b}_1 = (-sin(\frac{\pi}{2}(y-1))cos(\pi(x-\frac{1}{2})), 3cos(\frac{\pi}{2}(y-1))sin(\pi(x-\frac{1}{2})))$, $\nu_1 = 0.003$, $\omega_1 = 0.1$, and $\boldsymbol{b}_2 = \boldsymbol{b}_1$, $\nu_2 = 0.01$, $\omega_2 = 1$. We consider first a conforming grid in space. The mesh size over the interface is $h = 1/104$ and time step in Ω_1 is $k_1 = 1/128$, while in Ω_2, $k_2 = 1/94$. On Fig. 5, we observe, at final time $T = 1.5$, that the approximate solution computed using ten time windows and 3 iterations in each time window is close to the variational solution computed in one time window on the conforming finer space-time grid as shown on the error. We now consider nonconforming grids in space as shown on Fig. 4. The mesh size over the interface and time step in Ω_1 are $h_1 = 1/104$ and $k_1 = 1/128$, while in Ω_2, $h_2 = 1/81$ and $k_2 = 1/94$. On Fig. 6, we observe, at final time $T = 1.5$, that the approximate solution computed using 5 iterations in one time window is close to the variational solution computed on the conforming finer space-time grid. On Fig. 7 we observe the precision versus the mesh size and time step. The converged solution is such that the residual is smaller than 10^{-8}. A variational reference solution is computed on a time grid with 2,048 time steps and 384 mesh grid. The space-time nonconforming solutions are interpolated on the previous grid to compute the error. We start with a time grid with 32 time steps and 24 mesh size for the left domain and time steps 12 and 12 mesh size for the right domain and divide by 2 the time step and mesh size several times. Figure 7 shows the norms of the error in $L^2(I; L^2(\Omega_i))$ versus the time steps, for both subdomains. We observe the order 2 for the nonconforming space-time case, even though we have theoretical results only for the time semi-discretized case in [7].

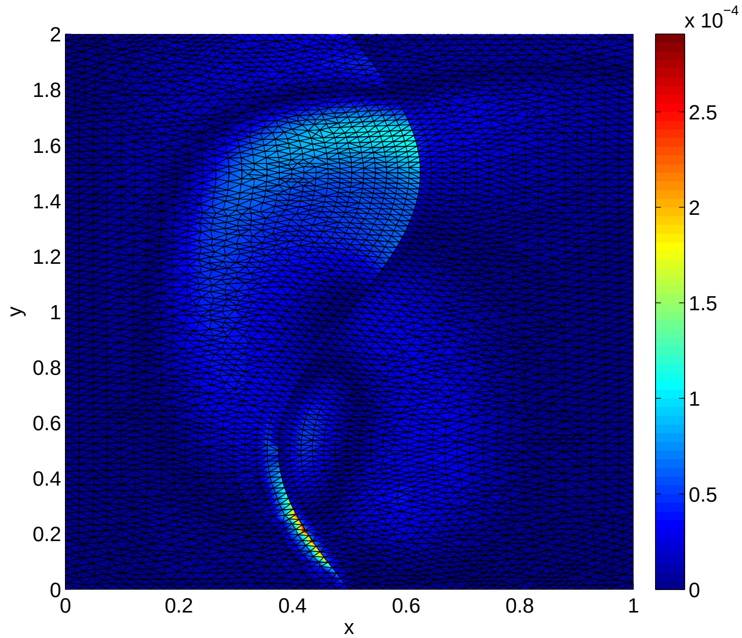

Fig. 5. Error between variational and DG-OSWR solutions, at final time, after 10 time windows and 3 iterations per window.

4 Conclusions

We have analyzed the continuous algorithm for variable discontinuous coefficients and general decompositions. We have shown numerically that the method preserves the order of the one domain scheme in the case of discontinuous variable coefficients, nonconforming grids in space and time and a curved interface. An analysis of the influence of the decomposition in time windows is in progress.

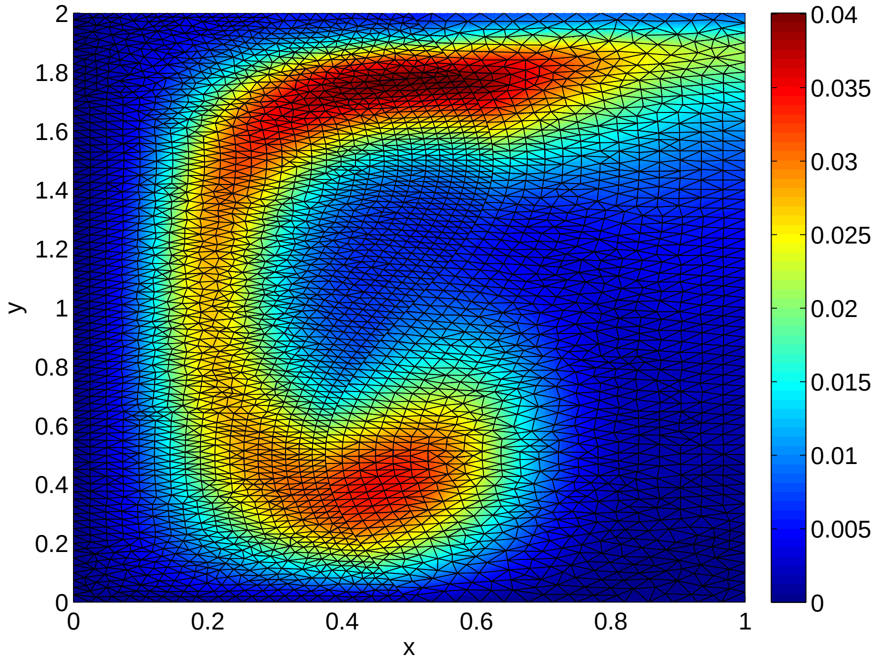

Fig. 6. DG-OSWR solution at final time, after 5 iterations.

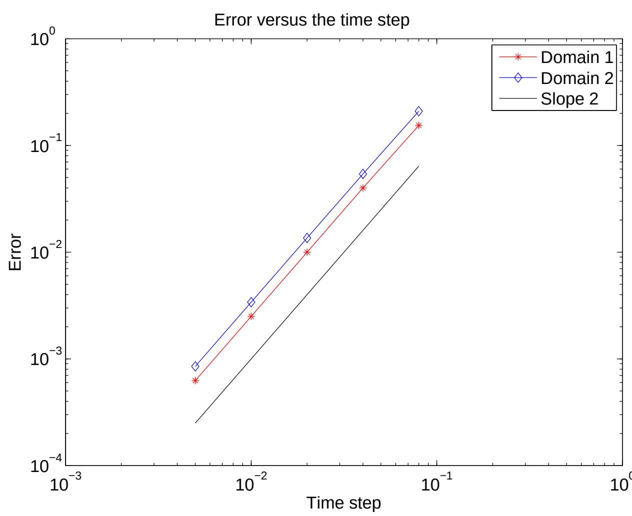

Fig. 7. Error curves versus the time step.

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